Weak Convergence to Stochastic Integrals

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Outline

1 Introduction

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Most of statistical inferences are often associated with limit theorems for random variables or stochastic processes.
Weak convergence of stochastic processes is a very important and foundational theory in probability and statistics.
Billingsley (1999) gave a systematic classical theory of weak convergence for stochastic processes.

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When the finite dimensional distributions of limiting processes are difficult to compute, this method can not be easily used. An alternative method is martingale convergence method. This method is based on the Martingale Problem of Semimartingales. When the limiting processes are semimartingales, martingale convergence method is effective.

The idea of martingale convergence method originate in the work of Stroock and Varadhan (Stroock and Varadhan (1969)). In Jacod and Shiryaev (2003), they gave a whole system of this method. When limiting process is a jump-process, their results are very powerful.

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Recently, martingale convergence method is usually used in the study of convergence of processes, such as discretized processes and statistics for high frequency data. (c.f. Jacod (2008), Aït-Sahalia and Jacod (2009), Fan and Fan (2011)).

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Recently, martingale convergence method is usually used in the study of convergence of processes, such as discretized processes and statistics for high frequency data. (c.f. Jacod (2008), Aït-Sahalia and Jacod (2009), Fan and Fan (2011)). However, it is rarely used in the contents of time series.

To our knowledge, Ibragimov and Phillips (2008) firstly introduced the martingale convergence method to the study of time series. They studied the weak convergence of various general functionals of partial sums of linear processes. The limiting process is a stochastic integral. Their result was used in the study for unit root theory.

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When the limiting processes are jump-processes, their methods may be useless, since they need that the residual in martingale decomposition is neglected, which may not be satisfied in the jump case.

Now, we extend Ibragimov and Phillips's results to two directions. One is about causal processes.

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Causal process $\{X_n, n \geq 1\}$ is defined by

$$X_n = g(\cdots, \varepsilon_{n-1}, \varepsilon_n),$$

where $\{\varepsilon_n; n \in Z\}$ is a sequence of i.i.d. r.v.'s with mean zero and g is a measurable function.

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where $\{\varepsilon_n; n \in Z\}$ is a sequence of i.i.d. r.v.'s with mean zero and g is a measurable function. It contains many important stochastic models, such as linear process, ARCH model, threshold AR (TAR) model and so on.

Wiener (1958) conjectured that for every stationary ergodic process $\{X_n, n \geq 1\}$, there exists a function g and i.i.d. $\varepsilon_n, n \geq 1$ such that

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We extend Ibregimov and Phillips's work to another direction: $\{X_n\}$ is a sequence of heavy-tailed random variables.

We discuss an important class of heavy-tailed random variables. A random variable X, the so-called regularly varying random variables with index $\alpha \in (0, 2)$ if there exists a positive parameter α such that

$$\lim_{x \to \infty} \frac{P(X > cx)}{P(X > x)} = c^{-\alpha}, \quad c > 0.$$

We study weak convergence of functionals of the i.i.d. regularly varying random variables. The limiting process is the stochastic integral driven by $\alpha-$stable Lévy process.

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However, if we want to extend Ibragimov and Phillips's results to the heavy-tailed random variables, the point process method can not be used easily, since the limiting process has jumps.

Because the limiting process has jumps, we have to modify the proof procedure in Ibragimov and Phillips (2008). Our method is not only effective for the jump case but also simpler than that of Ibragimov and Phillips (2008)(only for the continuous case).

We use the martingale approximation procedure, strong approximation of martingale and the martingale convergence method to prove our theorems.

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Martingale approximation procedure. Let $\{X_n\}$ be a sequence of random variables, $S_n = \sum_{k=1}^{n} X_k$. We can structure a martingale M_n, If the error $S_n - M_n$ is small enough in some sense, we can consider M_n instead of S_n.

