On Gaussian Fluctuations for Deformed Wigner Matrices

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This talk is to mainly study the following deformed random matrices:

$$H_{n,\alpha} = \frac{X_n}{\sqrt{n}} + \frac{1}{n^{\alpha/2}} \begin{pmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_n \end{pmatrix}$$

where X_n is standard Wigner matrix and is independent of the ξ_i's, and $0 < \alpha < 1$.

The focus is on the influence of the perturbation matrix upon the eigenvalues of X_n

Outline of the Talk

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2.3 Linear Eigenvalue Statistics (Main Results)

- 3.Sketch of the Proofs

1. Introduction

In this part we will quickly review some well-known results about the X_n without perturbation matrix.

Assume $X_n = (x_{ij})$, and denote the eigenvalues by $\lambda_1, \lambda_2, \cdots, \lambda_n$, i.e.,

$$\det(X_n - \lambda_i) = 0$$

In particular, each λ_i is a function of $(x_{ij}, 1 \leq i, j \leq n)$. Thus these λ_i are not independent of each other.

Question: What can we say about the eigenvalues λ_i?

For simplicity, we only consider the cases of GOE and GUE below, which are prototype of random matrix theory .

1.1 GOE and GUE

• GOE

Let $A = (\xi_{ij})_{n\times n}$, where

$$\xi_{ij} \sim N(0,1), \quad \text{all } \xi_{ij} \text{ are independent}$$

Define

$$X_n^{(1)} = \frac{1}{\sqrt{2}}(A + A')$$

Then $X_n^{(1)}$ is GOE

• GUE

Let $A = (\xi_{ij})_{n\times n}$, where

$$Re\xi_{ij}, Im\xi_{ij} \sim N(0,1), \quad \text{all } \xi_{ij} \text{ are independent}$$

Define

$$X_n^{(2)} = \frac{1}{2}(A + A^*)$$

Then $X_n^{(2)}$ is GUE

Let $\lambda_1^{(\beta)}, \cdots, \lambda_n^{(\beta)}$ be the real eigenvalues of $X_n^{(\beta)}$ where $\beta = 1, 2$. Then the j. p.d.f. due to Weyl is

$$p_n(x_1, \cdots, x_n) \propto \prod_{1 \le i < j \le n} |x_i - x_j|^\beta \cdot \prod_{i=1}^{n} e^{-\frac{\beta}{4} x_i^2}, \quad x_i \in R$$

This shows that the eigenvalues have a nice dependence structure.

The red part is the product of independent normal densities;

The blue part is a Van de Monde determinant which implies there exists pairwise interaction.
It is this Van de Monde determinant that cause both difficulty and interest in the RMT.

In fact, as we will see, the eigenvalues repel each other and so are arranged more regularly than the independent particles in the real line.

1.2 Wigner Semicircle Law

The first result of fundamental importance in RMT is as follows.

Let $F_n^{(\beta)}(x)$ be the empirical distribution of the eigenvalues defined by

$$F_n^{(\beta)}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{(\lambda_i^{(\beta)} \le \sqrt{n\beta}x)}$$

Then

$$F_n^{(\beta)} \xrightarrow{d} \rho_{sc}, \quad \text{in probability}$$

where ρ_{sc} is the so-called Wigner semicircle law

$$\rho_{sc}(x) = \frac{2}{\pi}\sqrt{1-x^2}, \quad |x| \le 1$$

1.3 Tracy-Widom Law

Let $\lambda_{(n)}^{(\beta)} = \max_i \lambda_i^{(\beta)}$. Then

$$\frac{\lambda_{(n)}^{(\beta)}}{\sqrt{n\beta}} \xrightarrow{P} 1, \qquad \text{Law of Large Numbers}$$

And moreover,

$$\sqrt{\beta} n^{1/6} (\lambda_{(n)}^{(\beta)} - \sqrt{n\beta}) \xrightarrow{d} F_\beta$$

where F_β is Tracy-Widom type distribution. In particular,

$$F_2(x) = e^{-\int_x^\infty (u-x) q^2(u) du}$$

and

$$F_1^2(x) = F_2(x) e^{-\int_x^\infty q(u) du}$$

• 1.4 CLT

Let φ be a certain smooth function, define a linear eigenvalue statistic

$$N_n^{(\beta)}(\varphi) = \sum_{i=1}^{n} \varphi\Big(\frac{\lambda_i^{(\beta)}}{\sqrt{n\beta}}\Big)$$

