# Topology of manifolds

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SMOOTH MANIFOLDS: Overlap functions are infinitely differentiable.
PL MANIFOLDS: Overlap functions are simplicial on some rectilinear subdivision.

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For surfaces and 3-manifolds all these arrows can be reversed. But in higher dimensions, in general, none can be reversed.

# PART I. HIGH DIMENSIONAL MANIFOLDS

## Sample High Dimensional Results

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*(Generalized Poincaré Conjecture): Suppose that* $\Sigma^n$ *is a closed topological, resp. PL, n-manifold homotopy equivalent to the n-sphere, for* $n \geq 5$. *Then* $\Sigma$ *is homeomorphic, resp. PL equivalent, to the n-sphere.*

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**Theorem**

*For each $n \geq 5$ the smooth n-manifolds (up to diffeomorphism) homotopy equivalent to $S^n$ form a finite abelian group under connected sum, which is in principle at least, computable; e.g. for $n = 7$ the group is $Z/28$.*

## Triangulating Topological Manifolds

**Theorem**

*If $M$ is a topological n-manifold, $n \geq 5$, there is one obstruction $\theta \in H^4(M; Z/2Z)$ to triangulating $M$.*

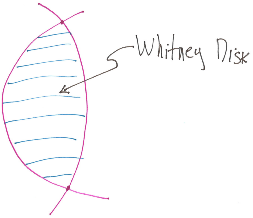
## Essential Point: Surgery theory

Given an embedded sphere $S^k \subset M$ with a trivial normal bundle, we remove a tubular neighborhood $S^k \times D^{n-k}$ and sew in $S^{n-k-1} \times D^k$. This is a surgery on the $k$-sphere. It kills the homology class of the sphere.

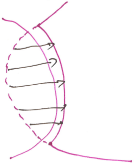
## Essential Point: Whitney trick

Suppose that $X^k$ and $Y^{n-k}$ are submanifolds of $M^n$ and $k$ and $n-k$ are both $> 2$. If $M$ is simply connected, then we can arrange that $X$ and $Y$ meet transversally and that the number of points of intersection is equal to the absolute value of the homological intersection number.

Whitney Trick




Push Across.



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By Whitney's trick there is a family of embedded 3-spheres whose homology classes are this basis and whose geometric intersection is equal to the algebraic intersection. Surgery on half this basis makes $W$ homotopy equivalent to $D^6$. Now we take a relative Morse function; cancel all handles except those of dimensions 3 and 4. These also can be cancelled by Whitney's trick.

# THE EXCEPTIONAL DIMENSIONS

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Much was known in dimension 3; 4 was extremely mysterious.

# PART II. 4-DIMENSIONAL MANIFOLDS

# Dimension 4

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HOMOTOPY THEORY OF 4-MANIFOLDS::

$M$ closed, simply connected 4-manifold is determined up to homotopy equivalence by $H_2(M; Z)$ with its intersection form. This form is integral and unimodular. There is an essentially complete classification of these forms (except for definite ones) determined by rank, signature, and whether or not the form is even. Theory of definite forms is complicated: first non-diagonalizable form is $E_8$ – the Cartan matrix of the exceptional Lie group $E_8$. It has rank 8, signature 8 and is unimodular and even.

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Which forms are realized? How many manifolds represent a given form?

## Examples of 4-Manifolds

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What about non-simply connected 4-manifolds?

## Smooth 4-manifolds: Donaldson theory

$P \to M^4$ principle $SU(2)$-bundle. Then $\mathcal{A}(P)$ is the space of connections; $\mathcal{G}$ group of gauge transformations. Acts on $\mathcal{A}(P)$, essentially freely, except for reducible connections where the stabilizers are $S^1$.

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This is a non-linear equation, elliptic modulo the action of $\mathcal{G}$. The formal dimension (the Fredholm index of the linearization) is

$$8c_2(P) - 3(1 - b_1(M) + b_2^+(M)).$$

For a generic metric it is a smooth manifold $\mathcal{M}(P)$ of this dimension in $\mathcal{A}(P)/\mathcal{G}$, except at the reducible connections where it is the quotient of a smooth manirfold by a semi-free $S^1$-action.

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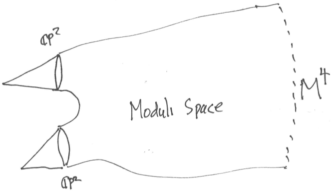
## Donaldson theory

Thus, $\mathcal{M}$ produces a smooth 5-dimensional bordism from $M$ to $\coprod_k \pm\mathbb{C}P^2$, where $k$ is the number of solutions, up to sign, to the equation $x^2 = -1$ in the form on $H_2(M)$. But this means that the number of pairs of solutions is at least $|signature(M)|$, and this implies that the form is diagonalizable over the integers.