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Strong approximation of martingale. Let M_n, $n = 1, 2, \cdots$, be a sequence of martingales. Under some conditions, we can find a Brownian motion (or Gaussian process) B such that $M_n - B \to 0$ a.s. with some rate.

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Theorem A (Jacod and Shiryaev (2003)) Let Y^n be a càdlàg process and M^n be a martingale on a same filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\geq 1}, P)$. Let M be a càdlàg process defined on the canonical space $(\mathbb{D}([0,1]), \mathscr{D}([0,1]), \mathbf{D})$. Assume that

(i) (M^n) is uniformly integrable;

(ii) $Y^n \Rightarrow Y$ for some Y with law $\widetilde{P} = \mathscr{L}(Y)$;

(iii)

$$M_t^n - M_t \circ (Y^n) \xrightarrow{P} 0, \quad 0 \leq t \leq 1$$

Then the process $M \circ (Y)$ is a martingale under $\widetilde{P}$.

Let M^n, $n = 1, 2, \cdots$, be a sequence of martingales. The predictable characteristic of M^n is a triplet (B_n, C_n, ν_n). If there exists limit (B, C, ν), then one can identify the limiting process M by (B, C, ν).

2 Convergence to Stochastic Integral Driven by Brownian Motion

Recall that $Z \in L^p$ $(p > 0)$ if $||Z||_p = [E(|Z|^p)]^{1/p} < \infty$ and write $||Z|| = ||Z||_2$.
To study the asymptotic property of the sums of causal process

$$X_n = g(\cdots, \varepsilon_{n-1}, \varepsilon_n),$$

martingale approximation is an effective method. We list the notations used in the following part:

- $\mathscr{F}_k = \sigma(\cdots, \varepsilon_{k-1}, \varepsilon_k)$.
- Projections $P_k Z = E(Z|\mathscr{F}_k) - E(Z|\mathscr{F}_{k-1})$, $Z \in L^1$.
- $D_k = \sum_{i=k}^{\infty} P_k X_i$, $M_k = \sum_{i=1}^{k} D_i$.

M_k is a martingale, we will use M_k to approximate sum S_k.

- $\theta_{n,p} = ||P_0 X_n||_p$, $\Lambda_{n,p} = \sum_{i=0}^{n} \theta_{i,p}$, $\Theta_{m,p} = \sum_{i=m}^{\infty} \theta_{i,p}$.
- B: standard Brownian motion.

Assumption 1. $X_0 \in L^q$, $q \geq 4$, and $\Theta_{n,q} = O(n^{-1/4}(\log n)^{-1})$.
Assumption 2.

$$\sum_{k=1}^{\infty} ||E(D_k^2|\mathscr{F}_0) - \sigma^2||_2 < \infty,$$

where $\sigma = ||D_k||$.
Assumption 3.

$$\sum_{k=0}^{\infty}\sum_{i=1}^{\infty} ||E(X_k X_{k+i}|\mathscr{F}_0) - E(X_k X_{k+i}|\mathscr{F}_{-1})||_4 < \infty,$$

and

$$||\sum_{k=0}^{\infty}\sum_{i=1}^{\infty} E(X_k X_{k+i}|\mathscr{F}_0)||_3 < \infty.$$

Remark 1 If we consider linear process to replace the causal process, Assumptions $1 \sim 3$ can easily be implied by the conditions in Ibragimov and Phillips (2008).

Theorem 1 Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function satisfying $|f'(x)| \leq K(1 + |x|^\alpha)$ for some positive constants K and α and all $x \in \mathbb{R}$. Suppose that X_t is a causal process satisfying Assumptions 1∼3. Then

$$\frac{1}{\sqrt{n}} \sum_{t=2}^{[n\cdot]} f(\frac{1}{\sqrt{n}} \sum_{i=1}^{t-1} X_i) X_t \Rightarrow \lambda \int_0^\cdot f'(B(v))dv + \sigma \int_0^\cdot f(B(v))dB(v), \tag{2.1}$$

where $\lambda = \sum_{j=1}^{\infty} EX_0 X_j$.

