Mathematical Roots

1848: Gauss-Bonnet Theorem

M closed Rem Manifold, curvature K

$$\int_M \frac{1}{2\pi} K = \chi(M)$$

1943: Allendorfer-Weil Version

M closed, oriented $2n$-dim Rem man.

Ω curvature, $Pf(A, \cdots, A)$, where

A skew symetric, the Pfaffian poly.

$$Pf\begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & a_n \\ & & & -a_n & 0 \end{pmatrix} = \prod_{i=1}^{n} a_i \qquad \boxed{P_f \text{ invariant under ad } SO(2n).}$$

$$P_\chi(\Omega^n) = P_\chi(\Omega \wedge \cdots \wedge \Omega) = \frac{1}{(2\pi)^n} Pf(\Omega \wedge \cdots \wedge \Omega)$$

Theorem: $\int_M P_\chi(\Omega^n) = \chi(M).$

proof required Triangulation + an iso imbedding of each $\triangle$ into R^N.

$\boxed{\text{Weil urged Chern to find intrinsic proof}}$

1944: Chern's proof:

Recall can find vector field V on M with isolated singularities at $\{m_1, \ldots, m_k\}$ with degrees $\{d_1, \ldots, d_k\}$. $\Rightarrow \chi(M) = \sum d_i$

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{i} & S(M) \\ & & \big\downarrow \pi \\ & & M \end{array}$$

tangent sphere bundle

$\pi^*(P_\chi(\Omega^n)) = d\,\tilde{P}_\chi(\theta, \Omega)$

$\int_{i_*(S^{2n-1})} \tilde{P}_\chi(\theta, \Omega) = 1$

θ is Riem connection

$$\int_M P_\chi(\Omega^n) = \int_{M-\{m_i\}} P_\chi(\Omega^n) = \int_{V_*(M-\{m_i\})} \pi^*(P_\chi(\Omega^n))$$

$$= \int_{V_*(M-\{m_i\})} d\,\tilde{P}_\chi(\theta,\Omega) = \int_{\text{closure}(V_*(M-\{m_i\}))} d\,\tilde{P}_\chi(\theta,\Omega) = \int_{\partial\,\text{closure}(\quad)} \tilde{P}_\chi(\theta,\Omega)$$

$$= \int_{\sum d_i S(M)_{m_i}} \tilde{P}_\chi(\theta,\Omega) = \sum d_i = \chi(M).$$

Q.E.D.

Keep your eye on $\tilde{P}_\chi$

(rhyme)

1945–1946 Chern–Weil Homomorphism

G lie group, $\mathfrak{g}$ lie alg, $I^\ell(G) =$ polynomials of deg ℓ on $\mathfrak{g}$ invariant under $\mathrm{Ad}\,G$. $I = \sum \oplus I^\ell$ is ring.

$$\begin{tikzcd} G \arrow[r] & E, \theta \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

Principal G-bundle over M
connection θ, curvature Ω
$P \in I^\ell(G)$

<u>Theorem</u>: $P(\Omega^\ell)$ is closed, G invariant, horizontal 2ℓ-form on E. Moreover (since $P(\Omega^\ell)$ may be considered closed form on M), its cohomology class $[P(\Omega^\ell)] \in H^{2\ell}(M, G)$ is independent of θ.

Proof: Direct calculation, using structural eq, $d\Theta = \Omega - [\Theta, \Theta]$ and invariance of P shows $d\, P(\Omega^\ell) = 0$. If (Θ_0, Ω_0) and (Θ_1, Ω_1) are two connection curvature pairs, can join Θ_0 to Θ_1 by curve of connections Θ_τ. Let Θ'_τ be derivative at τ. Note Θ'_τ is horizontal.

Thus, $P(\Theta' \wedge \Omega_\tau^{\ell-1})$ is horizontal and invariant so is the lift of a form on M.

Direct calculation shows

$$P(\Omega_1^\ell) - P(\Omega_0^\ell) = d\left(\ell \int_0^1 P(\Theta'_\tau \wedge \Omega_\tau^{\ell-1})\right)$$

Thus $[P(\Omega^\ell)] \in H^{2\ell}(M, R)$ independent of Θ

Q.E.D.