The following CLT holds: as $n \to \infty$

$$N_n^{(\beta)}(\varphi) - EN_n^{(\beta)}(\varphi) \xrightarrow{d} N(0, \sigma^2_{\varphi,\beta})$$

The centering constant $EN_n(\varphi)$ can be explicitly computed

$$\frac{1}{n}EN_n^{(1)}(\varphi) = \int_{-1}^{1} \varphi(x)\rho_{sc}(x)dx + \frac{1}{n}\text{error term}$$

and

$$\frac{1}{n}EN_n^{(2)}(\varphi) = \int_{-1}^{1} \varphi(x)\rho_{sc}(x)dx + \frac{1}{n^2}\text{error term}$$

A remarkable point is that there is no normalizing constant.
In comparison, recall the classical CLT for sums of i.i.d.r.v.'s:

$$\frac{1}{\sqrt{n}}(S_n - ES_n) \xrightarrow{d} N(0, \sigma^2)$$

where $S_n = \sum_{i=1}^{n} \xi_i$ and the ξ_i's are i.i.d. with $Var(\xi_i) = \sigma^2$.

This is mainly because that the eigenvalues are arranged more regularly than i.i.d.r.v.s on the real line.

There are also similar CLTs for logarithm of determinant and the number of eigenvalues in an interval after suitably scaled.

2. Deformed Random Wigner Matrices

- Basic Models

Let $X_n = (x_{ij})$ be a GUE or GOE matrix

$$x_{ij}, 1 \leq i \leq j \leq n, \quad \text{independent}$$

and

$$Ex_{ij} = 0, \quad Ex_{ii}^2 < \infty, \quad E|x_{ij}|^2 = 1$$

Let $\xi_n, n \geq 1$ be a sequence of i.i.d random variables,

$$E\xi_n = 0, \quad Var(\xi_n) = \sigma^2$$

Assume further that X_n and ξ_n's are independent of each other.
Define the deformed matrix

$$H_{n,\alpha} = \frac{X_n}{\sqrt{n\beta}} + \frac{1}{n^{\alpha/2}}\mathrm{diag}(\xi_1, \xi_2, \cdots, \xi_n), \quad 0 < \alpha < 1 \tag{1}$$

2.1 Average Spectral Distribution

• The Wigner semi-circle law still holds:

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the n real eigenvalues of $H_{n,\alpha}$ and define

$$F_{n,\alpha}(x) = \frac{1}{n} \sum_{k=1}^{n} 1_{(\lambda_k \leq x)}, \quad x \in \mathbb{R}$$

Then

$$F_{n,\alpha}(x) \xrightarrow{d} \rho_{sc}(x) \quad \text{in } P \tag{2}$$

This can be seen from a more general result about the perturbation of random matrix due to Pastur:

A simple approach to the global regim of Gaussian ensembles of random matrices, Ukranian Math. J. 2005

Define for any $z \in \mathbb{C} \setminus \mathbb{R}$ the Green functions

$$m_{n,\alpha}(z) = \int \frac{1}{x-z} dF_{n,\alpha}(x), \quad m(z) = \int_{-1}^{1} \frac{1}{x-z} \rho_{sc}(x) dx$$

then (2) can be equivalently expressed as

$$m_{n,\alpha}(z) \xrightarrow{P} m(z) \tag{3}$$

where

$$m(z) = -z + \sqrt{z^2 - 1}$$

The addition of a diagonal matrix has no influence upon the global limiting behaviors!

2.2 The Largest Eigenvalue Distribution

Let $\alpha = \frac{1}{3}$. Assume X_n is GUE model ($\beta = 2$) and $E|\xi_n|^7 < \infty$, then

$$n^{1/6}(\lambda_{(n)} - c_n) \xrightarrow{d} \xi + \eta, \quad n \to \infty \tag{4}$$

where $c_n \sim \sqrt{n}$ is a centering constant, ξ and η are independent,

$$\xi \sim F_2 \quad \text{Tracy-Widom law}, \quad \eta \sim N(0, \sigma^2)$$

Johansson, From Gumbel to Tracy-Widom, PTRF, 2007

The addition of a perturbation matrix does change the limiting distribution of largest eigenvalue of GUE.

We do not know any result about the perturbed GOE case yet.

2.3 The Linear Eigenvalue Statistics

In this part we shall see what changes the perturbation matrix will make in linear eigenvalue statistics.

- Rate of Convergence in Stieltjes Transformations
- The CLT for Linear Eigenvalue Statistics

It is known by (3) that

$$m_{n,\alpha}(z) \xrightarrow{P} m(z)$$

What is the rate of convergence?