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Let

$$Y_n = \alpha Y_{n-1} + X_n,$$

where $\{X_n\}_{n \geq 0}$ is a causal process, and want to estimate α from $\{Y_t\}$.

Let

$$\hat{\alpha} = \frac{\sum_{t=1}^{n} Y_{t-1} Y_t}{\sum_{t=1}^{n} Y_{t-1}^2}$$

denote the ordinary least squares estimator of α.
Let t_α be the regression $t-$statistic:

$$t_\alpha = \frac{(\sum_{t=1}^{n} Y_{t-1}^2)^{\frac{1}{2}} (\hat{\alpha} - 1)}{\sqrt{\frac{1}{n} \sum_{t=1}^{n} (Y_t - \hat{\alpha} Y_{t-1})}}.$$

Using Theorem 1 with $f(x) = x$, we get the asymptotic distribution of $n(\hat{\alpha} - 1)$ and t_α as follows.

Theorem 2
Under Assumptions 1-3, we have

$$n(\hat{\alpha}-1) \xrightarrow{d} \frac{\lambda+\sigma^2\int_0^1 B(v)dB(v)}{\sigma^2\int_0^1 B^2(v)dv}, \tag{2.1}$$

$$t_\alpha \xrightarrow{d} \frac{\lambda+\sigma^2\int_0^1 B(v)dB(v)}{(\int_0^1 B^2(v)dv)^{\frac{1}{2}}}. \tag{2.2}$$

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We can construct the confidence interval of α for the unit root testing.

3 Convergence to Stochastic Integral Driven by Lévy $\alpha-$Stable Process

Let $E = [-\infty, \infty] \backslash \{0\}$ and $M_p(E)$ be the set of Radon measures on E with values in $\mathbb{Z}_+$, the set of positive integers.

3 Convergence to Stochastic Integral Driven by Lévy $\alpha-$Stable Process

Let $E = [-\infty, \infty]\backslash\{0\}$ and $M_p(E)$ be the set of Radon measures on E with values in $\mathbb{Z}_+$, the set of positive integers.
For $\mu_n, \mu \in M_p(E)$, we say that μ_n vaguely converge to the measure μ, if

$$\mu_n(f) \to \mu(f)$$

for any $f \in C_K^+$, where C_K^+ is the class of continuous functions with compact support, denoted by $\mu_n \xrightarrow{v} \mu$.

Theorem 3 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable function such that

$$|f(x) - f(y)| \leq K|x - y|^a \tag{3.1}$$

for some positive constants K, a and all $x, y \in \mathbb{R}$. Suppose that $\{X_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables. Set

$$X_{n,j} = \frac{X_j}{b_n} - \mathbb{E}(h(\frac{X_j}{b_n})) \tag{3.2}$$

for some $b_n \to \infty$, where $h(x)$ is a continuous function satisfying $h(x) = x$ in a neighbourhood of 0 and $|h(x)| \leq |x|1_{|x|\leq 1}$.

Define ρ by

$$\rho((x,+\infty]) = px^{-\alpha}, \quad \rho([-\infty,-x)) = qx^{-\alpha} \tag{3.3}$$

for $x > 0$, where $\alpha \in (0,1)$, $0 < p < 1$ and $p + q = 1$. Then

$$\sum_{i=2}^{[n\cdot]} f(\sum_{j=1}^{i-1} X_{n,j})X_{n,i} \Rightarrow \int_0^{\cdot} f(Z_\alpha(s-))dZ_\alpha(s), \tag{3.4}$$

in $\mathbb{D}[0,1]$, where $Z_\alpha(s)$ is an $\alpha-$stable Lévy process with Lévy measure ρ iff

$$n\mathbb{P}[\frac{X_1}{b_n} \in \cdot] \xrightarrow{v} \rho(\cdot) \tag{3.5}$$

in $M_p(E)$.