The maps $W_{E,\Theta} : I(G) \to \Lambda^{\text{even}}_{\text{closed}}(M)$ and

$$W_E : I(G) \to H^{\text{even}}(M, \mathbb{R})$$

are ring homomorphisms. Moreover, they are natural in the categories of principal G-bundles, with or without connections. Thus we have

$$W_G : I(G) \to H^{\text{even}}(BG, \mathbb{R})$$

The collection of the above are all referred to as the Chern–Weil homomorphism(s)

Going back to $\ell\int_0^1 P(\theta_\tau' \wedge \Omega_\tau^{\ell-1}) \in \Lambda^{2\ell-1}(M)$

Suppose $\theta(s)$ is another curve joining θ_0, θ_1

obviously $d\left(\ell\int_0^1 P(\theta_\tau' \wedge \Omega_\tau^{\ell-1}) - \ell\int_0^1 P(\theta_s' \wedge \Omega_s^{\ell-1})\right) = 0$

<u>But in fact</u>

$$\ell\int_0^1 P(\theta_\tau' \wedge \Omega_\tau^{\ell-1}) - \ell\int_0^1 P(\theta_s' \wedge \Omega_s^{\ell-1}) \quad \text{is } \underline{\text{exact}}.$$

Thus can define

$$\boxed{CSP(\theta_0, \theta_1) = \ell\int_0^1 P(\theta_\tau' \wedge \Omega_\tau^{\ell-1}) \in \Lambda^{2\ell-1}(M) / \Lambda^{2\ell-1}_{\text{exact}}(M)}$$

$$\begin{array}{c} (E, \theta) \\ \downarrow \pi \\ M \end{array} \qquad \text{can lift back to } E \qquad \begin{array}{c} (\pi^*(E), \pi^*(\theta)) \\ \downarrow \\ E \end{array}$$

But

$$G \times E \cong \pi^*(E)$$
$$(g, e) \to R_g(e)$$

Thus $\pi^*(E)$ has product connection σ

Set $TP(\theta) = CSP(\sigma, \pi^*(\theta)) \in \Lambda^{2\ell-1}(E) / \Lambda^{2\ell-1}_{exact}(E)$

Since curvature of $\sigma = 0$

$$d\,TP(\theta) = \pi^*(P(\Omega^\ell))$$

By calculation

$$TP(\theta) = \sum_{i=0}^{\ell-1} B_i\, P(\theta \wedge [\theta \wedge \theta]^i \wedge \Omega^{\ell-i-1}) \quad \Big/ \Lambda^{2\ell-1}_{exact}(E)$$

$B_i = (-1)^i \ell!\,(\ell-1)! \,/\, 2^i (\ell+i)!\,(\ell-i-1)!$

<u>Note</u>: $d(TP(\theta)|G) = (d\,TP(\theta))|G = \pi^*(P(\Omega^\ell))|G = 0$

Thus $[TP(\theta)|G] \in H^{2\ell-1}(G, R)$.

Pontryagin Polynomials

$$\begin{array}{c} O(n) \longrightarrow (E, \theta) \\ \Big\downarrow \pi \\ M \end{array}$$

$$[P_k(\Omega^{2k})] \in \text{real}(H^{4k}(M, \mathbb{Z}))$$

$$\left[\tfrac{1}{2} TP_k(\theta)\right]\Big|\, O(n) \in \text{real}\left(H^{4k-1}(O(n), \mathbb{Z})\right)$$

$$P_1(A, B) = -\frac{1}{8\pi^2}\, tr(AB) \qquad P_1(\Omega^2) = -\frac{1}{8\pi^2}\, tr(\Omega \wedge \Omega)$$

$$TP_1(\theta) = \frac{1}{16\pi^2}\left(tr(\theta \wedge \theta \wedge \theta) - 2\,\theta \wedge \Omega\right)$$

1971 Conformal Invariant for 3-Manifolds

M a closed, oriented, Riemannian 3-manifold

$$
\begin{array}{ccc}
SO(3) \xrightarrow{i} & (F(M), \theta) & \\
& \Big\downarrow & \\
& M &
\end{array}
$$

oriented Frame bundle with Riem connection.

Since all such are parallizable, can find

X-section $f: M \longrightarrow F(M)$.

$$\Phi(M) = \int_{f(M)} \tfrac{1}{2} TP_1(\theta) \mod \mathbb{Z}$$

If h another X-section, homologically

$$h(M) = f(M) + n\, i(SO(3)) + \text{torsion}$$

By integrality of $\frac{1}{2}TP_1$ on the fiber, can see $\Phi(M)$ well defined.

<u>Theorem</u> $\Phi(M)$ is a conformal inv.

