RTT Realisation of the Yangian for the Hubbard Chain

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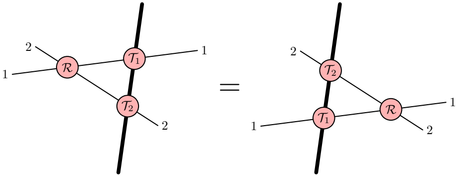


work with M. de Leeuw; to appear

Introduction and Overview

This talk is about a **curious R/S-matrix** $\boldsymbol{\mathcal{R}(u_1, u_2)}$ for

- the integrable structure of the one-dimensional Hubbard model,
- scattering on the worldsheet of $AdS_5 \times S^5$ strings,
- scattering in massive supersymmetric Chern–Simons theories.

An Open Problem:

- Classification of R-matrices via Yangian and quantum affine algebra.
- Conventional R-matrices of difference form $\mathcal{R}(u_1, u_2) = \mathcal{R}(u_1 - u_2)$.
- Above R-matrix not of difference form: $\mathcal{R}(u_1, u_2) \neq \mathcal{R}(u_1 - u_2)$!
- It escapes the standard classification of quantum algebras.

Question to Address: What is the algebraic origin of $\mathcal{R}(u_1, u_2)$?

Overview:

- Hubbard chain, AdS/CFT worldsheet scattering and symmetries,
- Drinfeld (I) realisation of the Yangian,
- RTT realisation of the Yangian.

I. A Quantum Algebra for the Hubbard Chain?

Hubbard Chain

The one-dimensional Hubbard model is very special: [Essler, Frahm Göhmann Klümper, Korepin]

- four-state spin chain,
- nearest-neighbour Hamiltonian $\mathcal{H}$,
- non-trivial coupling constant U,
- quantum integrable model,
- solved by Bethe ansatz: Lieb–Wu equations, [Lieb, Wu Phys. Rev. Lett. 20, 1445 (1968)]
- R-matrix $\mathcal{R}$ found by Shastry, [Shastry PRL 56,2453]
- $\mathcal{R}(u_1, u_2)$ depends on two independent spectral variables.

Model for spin-$\frac{1}{2}$ electrons on lattice of atoms.
4 states per site:

$$|0\rangle = |\underline{\;\;\;}\rangle, \quad c_1^\dagger|0\rangle = |\underline{\uparrow\;}\rangle, \quad c_2^\dagger|0\rangle = |\underline{\;\downarrow}\rangle, \quad c_1^\dagger c_2^\dagger|0\rangle = |\underline{\uparrow\downarrow}\rangle.$$

Simple NN hopping interaction and on-site repulsion:

$$\mathcal{H}: \begin{array}{l} |\underline{\uparrow\updownarrow}, \underline{\;\updownarrow}\rangle \leftrightarrow |\underline{\;\updownarrow}, \underline{\uparrow\updownarrow}\rangle, \\ |\underline{\updownarrow\downarrow}, \underline{\updownarrow\;}\rangle \leftrightarrow |\underline{\updownarrow\;}, \underline{\updownarrow\downarrow}\rangle, \end{array} \qquad \mathcal{H}|\underline{\uparrow\downarrow}\rangle = U|\underline{\uparrow\downarrow}\rangle.$$

Hubbard Chain Symmetries

Want to understand quantum algebra origin of integrable Hubbard chain.
Lie Symmetries of Hubbard Hamiltonian?

- number of up spins conserved,
- number of down spins conserved,
- $\mathfrak{sl}(2)$ spin symmetry, fundamental rep.: $|{\uparrow}\rangle \leftrightarrow |{\downarrow}\rangle$,
- $\mathfrak{sl}(2)$ "eta-pairing" symmetry, fundamental rep.: $|\,\rangle \leftrightarrow |{\uparrow\downarrow}\rangle$
  (symmetry in bulk, broken by boundary conditions for odd L).

Suitable **Quantum Algebras** for integrability?