Remark 3 Condition (3.5) implies that X_i is regularly varying random variable. b_n is the the normalization factor, it is determined by the quantile of X_i.

Remark 4 To discuss the weak convergence of heavy-tailed random variables, X_1 is usually assumed to be symmetric, but we don't have such assumption. In order to use the martingale convergence method, we study the asymptotic properties of

$$X_{n,j} = \frac{X_j}{b_n} - \mathbb{E}(h(\frac{X_j}{b_n}))$$

instead of $\frac{X_j}{b_n}$.

Remark 5 The limiting process in Theorem 2 is stochastic integral driven by $\alpha-$stable Lévy process. The main difference between this result and Theorem 1 is the continuity of the limiting process. For the heavy-tailed case, the limiting process is discontinuous. It will be more complex than the continuous case, we should modify the martingale convergence method.

Remark 6 When X_n is a stationary sequence instead of an i.i.d. sequence, we have the following theorem.

Theorem 4 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable function satisfying (3.1), for some constants $K > 0$, $a > 0$ and all $x, y \in \mathbb{R}$. Suppose that $\{X_n\}_{n\geq 1}$ is a sequence of stationary random variables, defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Set

$$X_{n,j} = \frac{X_j}{b_n} - \mathbb{E}(h(\frac{X_j}{b_n})|\mathscr{F}_{j-1}) \tag{3.6}$$

for some $b_n \to \infty$. Define ρ as (3.3) for $\alpha \in (0, 1)$.

Then

$$\sum_{i=2}^{[n\cdot]} f\Big(\sum_{j=1}^{i-1} X_{n,j}\Big) X_{n,i} \Rightarrow \int_0^{\cdot} f(Z_\alpha(s-))dZ_\alpha(s), \tag{3.7}$$

in $\mathbb{D}[0,1]$, where $Z_\alpha(s)$ is an $\alpha-$stable Lévy process with Lévy measure ρ if

$$\sum_{j=1}^{[nt]} \mathbb{P}[X_{n,j} > x | \mathscr{F}_{j-1}] \xrightarrow{\mathbb{P}} t\rho(x,\infty) \quad \text{if} \quad x > 0 \tag{3.8}$$

and

$$\sum_{j=1}^{[nt]} \mathbb{P}[X_{n,j} < x | \mathscr{F}_{j-1}] \xrightarrow{\mathbb{P}} t\rho(-\infty,x) \quad \text{if} \quad x < 0. \tag{3.9}$$

Remark 7 The condition (3.5) is crucial for the proof of Theorem 3, it depicts the convergence of compensator jump measure. Conditions (3.8), and (3.9) also depict the the convergence of compensator jump measure respectively, so the proofs of Theorem 3 is similar based on the convergence of compensator jump measure.

The outline of identifying the limiting process

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Set

$$Y_n(t) = \sum_{i=2}^{[nt]} f(\sum_{j=1}^{i-1} X_{n,j}) X_{n,i},$$

$$Y(t) = \int_0^t f(Z_\alpha(s-)) dZ_\alpha(s),$$

$$S_n(t) = \sum_{i=1}^{[nt]} X_{n,i}.$$

We set

$$\mu_n(\omega; ds, dx) = \sum_{i=1}^{n} \varepsilon_{(\frac{i}{n}, \frac{X_i(\omega)}{b_n})}(ds, dx),$$

then

$$\nu_n(\omega; ds, dx) := \sum_{i=1}^{n} \varepsilon_{(\frac{i}{n})}(ds)\mathbb{P}(\frac{X_i}{b_n} \in dx)$$

is the compensator of μ_n by the independence of $\{X_i\}_{i\geq 1}$. Set

$$\zeta_n(\omega; ds, dx) = \sum_{i=1}^{n} \varepsilon_{(\frac{i}{n}, \frac{X_i(\omega)}{b_n} - c_n)}(ds, dx),$$

we have

$$\varphi_n(\omega; ds, dx) := \sum_{i=1}^{n} \varepsilon_{(\frac{i}{n})}(ds)\mathbb{P}(\frac{X_i}{b_n} - c_n \in dx)$$

is the compensator of $\zeta_n(\omega; ds, dx)$, where $c_n = \mathbb{E}[h(\frac{X_1}{b_n})]$.