<u>Theorem</u> A necessary condition that M be conformally immersable in R^4 is that $\Phi(M) = 0$

e.g. $\Phi(RP^3) = \frac{1}{2}$ and thus RP^3 cannot conformally immerse in R^4.

<u>Note</u>: RP^3 can be non-conformally immersed in R^4, and locally isometricly immersed.

Let $C(M)$ denote space of conformal structures on M

$\Phi : C(M) \longrightarrow \mathbb{R}/\mathbb{Z}$

<u>Theorem</u> $g \in C(M)$ is a critical point of $\Phi \Longleftrightarrow \{M, g\}$ is locally conformally flat

Above Theorems, along with various generalization are <u>joint with Chern</u>

More $\mathbb{R}/\mathbb{Z}$ functions

Back to the Euler Class

$P_\chi \in I^n(SO(2n))$; $W(P_\chi) \in \text{real}\left(H^{2n}(BSO(2n), \mathbb{Z})\right)$

$$\begin{array}{c} (V^{2n}, \theta) \\ \downarrow \\ M \end{array}$$

Riemannian vector bundle with θ preserving $\langle , \rangle$

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{i} & S(V^n) \\ & & \downarrow \pi \\ & & M \end{array}$$

Associated Sphere bundle

$[P_\chi(\Omega^n)] \in \text{real}\left(H^{2n}(M, \mathbb{Z})\right)$

$$\pi^*(P_\chi(\Omega^n)) = d\tilde{P}_\chi(\theta) \qquad \int_{S^{2n-1}} \tilde{P}_\chi(\theta) = 1$$

$$0 \longrightarrow H_{2n-1}(S^{2n-1}) \xrightarrow{i_*} H_{2n-1}(S(V^{2n})) \xrightarrow{\pi_*} H_{2n-1}(M) \longrightarrow 0$$

Let $\mu \in Z_{2n-1}(M)$, a smooth cycle.

By sequence above can find

$\mu' \in Z_{2n-1}(S(V^{2n}))$ with

$\pi^*(\mu') = \mu + \partial\alpha$. Set

$$\boxed{\hat{\chi}(\mu) = \int_{\mu'} \tilde{P}_\chi(\theta) - \int_\alpha P_\chi(\Omega^n) \mod \mathbb{Z}}$$

From the exact sequence and the integrality of $P_\chi(\Omega^n)$ and $\tilde{P}_\chi(\theta)\big|_{S^{2n-1}}$, can show $\hat{\chi}(\mu)$ well defined.

$$\hat{\chi} : Z_{2n-1}(M) \to \mathbb{R}/\mathbb{Z}$$ and

$$\hat{\chi}(\partial\alpha) = \int_\alpha P_\chi(\Omega^n) \mod \mathbb{Z}$$

Since knowing the mod $\mathbb{Z}$ reduction of the integral of a form over every smooth chain determines the form, we see $\boxed{\hat{\chi} \text{ determines } P_\chi(\Omega^n)}$

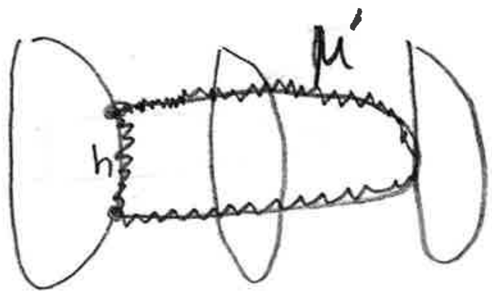
Ex: $n=1$. Then have S^1 bundle over M. Suppose μ is a closed loop. Claim:

$$\boxed{\hat{\chi}(\mu) = \frac{1}{2\pi}\left[\text{angle of holonomy around } \mu\right]}$$

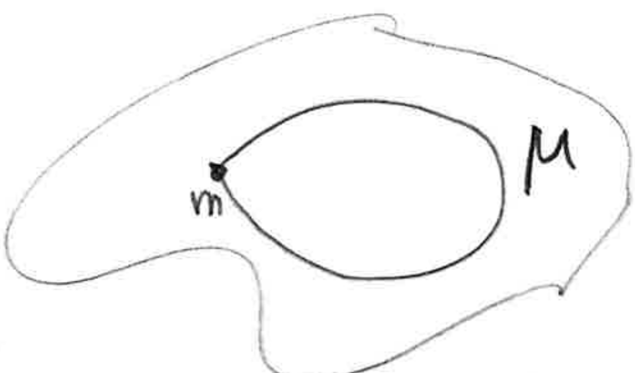
Proof:

In this case $P_\chi(\Omega^n) = \frac{1}{2\pi}\Omega$ and

$\tilde{P}_\chi(\theta) = \frac{1}{2\pi}\theta$



$\bar{\mu}$ is horizontal except over m where it is verticle thus



$$\hat{\chi}(\mu) = \int_{\mu'} \frac{1}{2\pi}\theta = \frac{1}{2\pi} h$$

Q.E.D.