We only consider the $X_n^{(1)}$ case (GOE case) below

(i) $\alpha = 1$.

$$\begin{aligned} H_{n,\alpha} &= \frac{X_n^{(1)}}{\sqrt{n}} + \frac{1}{\sqrt{n}} \begin{pmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_n \end{pmatrix} \\ &= \begin{pmatrix} x_{11} + \xi_1 & \boldsymbol{x}_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} + \xi_2 & \cdots & x_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ \cdots & \cdots & \cdots & x_{nn} + \xi_n \end{pmatrix} \end{aligned}$$

This was studied by Khorunzhy, Khoruzhenko and Pastur:

On Asymptotic properties of large random matrices with independent entries (1996)

$$Em_{n,1}(z) = m(z) + \frac{m^3(z)}{(1-m^2(z))^2} \cdot \frac{1}{n} + O(n^{-3/2})$$

and

$$Var(m_{n,1}(z)) = \frac{|m^2(z)|^2}{|1-m^2(z)|^2} \cdot \frac{1}{n^2} + o(n^{-2})$$

This shows that the extra diagonal matrix does not affect the rate of convergence.

(ii) $0 < \alpha < 1$, We prove

Theorem

Assume that $X_n^{(1)}$ is GOE,
$\xi_n, n \geq 1$ is i.i.d. $E\xi_n = 0$, $Var(\xi_n) = \sigma^2 > 0$, and
$E|\xi_n|^{q+2} < \infty$ where $q\alpha > 2$.
Then for each $z = E + i\eta$, $\eta \neq 0$

$$Em_{n,\alpha}(z) = m(z) + \frac{m^3(z)}{1 - m^2(z)} \cdot \frac{\sigma^2}{n^\alpha} + O(n^{-\min(3\alpha,2)/2})$$

and

$$Var(m_{n,\alpha}) = \frac{|m^2(z)|^2}{|1 - m^2(z)|} \cdot \frac{\sigma^2}{n^{1+\alpha}} + O(n^{-\min(1+3\alpha,3+\alpha)/2})$$

The addition of a diagonal matrix deteriorates the precision of estimating $m(z)$ by $m_{n,\alpha}(z)$.

• The CLT for Linear Eigenvalue Statistics

We can also prove $m_{n,\alpha}(z) - Em_{n,\alpha}(z)$ follows after properly scaled the normal distribution. More generally, the CLT for linear statistics for deformed matrices still holds.

For a roughly q-integrable test function ϕ, define

$$N_{n,\alpha}(\phi) = \sum_{k=1}^{n} \phi(\lambda_k)$$

Then we have

Theorem

$$\frac{1}{n^{(1-\alpha)/2}}(N_{n,\alpha}(\phi) - EN_{n,\alpha}(\phi)) \xrightarrow{d} N(0, \sigma_\phi^2)$$

where σ_ϕ^2 can be given explicitly.

The normalizing factor is no longer a constant.

Remarks:

(i) The case without diagonal matrix was studied by Lyvota and Pastur:

Central limit theorem for linear eigenvalue statistics of random matrices with independent entries. Ann. Probab. (2009)

They proved

$$N_n(\phi) - EN_n(\phi) \xrightarrow{d} N(0, \sigma_\phi^2)$$

with a different σ_ϕ^2.

(ii) In the case $\alpha < 1$, there is $\frac{1}{n^{(1-\alpha)/2}}$ normalization, which is between constant and $\frac{1}{\sqrt{n}}$.

3. Ideas of the Proofs

- The proof of rate of convergence

A basic tool is the Stein equation

(i) Assume that $\xi \sim N(0, \sigma^2)$, and f a differentiable function. Then

$$E\xi f(\xi) = \sigma^2 Ef'(\xi)$$

(ii) Assume that ξ is a random variable with finite $(q+2)$-th moment, and f is a differentiable function of order $(q+1)$. Then

$$E\xi f(\xi) = E\xi Ef(\xi) + Var(\xi)Ef'(\xi) + \cdots + \frac{\kappa_{q+1}}{q!}Ef^{(q)}(\xi) + \varepsilon_q$$

where κ_l is the $(l+1)$-th cumulant of ξ, ε_q is an error term.

How do we apply the Stein equation?