For $S_n(t)$,

$$
\begin{aligned}
S_n(t) &= \int_0^t \int h(x)(\mu_n(ds, dx) - \nu_n(ds, dx)) + \sum_{i=1}^{[nt]} (\frac{X_i}{b_n} - h(\frac{X_i}{b_n})) \\
&=: \widetilde{S}_n(t) + \sum_{i=1}^{[nt]} (\frac{X_i}{b_n} - h(\frac{X_i}{b_n})).
\end{aligned}
$$

The predictable characteristics of $\widetilde{S}_n(t)$ are

$$
B_n^2(t) = 0,
$$

$$
C_n^{22}(t) = \int_0^t \int h^2(x)\nu_n(ds, dx) - \sum_{s \leq t} (\int h(x)\nu_n(\{s\}, dx))^2.
$$

For $Y_n(t)$,

$$\begin{aligned}
& Y_n(t) \\
= \quad & \int_0^t \int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)(\mu_n(ds,dx) - \nu_n(ds,dx)) \\
& + \int_0^t \int (h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x) - f(\sum_{j=1}^{[ns]-1} X_{n,j})h(x))\nu_n(ds,dx) \\
& + \sum_{i=2}^{[nt]} (f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n} - h(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n})) \\
=: \quad & \widetilde{Y}_n(t) + \sum_{i=2}^{[nt]} (f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n} - h(f(\sum_{j=1}^{i-1} X_{n,j})\frac{X_i}{b_n})).
\end{aligned}$$

The predictable characteristics of $\widetilde{Y}_n(t)$ are

$$B_n^1(t) = \int_0^t \int (h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x) - f(\sum_{j=1}^{[ns]-1} X_{n,j})h(x))\nu_n(ds, dx),$$

$$C_n^{11}(t) = \int_0^t \int h^2(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)\nu_n(ds, dx) - \sum_{s \le t}(\int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)$$

$$\begin{aligned} C_n^{12}(t) &= C_n^{21}(t) \\ &= \int_0^t \int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)h(x)\nu_n(ds, dx) \\ &\quad - \sum_{s \le t}(\int h(f(\sum_{j=1}^{[ns]-1} X_{n,j})x)\nu_n(\{s\}, dx))(\int h(x)\nu_n(\{s\}, dx)) \end{aligned}$$

We need to prove
$(B_n, C_n, \lambda_n) - (B \circ S_n(t), C \circ S_n(t), \lambda \circ S_n(t)) \xrightarrow{\mathbb{P}} 0$(c.f.Theorem A), where λ_n is the compensated jump measure of $\widetilde{Y}_n$. It implies the tightness of $\widetilde{Y}_n$, which means that the subsequence of distribution of $\widetilde{Y}_n$ weakly converges to a limit. The predictable characteristics of different limiting processes are (B, C, λ) by the Theorem A. Furthermore, since (3.1), the martingale problem $\varsigma(\sigma(Y_0), Y|\mathcal{L}_0, B, C, \lambda)$ has unique solution, $\widetilde{\mathbb{P}}$, by Theorem 6.13 in Applebaum (2009). We obtain the limiting process is unique, On the other hand, the predicable characteristics of $\int_0^{\cdot} f(Z_\alpha(s-))dZ_\alpha(s)$ under $\mathbb{P}$ are (B, C, λ). We can identify the limiting process, $\int_0^{\cdot} f(Z_\alpha(s-))dZ_\alpha(s)$, under $\mathbb{P}$.

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Thanks !