Since could have $\Omega \equiv 0$, see that in general $\hat{\chi}$ carries more info than does $P(\Omega^n)$

1973 Differential Charactors (with Cheeger)

$\omega \in \Lambda^k(M)$ and $\alpha \in C_k(M)$, a smooth k-chain, we write $\overline{\omega}(\alpha) = \int_\alpha \omega \mod \mathbb{Z}$.

Let $Z_k(M)$ the abelian group of smooth k-cycles.

$$\boxed{\hat{H}^k(M) = \left\{ f \in \mathrm{Hom}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z}) \;\middle|\; f \circ \partial = \overline{\omega}_f \right\}}$$

for some $\omega_f \in \Lambda^k(M)$

$\hat{H}^*(M)$ called <u>differential characters</u>

Easily seen
1) ω_f closed
2) ω_f has integral periods
3) ω_f uniquely determined by f

Let $\overline{}$ denote reduction mod $\mathbb{Z}$

Given $f \in \hat{H}^k$, since $\mathbb{R}$ is divisible can find $T \in C^{k-1}(M, \mathbb{R})$ with $\overline{T}|Z_{k-1}(M) = f$.

$\overline{\delta T} = \delta \overline{T} = f \circ \partial = \overline{\omega_f} \Rightarrow$

$\overline{\omega_f - \delta T} = 0 \Rightarrow \omega_f - \delta T = c \in C^k(M, \mathbb{Z})$.

Moreover, $\delta c = \delta \omega_f = d\omega_f = 0 \Rightarrow$

$c \in Z^k(M, \mathbb{Z})$, an integral cocycle.

Easily seen that $[c] \in H^k(M, \mathbb{Z})$ is independent of choice of T.

Set $\eta_f = [c]$. $\boxed{\text{real}(\eta_f) = [\omega_f]}$

$\Lambda^k_{\mathbb{Z}}(M)$ denotes closed forms with integral periods. Then

$$\boxed{\begin{array}{ll} \delta_1 : \hat{H}^k \longrightarrow \Lambda^k_{\mathbb{Z}} & \delta_1(f) = \omega_f \\ \delta_2 \ \hat{H}^k \longrightarrow H^k(M, \mathbb{Z}) & \delta_2(f) = \mu_f \end{array}}$$

$\rho \in \Lambda^{k-1}(M) \qquad \alpha \in Z^{k-1}(M) \qquad \bar{\rho}(\alpha) = \overline{\int_\alpha \rho}$

$\bar{\rho} \circ \partial = \overline{d\rho}$. Thus $\bar{\rho} \in \hat{H}^k(M)$.

$\bar{\rho} = 0 \iff \rho \in \Lambda^{k-1}_{\mathbb{Z}}(M)$. Also

note $H^{k-1}(M, R/\mathbb{Z}) \subseteq \hat{H}^k$ Thus

$$\boxed{\begin{array}{ll} i_1 : H^{k-1}(M, R/\mathbb{Z}) \hookrightarrow \hat{H}^k(M) \\ i_2 : \Lambda^{k-1}_{\mathbb{Z}}(M) \hookrightarrow \hat{H}^k(M) \end{array}}$$

$\mathrm{Im}(i_1) = \ker(\delta_1)$, $\mathrm{Im}(i_2) = \ker(\delta_2)$, δ_1, δ_2 onto.

$$
\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{k-1}(M, \mathbb{R}/\mathbb{Z}) \arrow[rr, "\mathrm{Bock}"] \arrow[dr, "i_1"] & & H^k(M, \mathbb{Z}) \arrow[ur] \arrow[dr, "\mathrm{real}"] & \\
H^{k-1}(M, \mathbb{R}) \arrow[ur, "p"] \arrow[dr, "\mathrm{deRh}"'] & & \hat{H}^k(M) \arrow[ur, "\delta_2"] \arrow[dr, "\delta_1"] & & H^k(M, \mathbb{R}) \\
& \Lambda^{k-1}(M) / \Lambda^{k-1}_{\mathbb{Z}}(M) \arrow[ur, "i_2"] \arrow[rr, "d"] & & \Lambda^k_{\mathbb{Z}}(M) \arrow[ur, "\mathrm{deRh}"] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}
$$

Commutative Diagram
of Functors + Nat. Trans.