- Lie algebras $\mathfrak{sl}(2)$, $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) = \mathfrak{so}(4)$?
- Yangians $\mathrm{Y}[\mathfrak{sl}(2)]$, $\mathrm{Y}[\mathfrak{sl}(2)] \times \mathrm{Y}[\mathfrak{sl}(2)]$, $\mathrm{Y}[\mathfrak{so}(4)]$, ...?
- quantum deformations $\mathrm{U}_q[\mathfrak{sl}(2)]$, ...?
- quantum affine algebras $\mathrm{U}_q[\mathfrak{sl}(2)^{(1)}]$, $\mathrm{U}_q[\mathfrak{so}(4)^{(1)}]$, $\mathrm{U}_q[\mathfrak{so}(4)^{(2)}]$, ...?!
- ...?!

Hubbard chain way more complicated than anything these algebras predict.
E.g. predicted R-matrices have difference form.

Supersymmetric Hubbard Chain

4 states: 2 bosonic $|\underline{}\rangle$, $|\underline{\uparrow\downarrow}\rangle$ and 2 fermionic $|\underline{\downarrow}\rangle$, $|\underline{\uparrow}\rangle$.
Two Lie superalgebras with $2|2$-dimensional irrep.:

- $\mathfrak{osp}(2|2)$ fundamental representation ($\simeq \mathfrak{sl}(2|1)$ minimal typical),
- $\mathfrak{sl}(2|2)$ fundamental representation.

$\mathfrak{sl}(2|2)$ Essler–Korepin–Schoutens model

[Essler Korepin Schoutens]

- two manifest $\mathfrak{sl}(2)$'s,
- some Hubbard-like interaction terms,

but:

- $\mathfrak{sl}(2|2)$ manifest supersymmetry,
- R-matrix of difference form,
- NN Hamiltonian is simple graded permutation,
- no coupling constant.

Several other "generalised" Hubbard models proposed;
standard $\mathfrak{so}(4)$, $\mathfrak{so}(5)$, $\mathfrak{sl}(2|1) = \mathfrak{osp}(2|2)$, $\mathfrak{sl}(2|2)$ quantum algebras.
Do not possess special features of Hubbard model.

Central Extension

New clues from AdS/CFT string/gauge duality. [NB et al. 1012.3982]
Quantum algebra related to peculiar feature of superalgebra $\mathfrak{psl}(2|2)$.

Non-trivial extensions of simple superalgebras:

- $\mathfrak{psl}(n|n)$ have non-trivial central extension: $\mathfrak{sl}(n|n)$, and a $\mathfrak{gl}(1)$ outer automorphism: $\mathfrak{pgl}(n|n)$. Both extensions coexist in $\mathfrak{gl}(n|n)$.
- $\mathfrak{psl}(2|2)$ has a triple central extension: $\mathfrak{psl}(2|2) \ltimes \mathbb{C}^3$, and a $\mathfrak{sl}(2)$ outer automorphism: $\mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2)$. Each one originates from exceptional $\mathfrak{d}(2,1;\alpha)$ for $\alpha \to 0$. Both extensions coexist in $\mathfrak{h} := \mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2) \ltimes \mathfrak{gl}(1)^3$.

Integrable structure of Hubbard chain appears to arise from

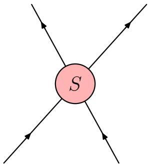
- subalgebra of Yangian $\mathrm{Y}[\mathfrak{h}]$ or [NB hep-th/0511082] [Gomez Hernández] [Plefka Spill Torrielli] [NB nlin/0610017] [NB 0704.0400]
- from a non-trivial deformation of Yangian $\mathrm{Y}[\mathfrak{gl}(2|2)]$. [Moriyama Torrielli] [Beisert Spill]