We need another basic tool:

Fix z with $Imz \neq 0$. Let $G_n =: (G_{ij}(z)) = \frac{1}{H_{n,\alpha} - z}$. Then

(i) resolvent identity:

$$G_n = -\frac{1}{z} + \frac{1}{z} G_n H_{n,\alpha} \quad \text{recursive relation}$$

$$G_{ij}(z) = -\frac{\delta_{ij}}{z} + \frac{1}{z} \sum_{k=1}^{n} G_{ik}(z) H_{kj}$$

(ii) differentiable formula

$$\frac{\partial G_{pq}(z)}{\partial H_{ii}} = -G_{pi}(z) G_{qi}(z),$$

and

$$\frac{\partial G_{pq}(z)}{\partial H_{ij}} = -G_{pi}(z) G_{qj}(z) - G_{pj}(z) G_{qi}(z), \quad i \neq j$$

By the resolvent identity,

$$Em_{n,\alpha}(z) = \frac{1}{n}ETrG_n = -\frac{1}{z} + \frac{1}{nz}\sum_{i,k} EG_{ik}(z)H_{ki}$$

Thus the stein equation and differentiable formula are applicable. The result is very clean and simple under the case: y_n's are normal. we obtain

$$\begin{aligned} Em_{n,\alpha}(z) &= -\frac{1}{z} - \frac{1}{z}(Em_{n,\alpha}(z))^2 - \frac{1}{z}Var(m_{n,\alpha}(z)) \\ &\quad -\frac{\sigma^2}{zn^{1+\alpha}}\sum_i E(G_{ii}(z))^2 - \frac{1}{zn^2}E\sum_{i,k}(G_{ii}(z))^2 \end{aligned}$$

$$Em_{n,\alpha}(z) = -\frac{1}{z} - \frac{1}{z}(Em_{n,\alpha}(z))^2 + \text{error term}$$

It suffice to figure out a precise upper bound for $Var(m_{n,\alpha}(z))$ and $\sum_i E(G_{ii}(z))^2$.

Repeating the above argument!

• The proof of the CLT

The proof is basically along the idea of Lyvota and Pastur (2009). Write

$$N_n^0 = \frac{1}{n^{(1-\alpha)/2}}(N_{n,\alpha}(\phi) - EN_{n,\alpha}(\phi))$$

We shall prove that for every x

$$Z_n(x) =: Ee^{ixN_n^0} \to e^{-\frac{V_\phi x^2}{2}}, \quad n \to \infty$$

This is in turn proved using the subsequence technique. Namely,

(i) $Z_n(x), Z_n'(x), n \geq 1$ are relatively compact uniformly in x

(ii) If $Z_n(x) \to Z(x)$, then $Z_n'(x) \to -V_\phi x Z(x)$.

Then we have

$$Z'(x) = -V_\phi x Z(x) \quad \Rightarrow \quad Z(x) = e^{-\frac{V_\phi x^2}{2}}$$

Write

$$\phi(\lambda) = \int_{-\infty}^{\infty} e^{it\lambda}\hat{\phi}(t)dt$$

where $\hat{\phi}(t)$ is the Fourier transform of ϕ.
Then

$$\begin{aligned} N_{n,\alpha}(\phi) = \sum_{k=1}^{n} \phi(\lambda_k) &= \int_{-\infty}^{\infty} \sum_{k=1}^{n} e^{it\lambda_k}\hat{\phi}(t)dt \\ &= \int_{-\infty}^{\infty} Tre^{itH_{n,\alpha}}\hat{\phi}(t)dt \end{aligned}$$

Let

$$U(t) = e^{itH_{n,\alpha}}, \quad t \in \mathbb{R}$$

We need the following basic facts.

(i)

$$|U(t)_{ij}| \leq 1, \quad \sum_k U(s)_{jk} U(t)_{kj} = U(s+t)_{jj}$$

(ii) Duhamel identity

$$U(t) = 1 + i \int_0^t U(s) H_n ds \quad \text{recursive relation}$$

In particular,

$$U(t)_{jl} = \delta_{jl} + i \sum_k \int_0^t U(s)_{jk} (H_n)_{kl} ds$$

(iii) differential formula

$$\frac{\partial U(t)_{pq}}{\partial H_{jj}} = iU_{pj} * U_{qj}(t)$$

and

$$\frac{\partial U(t)_{pq}}{\partial H_{jk}} = i(U_{pj} * U_{qk}(t) + U_{pk} * U_{qj}(t)), \quad j \neq k$$

(iv)

$$\frac{\partial Tr\phi(U(H_n))}{\partial H_{jj}} = \phi'(H_n)_{jj}, \quad \frac{\partial Tr\phi(U(H_n))}{\partial H_{jk}} = 2\phi'(H_n)_{jk}, \quad j \neq k$$

The End!