## Moduli space as a bordism



## Donaldson theory

### Theorem

*(Donaldson) $E_8 \oplus E_8$ does not occur as the intersection form of a s.c. smooth 4-manifold.*

## Donaldson theory

For s. c. smooth 4-manifolds that are not negative definite, the moduli space $\mathcal{M}(P)$ will be a smooth manifold of dimension $8c_2(P) - 3(1 + b_2^+)$ sitting in $\mathcal{A}^{\mathrm{irr}}(P)/\mathcal{G}$. There is a natural map

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These can be used as invariants to show smooth manifolds are not diffeomorphic.

**Theorem**

*There are infinitely many pairwise non-diffeomorphic, s.c. algebraic surfaces all homotopy equivalent to $\mathbb{C}P^2$ blown up at 9 points.*

## Seiberg-Witten theory

This is another gauge theory inspired by physics whose moduli space of classical solutions are be used to give invariants of 4-manifolds, the Seiberg-Witten invariants. It turns out that these invariants carry equivalent information to the Donaldson polynomial invariants but are simpler to work with.

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**Theorem**

*(Thom Conjecture) A smooth algebraic curve of degree $d$ in $\mathbb{C}P^2$ has minimal genus in its homology class.*

## SUMMARY OF SMOOTH 4-DIMENSIONAL MANIFOLDS

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NICE INVARIANTS SHOWING THINGS ARE COMPLICATED

LOTS OF QUESTIONS – FOR MOST WE DO NOT HAVE EVEN CONJECTURAL ANSWERS

## Topological 4-Manifolds

Recall the Whitney disk idea. Given two $n$-dimensional submanifolds of a $2n$-dimensional manifold with excess intersection points. embed a 2-disk with boundary arcs connecting a pair of intersection points and use it to deform one manifold so as to remove the pair of intersection points.

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Freedman used this to make an infinitely recursive construction that leads to an embedded topological, but highly non-smooth, 2-disk as required. Thus, he was able to push high dimensional techniques down to dimension 4.

## Sample Topological Results

**Theorem**

*(Freedman) 1. Any unimodular intersection form occurs for exactly one or two homeomorphism classes simply connected 4 manifolds. If there are two homeomorphism classes only one is stably smoothable (i.e., it times $S^1$ has a smooth structure).*
*2. Every homology class in $H_2(\mathbb{C}P^2)$ is represented by a topologically embedded locally flat 2-sphere.*

## The Dichotomy

Freedman's theory and Donaldson theory are widely at odds. This gives many striking consequences in dimension 4 unlike any other dimension:

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**Theorem**

*There are compact simply connected 4-manifolds with infinitely many differentiably distinct smooth structures.* $\mathbb{R}^4$ *has uncountably many differentiably distinct smooth structures. (For all other n* $\mathbb{R}^n$ *has a unique smooth structure up to diffeomorhism.)*

# 3-DIMENSIONAL MANIFOLDS

## 3-Dimensional Manifolds

**Theorem**

*For 3-dimensional manifolds* $SMOOTH = PL = TOPOLOGICAL$

## Basic Ingredients

Fundamental group is a central feature.

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Homogeneous Geometry is a central feature.

## Surfaces

All compact surfaces are uniformizable: either they are finitely covered by $S^2$, by a 2-torus, or they are hyperbolic: quotient of the upper half-plane by a discrete subgroup of $PSL(2, Z)$ acting freely. The plane, the 2-sphere, and the hyperbolic plane have *homogeneous geometries* – the group of isometries acts transitively. Homogeneous manifolds are of the form $G/H$ where $H$ is a compact subgroup of $G$.

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Thus, every surface is the quotient of $S^2$, $\mathbb{R}^2$, or $\mathbb{H}^2$ by a discrete group of symmetries preserving a homogeneous metric (round, flat, or constant curvature $-1$). These are *locally homogeneous* manifolds – covered by homogeneous manifolds with covering transformations being isometries.

## Homogeneous 3-dimensional geometries

There are 8 homogeneous geometries in dimension 3:
Three of constant sectional curvature – round, flat, hyperbolic.
Products – $S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$.
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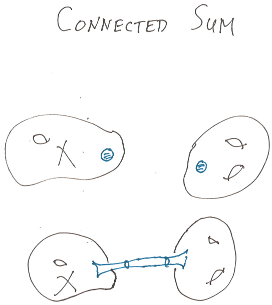
These give us 8 classes of locally homogeneous 3-manifolds.
Round, flat, and hyperbolic
Hyperbolic times $S^1$, or $S^2 \times S^1$
Nil, solv, or $S^1$-bundles over hyperbolic surfaces.