From the Diagram can show

$$0 \longrightarrow H^{k-1}(M,\mathbb{R})\big/ H^{k-1}(M,\mathbb{Z}) \longrightarrow \hat{H}^k(M) \xrightarrow{\delta_1 \times \delta_2} H^k(M,\mathbb{Z}) \times \Lambda^k_{\mathbb{Z}}(M)$$

Thus, $f \in \hat{H}^k(M)$ is determined by η_f and ω_f up to elements of a real Torus of dim $(k-1)$ Betti number.

<u>Examples</u>: $\hat{H}^1(M) = C^\infty(M, \mathbb{R}/\mathbb{Z})$

$\hat{H}^2(M)$ = iso classes of S^1 bundles with connection

Lift of Chern-Weil Homomorphism

$$\begin{array}{ccc} G \to (E, \Theta) & \quad & \eta \in H^{2\ell}(BG, \mathbb{Z}) \Rightarrow \\ \downarrow & & \eta(E) \in H^{2\ell}(M, \mathbb{Z}) \\ M & & \end{array}$$

Define

$$[H^{2\ell}(BG, \mathbb{Z}), I^{\ell}(G)] = \{ (\eta, P) \mid \mathrm{real}(\eta) = W(P) \}$$

<u>Theorem</u>: There is a unique Natural map

$$\hat{W} : [H^{2\ell}(BG, \mathbb{Z}), I^{\ell}(G)] \longrightarrow \hat{H}^{2\ell}(M)$$

with

$$\delta_2 \circ \hat{W}(\eta, P) = \eta(E)$$
$$\delta_1 \circ \hat{W}(\eta, P) = P(\Omega^{\ell})$$

$\hat{W}$ is ring homomorphism

$SO(2n)$ $\qquad \chi \in H^{2n}(BSO(2n), \mathbb{Z})$ $\qquad$ Euler class

$$\hat{W}(\chi, P_\chi) = \hat{\chi} \in \hat{H}^{2n-1}$$

$U(n)$ $\qquad c_i$ integral chern class an

C_i corresponding polynomial

$$\hat{W}(c_i, C_i) = \hat{c}_i \in \hat{H}^{2i-1}$$

$O(n)$ $\qquad p_i$ integral pontryagin class

$$\hat{W}(p_i, P_i) = \hat{p}_i \in \hat{H}^{4i-1}$$

Using that $\hat{W}$ is ring homomorphism can get useful applications to conformal immersion theorem, eta type invariants, and flat bundles.

Flat Bundles

$$\begin{array}{ccc} G & \to & (E,\theta) \\ & & \downarrow \\ & & M \end{array}$$

A bundle with connection is called <u>flat</u> if $\Omega \equiv 0$. All such derive from representations in G of $\pi_1(M)$.

Since $\Omega = 0$, $\hat{W}(\eta, P) \in H^{odd}(M, \mathbb{R}/\mathbb{Z})$.

By naturality, these lead to $\mathbb{R}/\mathbb{Z}$ classes in BG_0, the discretized version of G. These all turn out to be <u>Borel Classes</u>, i.e. represented by measurable classes in the <u>bar resolution</u>

Recall,

A k-simplex in the bar resolution is $(g_0, \dots, g_k)$, under $\sim$

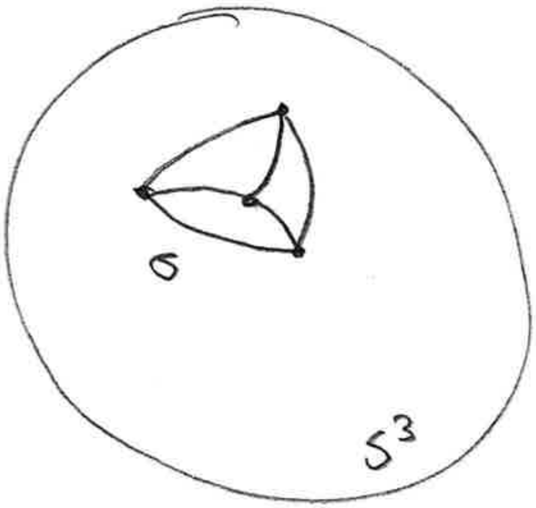
$$(g_0, \dots, g_k) \sim (g g_0, \dots, g g_k).$$

$$\partial(g_0, \dots, g_k) = \sum_{i=0}^{k} (-1)^i (g_0, \dots, \hat{g}_i, \dots, g_k)$$

$$\boxed{\text{We now express } \hat{\chi} \in H^{2n-1}_{\text{Borel}}(BSO(2n)_0)}$$

Fix $e \in S^{2n-1}$, and call $(g_0, \dots, g_{2n-1})$ generic if $\{g_i(e)\}$ are linearly independent. Such generic simplices form a dense open set.