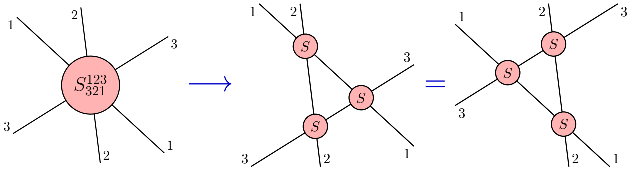
II. Worldsheet S-Matrix

Worldsheet Scattering

$AdS_5 \times S^5$ string worldsheet scattering picture:

- infinitely extended two-dimensional worldsheet,
- 8 bosons, 8 fermions,
- $\mathfrak{sl}(2|2) \oplus \mathfrak{sl}(2|2)$ residual symmetry,
- some deformed relativistic dispersion relation,
- 2-particle scattering matrix,
- integrability: factorised multi-particle scattering, YBE.





- S-matrix splits into two equivalent factors $S = \mathcal{R} \otimes \mathcal{R}$.
- each factor $\mathcal{R}$: 2 bosons, 2 fermions; $\mathfrak{sl}(2|2)$ symmetry.
- S-matrix factor equivalent to Shastry's R-matrix for Hubbard chain.

Extended $\mathfrak{sl}(2|2)$ Algebra

$\mathfrak{psl}(2|2)$ as algebra of projective supermatrices ($\mathrm{sdet} = 1$, modulo trace)

$$\mathrm{J}^A{}_B = \left(\begin{array}{c|c} \mathrm{L}^a{}_b & \mathrm{Q}^\alpha{}_b \\ \hline \bar{\mathrm{Q}}^a{}_\beta & \tilde{\mathrm{L}}^\alpha{}_\beta \end{array}\right).$$

Note: both susys Q, $\bar{\mathrm{Q}}$ are spin $(\frac{1}{2}, \frac{1}{2})$ of $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Can mix!

Centrally Extended $\mathfrak{psl}(2|2)$: $a, b, \ldots = 1, 2$, $\alpha, \beta, \ldots = 3, 4$

$\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$: $\mathrm{L}^a{}_b, \tilde{\mathrm{L}}^\alpha{}_\beta$, susy: $\mathrm{Q}^\alpha{}_b, \bar{\mathrm{Q}}^a{}_\beta$, centre: $\mathrm{H}, \mathrm{C}, \bar{\mathrm{C}}$.

Algebra: like $\mathfrak{sl}(2|2)$, but additional $\{\mathrm{Q}, \mathrm{Q}\}$ and $\{\bar{\mathrm{Q}}, \bar{\mathrm{Q}}\}$ susy relations

$$\begin{aligned}
\{\mathrm{Q}^\alpha{}_b, \mathrm{Q}^\gamma{}_d\} &= \varepsilon^{\alpha\gamma}\varepsilon_{bd}\mathrm{C}, \\
\{\mathrm{Q}^\alpha{}_b, \bar{\mathrm{Q}}^c{}_\delta\} &= \delta^c_b\tilde{\mathrm{L}}^\alpha{}_\delta + \delta^\alpha_\delta \mathrm{L}^c{}_b + \tfrac{1}{2}\delta^\alpha_\delta\delta^c_b\mathrm{H}, \\
\{\bar{\mathrm{Q}}^a{}_\beta, \bar{\mathrm{Q}}^c{}_\delta\} &= \varepsilon^{ac}\varepsilon_{\beta\delta}\bar{\mathrm{C}}.
\end{aligned}$$

Extended $\mathfrak{sl}(2|2)$ Quantum Algebra

Want to construct a quantum algebra: Hopf algebra on enveloping algebra. Procedure for simple algebras clear. Centre? Inspiration from AdS/CFT:

Coalgebra: Trivial, but susy coproducts braided by extra central U

$$\begin{aligned} \Delta(\mathrm{L}) &= \mathrm{L}\otimes 1 + 1\otimes \mathrm{L}, & \Delta(\mathrm{Q}) &= \mathrm{Q}\otimes 1 + \mathrm{U}^{+1}\otimes \mathrm{Q},\\ \Delta(\tilde{\mathrm{L}}) &= \tilde{\mathrm{L}}\otimes 1 + 1\otimes \tilde{\mathrm{L}}, & \Delta(\bar{\mathrm{Q}}) &= \bar{\mathrm{Q}}\otimes 1 + \mathrm{U}^{-1}\otimes \bar{\mathrm{Q}},\\ \Delta(\mathrm{H}) &= \mathrm{H}\otimes 1 + 1\otimes \mathrm{H}, & \Delta(\mathrm{U}) &= \mathrm{U}\otimes \mathrm{U}. \end{aligned}$$

Problem:

- Need consistent bialgebra structure, e.g. $\Delta(\mathrm{C}) = \mathrm{C}\otimes 1 + \mathrm{U}^{+2}\otimes \mathrm{C}$.
- Invariant R-matrix requires cocommutative centre $\mathrm{C}, \bar{\mathrm{C}}$.

Solution: constrain extra central elements [NB hep-th/0511082] [Gomez Hernández] [Plefka Spill Torrielli]

$$\mathrm{C} = \hbar^{-1}(\mathrm{U}^{+2} - 1), \qquad \bar{\mathrm{C}} = \hbar^{-1}(1 - \mathrm{U}^{-2}).$$

- Two central elements (H, U) instead of three $(\mathrm{H}, \mathrm{C}, \bar{\mathrm{C}})$;
- one coupling constant $\hbar$.
- Representations of non-extended $\mathfrak{sl}(2|2)$ recovered for $\mathrm{U} \simeq \pm 1$.

Extended $\mathfrak{sl}(2|2)$ Yangian

Integrability usually related to infinite-dimensional algebra: Yangian [Drinfel'd 1985]

- based on polynomial loop algebra J_n, $n \geq 0$,
- quantum algebra: deformation of UEA.

Drinfeld Realisation: (Drinfeld I, old, original, ...)
Start with extended $\mathfrak{sl}(2|2)$ quantum algebra generated by J^A (level-zero)

$$[\mathrm{J}^A, \mathrm{J}^B\} = f^{AB}{}_C \mathrm{J}^C, \qquad \Delta(\mathrm{J}^A) = \mathrm{J}^A \otimes 1 + \mathrm{U}^{[A]} \otimes \mathrm{J}^A.$$

Introduce level-one generators $\widehat{\mathrm{J}}^A$. Adjoint/coproduct/Serre: [NB 0704.0400]

$$[\mathrm{J}^A, \widehat{\mathrm{J}}^B\} = f^{AB}{}_C \widehat{\mathrm{J}}^C,$$

$$\Delta(\widehat{\mathrm{J}}^A) = \widehat{\mathrm{J}}^A \otimes 1 + \mathrm{U}^{[A]} \otimes \widehat{\mathrm{J}}^A + \tfrac{1}{2}\hbar f^A{}_{BC} \mathrm{J}^B \mathrm{U}^{[C]} \otimes \mathrm{J}^C,$$

$$\big[[\mathrm{J}^A, \widehat{\mathrm{J}}^B\}, \widehat{\mathrm{J}}^C\big\} + 2 \text{ cyclic} = \mathcal{O}(\hbar^2).$$

Coalgebra consistent provided that

$$\widehat{\mathrm{C}} = -\tfrac{1}{2}(1 + \mathrm{U}^{+2})\mathrm{H}, \qquad \widehat{\overline{\mathrm{C}}} = -\tfrac{1}{2}(1 + \mathrm{U}^{-2})\mathrm{H}.$$