## Connected Sum

Connected sum of manifolds.



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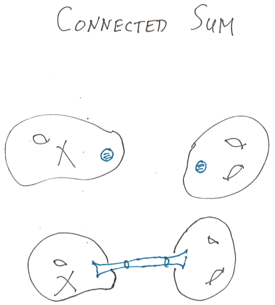


## No direct analogue of uniformization

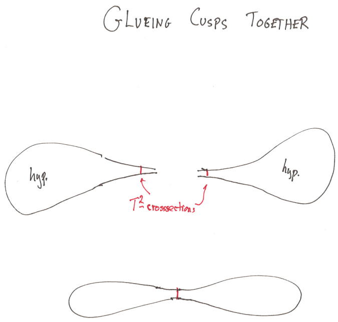
**Theorem**

*Except for $\mathbb{R}P^3 \# \mathbb{R}P^3$, no non-trivial connected sum of closed 3-manifolds can be given a locally homogeneous metric.*

Pf. The connecting sum sphere is a non-trivial element in $\pi_2$. Thus, $\pi_2(G/H) \neq 0$, and hence $G/H = S^2 \times \mathbb{R}$.

## No direct analogue of uniformization

Non-compact hyperbolic manifolds of finite volume have ends that are cusps: topologically $T^2 \times [0, \infty)$. We could glue two such together to produce a new manifold that does not carry a metric modeled on one of these 8.



## GEOMETRIATION CONJECTURE

Any closed 3-manifold has a two-fold decomposition: First is connected sum decomposition (along a family of $S^2$s) into its prime factors. The second is cutting open along 2-tori (whose fundamental groups inject into the manifold.) The result is a collection of compact and open pieces, each of which has a homogeneous metric of finite volume.

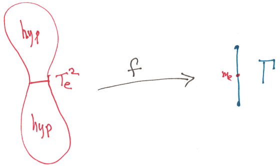
# GEOMETRIATION CONJECTURE

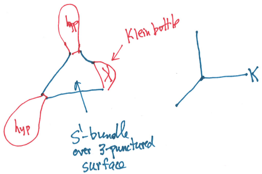
## Conjecture

*For any prime 3-manifold $M$ there is a finite graph $\Gamma$ with some of the vertices of order 1 marked with $K$ and a map $f : M \to \Gamma$ transverse to the midpoints of the edges of $\Gamma$ such that:*

1. *For each midpoint $m_e$ of an edge $e$ $f^{-1}(m_e)$ is a torus $T_e$.*
2. *For every edge $e$ the map $\pi_1(T_e) \to \pi_1(M)$ is injective.*
3. *The components of $M \setminus \cup_e T_e$ are bijective with the vertices of $\Gamma$.*
4. *The components corresponding to a vertex marked $K$ are twisted interval bundles over the Klein bottle.*
5. *Every other component has a locally homogeneous metric of finite volume based on one of the 8 geometries.*

# GEOMETRIATION CONJECTURE



## GEOMETRIATION CONJECTURE

**Theorem**

*(Perelman, using Hamilton's Ricci flow) The Geometrization Conjecture is true.*

## Method of Proof: Ricci flow with surgery

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$$\frac{\partial g}{\partial t} = -2Ric(g(t)).$$

Parabolic equation (modulo the action of the diffeomorphism group).

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Hamilton proved short-time existence and uniqueness of solutions when the manifold is compact.

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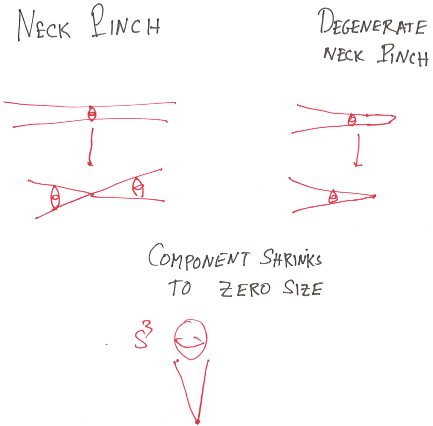
Perelman gave a qualitative description of these in dimension 3 and established some geometric control near where the siingularities are occurring – namely regions of sufficiently high curvature..

## Ricci flow with surgery



## Ricci flow with surgery

One cuts away the singularity regions contained in necks and degenerate necks, glues in 3-balls, and one removes the round components shrinking to points. After these surgeries, one continues (or more precisely restarts) the flow. This forms a Ricci flow with surgery defined for all $t \in [0, \infty)$.