Since generic, $\{g_i(e)\}$ determine a unique, convex, oriented geodesic simplex in S^{2n-1}, $\sigma(g_0, \ldots, g_{2n-1})$



$$\boxed{\hat{\chi}(g_0, \ldots, g_{2n-1}) = \frac{\mathrm{Vol}(\sigma)}{\mathrm{Vol}(S^{2n-1})} \mod \mathbb{Z}}$$

Easy to see that $\hat{\chi}$ is a cocycle and is Borel.

$\hat{\chi}: H_{2n-1}(BSO(2n), \mathbb{Z}) \to R/\mathbb{Z}$.

$\boxed{\text{Range of } \hat{\chi} \text{ is interesting}}$

Easy Fact : Range$(\hat{\chi}) \supseteq \mathbb{Q}/\mathbb{Z}$

Any thing else?

A geodesic k-simplex has $\frac{(k+1)k}{2}$ dihedral angles (between adjacent top dim faces) and these determine its geometry up to a ridyid motion of the sphere

S^2 

$$\frac{\mathrm{Vol}(\sigma)}{\mathrm{Vol}(S^2)} = \frac{1}{2} - \frac{1}{2}\left(\frac{\alpha_1}{2\pi} + \frac{\alpha_2}{2\pi} + \frac{\alpha_3}{2\pi}\right)$$

a nice rational function. However

the volome function of simplices in S^3 is trancendental function of dihedral $\angle$s.

Call a simplex <u>rational</u> if each dihedral $\angle$ is rational multiple of 2π

<u>Theorem (Thurson)</u> In S^3, for all but a finite number of rational simplices σ can find integer m with

$$m \frac{\mathrm{Vol}(\sigma)}{\mathrm{Vol}(S^3)} \bmod \mathbb{Z} \in \mathrm{Range}(\hat{\chi}).$$

Thus $\mathrm{Range}(\hat{\chi})$ contains irrational values unless all but a finite number of rational simplices have rational relative volume. So far <u>not a single example</u> of rational simplex with irrational vol. !

Uniqueness of $\hat{H}$

Other functors in the smooth category have been developed and shown isomorphic to $\hat{H}$. Recently by Harvey, Lawson and Zweck, for example, and, as it turns out, Delign cohomology in the smoot cat. is also equivalent to $\hat{H}$. Of course all satisfy the character diagram

It turns out that the diagram itself completely determines the functor

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{k-1}(M,\mathbb{R}/\mathbb{Z}) \arrow[rr, "\text{Bock}"] \arrow[dr, "x_1"] & & H^k(M,\mathbb{Z}) \arrow[ur] \arrow[dr, "\text{real}"] & \\
H^{k-1}(M,\mathbb{R}) \arrow[ur, "p"] \arrow[dr, "\text{deRh}"'] & & \hat{G}^k(M) \arrow[ur, "y_2"] \arrow[dr, "y_1"] & & H^k(M,\mathbb{R}) \\
& \Lambda^{k-1}(M)/\Lambda^{k-1}_{\mathbb{Z}}(M) \arrow[ur, "x_2"] \arrow[rr, "d"] & & \Lambda^k_{\mathbb{Z}}(M) \arrow[ur, "\text{deRh}"] \arrow[dr, dash] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

<u>Theorem</u> (with Sullivan) In the smooth category, any $\mathbb{Z}$-graded functor together with maps, $\{\hat{G}, x_1, x_2, y_1, y_2\}$, which satisfies the above diagram is equivalent via a unique natural $\Phi : \hat{G} \to \hat{H}$ which commutes via the identity map on all other functors in the diagram.

i.e. $\Phi \circ x_1 = i_1$, $\Phi \circ x_2 = i_2$, $\delta_1 \circ \Phi = y_1$, $\delta_2 \circ \Phi = y_2$

Hopkins-Singer (2005)

Showed that similar ^ functors could be constructed for all generalized cohomology theories.

Tomorrow, Sullivan and I will construct two (equivant) such functors for K^{even}.