Fundamental Representation

Ansatz for fundamental representation on $2+2$ states $|a\rangle$, $|\alpha\rangle$

$$\mathrm{Q}^{\alpha}{}_{b}|c\rangle = a\,\delta^{c}_{b}|\alpha\rangle, \qquad \mathrm{Q}^{\alpha}{}_{b}|\gamma\rangle = b\,\varepsilon^{\alpha\gamma}\varepsilon_{cd}|d\rangle,$$
$$\bar{\mathrm{Q}}^{a}{}_{\beta}|c\rangle = c\,\varepsilon^{ac}\varepsilon_{\beta\delta}|\delta\rangle, \qquad \bar{\mathrm{Q}}^{a}{}_{\beta}|\gamma\rangle = d\,\delta^{\gamma}_{\beta}|a\rangle.$$

Yangian evaluation representation:

$$\widehat{\mathrm{J}}^{A} \simeq u\,\mathrm{J}^{A}.$$

7 parameters (a, b, c, d, H, U, u), 5 constraints: 2-parameter family

- evaluation parameter u,
- normalisation parameter (equivalent representations).

Higher Representations: [NB nlin.SI/0610017]

- constructible from tensor products of fundamentals using coproduct Δ;
- analogous to standard $\mathfrak{sl}(2|2)$ representation theory.

Fundamental R-Matrix

Fundamental R-matrix $\mathcal{R}(u_1, u_2) : \mathbb{C}^{2|2} \otimes \mathbb{C}^{2|2} \to \mathbb{C}^{2|2} \otimes \mathbb{C}^{2|2}$ [NB hep-th/0511082]

$$
\begin{aligned}
\mathcal{R}|ab\rangle &= \tfrac{1}{2}(A+B)|ab\rangle + \tfrac{1}{2}(A-B)|ba\rangle + \tfrac{1}{2}C\varepsilon^{ab}\varepsilon_{\gamma\delta}|\gamma\delta\rangle,\\
\mathcal{R}|\alpha\beta\rangle &= -\tfrac{1}{2}(D+E)|\alpha\beta\rangle - \tfrac{1}{2}(D-E)|\beta\alpha\rangle - \tfrac{1}{2}F\varepsilon^{\alpha\beta}\varepsilon_{cd}|cd\rangle,\\
\mathcal{R}|a\beta\rangle &= G|\beta a\rangle + H|a\beta\rangle,\\
\mathcal{R}|\alpha b\rangle &= K|b\alpha\rangle + L|\alpha b\rangle.
\end{aligned}
$$

Coefficient functions $A, \ldots, L$ uniquely determined by cocommutativity

$$
\Delta_{\mathrm{op}}(\mathrm{J}) = \mathcal{R}^{-1}\Delta(\mathrm{J})\mathcal{R}, \qquad \Delta_{\mathrm{op}}(\widehat{\mathrm{J}}) = \mathcal{R}^{-1}\Delta(\widehat{\mathrm{J}})\mathcal{R}.
$$

R-matrix automatically satisfies YBE $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.
R-matrix equivalent to Shastry's R-matrix for Hubbard chain. [NB nlin.SI/0610017]
Elements have branch cuts: each u lives on elliptic surface.

Higher Representations: [Chen Dorey Okamura] [Arutyunov Frolov]

- R-matrices apparently exist and are determined uniquely. [de Leeuw 0804.1047] [Arutyunov de Leeuw Torrielli]

Further Developments

Many results obtained on R-matrix:

Crossing Equation. [Janik hep-th/0603038]

$$\mathcal{R}(u_1, u_2) \sim \mathcal{R}^{\mathsf{ST}\otimes 1}(u_1', u_2).$$

- Exact equality constrains overall phase factor.
- Required for consistency of fusion relations.

Classical Limit. [Torrielli hep-th/0701281] [NB Spill]

- Simpler quasi-triangular Lie bialgebra with classical r-matrix.
- $\mathfrak{sl}(2|2)$ enlarged by $\mathfrak{u}(1)$ automorphism: deformation of $\mathfrak{gl}(2|2)$.

q-deformation. [NB Koroteev] [NB 1002.1097] [NB Galleas Matsumoto]

- Deformed Hubbard chain (Alcaraz–Bariev), one additional parameter.
- Quantum affine algebra.
- Generators H and U appear more symmetrically.
- Some non-standard commutation relations.