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There are regions where the volume, rescaled by $t^{-3/2}$, does not collapse.
These are converging geometrically to complete hyperbolic manifolds of finite volume.

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Surgeries do connected sum decomposition and remove components that have metrics of constant positive curvature.

Perelman was then able to analyze the long-term nature of the solution.

There are regions where the volume, rescaled by $t^{-3/2}$, does not collapse.

These are converging geometrically to complete hyperbolic manifolds of finite volume.

The rest of the manifold is volume collapsing.

## Collapsed Regions

Basic Idea: Collapsed regions are close to lower dimensional spaces with curvature bounded below (Alexandrov spaces) and the theory of Alexandrov spaces can be used to understand the topology of these regions.

## Gromov-Hausdorff convergence

Two compact metric spaces $X, Y$ are within $\epsilon$ in the G-H distance if there is a metric on $X \coprod Y$ extending the given metrics on $X$ and $Y$ so that $X$ is in the $\epsilon$-neighborhood of $Y$ and $Y$ is is the $\epsilon$-neighborhood of $X$.

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If the G-H distance from $X$ to $Y$ is 0 then $X$ and $Y$ are isometric.
We say that a sequence $(X_n, x_n)$ converges in the G-H sense to $(Y, y)$ if the balls closed balls $\overline{B(x, R + \epsilon_n)}$ converge in the G-H sense to $\overline{B(y, R)}$

## Alexandrov spaces with curvature $\geq k$

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Given any 3 points $x, y, z$ construct in the surface of constant curvature $k$ points $\tilde{x}, \tilde{y}, \tilde{z}$ with the same pairwise distances. Then $\tilde{\angle} xyz$ is defined to be $\angle \tilde{x}\tilde{y}\tilde{z}$. If every time we have $p; a, b, c$ in $X$ and $\tilde{\angle} apb + \tilde{\angle} bpc + \tilde{\angle} cpa \leq 2\pi$, then $X$ is said to be an Alexandrov space with curvature $\geq k$.

## Alexandrov spaces

An Alexandrov space has a tangent cone at every point and scalings of $(X, x)$ tending to infinity converge in the Gromo-Hausdorff sense to the tangent cone.

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An Alexandrov space has a dimension (its Hausdorff dimension). It is an integer and there is an open dense set that is a manifold of that dimension.

## Alexandrov spaces as limits

**Theorem**

*Let $M_i^n$ be a sequence of complete Riemannian manifolds of dimension $n$ with sectional curvature $\geq k$. Then after passing to a subsequence there is a G-H limit. This limit is an Alexandrov space of dimension $\leq n$ and curvature $\geq k$.*

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### Theorem

*Let $M_i^n$ be a sequence of complete Riemannian manifolds of dimension $n$ with sectional curvature $\geq k$. Then after passing to a subsequence there is a G-H limit. This limit is an Alexandrov space of dimension $\leq n$ and curvature $\geq k$.*
*If the volumes of the manifolds in the sequence are tending to zero, then the limit is lower dimensional*

## Alexandrov limits of balls in Ricci flow

In the collapsing regions of Ricci flow, we rescale at the negative curvature scale, so that at each point we have $B(x,1)$ with sectional curvature $\geq -1$. Any sequence of these balls then have a G-H limit which is an Alexandrov ball of dimension 1 or 2.

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When the Alexandrov space limit is 1-dimensional, it is an interval and the manifold fibers over either an interval or a circle with fiber a $T^2$ or $S^2$.

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When the Alexandrov space limit is 1-dimensional, it is an interval and the manifold fibers over either an interval or a circle with fiber a $T^2$ or $S^2$.
These regions are understood topologically and from this information one can establish the geometrization conjecture.

## Ricci flow and Geometrization

In fact Ricci flow performs exactly the topological steps required by Geometrization:

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The finite-time singularities perform the connected sum decomposition and remove the components with postively curved metrics.
The division between collapsed and non-collapsed at infinity is the torus division of the hyperbolic pieces from the other geometric pieces.
The tori dividing up the other geometric peices are not produced by Ricci flow: here one uses *a priori* topological knowledge.

## Invariants of 3-manifolds

Geometrization is not the end of the story for 3-manifolds. There are many invariants defined for 3-manifolds and for knots in them. So are combinatorially defined (Jones Polynomial, various algebraic generalizations of the Jones polynomial, Khovanov homology), some come from physics (Witten's generalization of the Jones polynomial), ASD and SW Floer homology, and some come from topology (Heegaard Floer homology).

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The relationship of these invariants to the classification of 3-manifolds is a mystery.