Open Questions

Understanding is far from complete:

Extra Central Elements.

- Role of central element U?
  Reshetikhin twist? Partial q-deformation of algebra?

Spectral Parameter Surfaces.

- Why (representation-dependent!) elliptic surfaces for u?
  Elliptic surface from algebra? (not visible in Drinfeld realisation).
  Some relationship to XYZ-type algebras?

Completion of Algebra.

- What further generators are needed?
- Yangian Double?

Algebraic Properties.

- Universal R-matrix?
- Quasi-triangularity?

Difference Form.

- Why does $\mathcal{R}(u_1, u_2)$ not have difference form?

III. RTT Realisation

Idea of RTT Realisation

Start with some concrete R-matrix $\mathcal{R}(u_1, u_2)$ acting on the space $\mathbb{V}$

$$\mathcal{R} : \mathbb{C}^2 \to \mathrm{End}(\mathbb{V} \otimes \mathbb{V}); \qquad \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Then construct a Hopf algebra A based on it as follows:
Space. Consider some abstract objects $\mathcal{T}(u)$ of the type

$$\mathcal{T} : \mathbb{C} \to \mathrm{End}(\mathbb{V}) \otimes \mathrm{A};$$

A is the space of polynomials in the matrix elements of $\mathcal{T}(u)$ in space 1.

$$\mathcal{T}^a{}_b(u)\mathcal{T}^c{}_d(v) \cdots \mathcal{T}^x{}_y(z) \in \mathrm{A}.$$

Algebra. Identify polynomials according to the RTT relations

$$\mathcal{R}^{cd}{}_{ab}(u,v)\mathcal{T}^e{}_c(u)\mathcal{T}^f{}_d(v) = \mathcal{T}^d{}_b(v)\mathcal{T}^c{}_a(u)\mathcal{R}^{ef}{}_{cd}(u,v).$$

Coalgebra. Coproduct imposes fusion relation

$$\Delta(\mathcal{T}^a{}_b(u)) = \mathcal{T}^a{}_c(u) \otimes \mathcal{T}^c{}_d(u).$$

Quotient, Double. Quotient out ideals, construct quantum double . . .

Relationship to Drinfeld Realisation

Question. How is the RTT realisation related to the Drinfeld realisation?

Example. RTT realisation of Yangian $\mathrm{Y}[\mathfrak{gl}(n)]$.
Start with well-known R-matrix ($E^a{}_b$ is matrix with 1 at position a, b)

$$\mathcal{R}(u,v) = r_0(u,v)\big((u-v)E^a{}_a \otimes E^b{}_b + \hbar E^a{}_b \otimes E^b{}_a\big).$$

Expansion of $\mathcal{T}(u)$ around $u = \infty$ yields Drinfeld generators $\mathrm{J}^a{}_b$, $\widehat{\mathrm{J}}^a{}_b$

$$\mathcal{T}(u) = \exp\big(\hbar u^{-1} E^a{}_b \otimes \mathrm{J}^b{}_a + \hbar u^{-2} E^a{}_b \otimes \widehat{\mathrm{J}}^b{}_a + \ldots\big) = E^a{}_b \otimes \mathcal{T}^b{}_a(u).$$

RTT relations yield Yangian algebra and coproducts in Drinfeld realisation.

Representation. Fundamental evaluation representation with parameter v

$$\mathcal{T}^a{}_b(u) \simeq \mathcal{R}^{ac}{}_{bd}(u,v) E^d{}_c.$$

Quotient. Can quotient out traces $\mathrm{J}^a{}_a$, $\widehat{\mathrm{J}}^a{}_a$ for $\mathrm{Y}[\mathfrak{sl}(n)]$.

Procedure for Extended $\mathrm{Y}[\mathfrak{sl}(2|2)]$

We have a concrete R-matrix $\mathcal{R}(u,v)$ on $\mathbb{V} = \mathbb{C}^{2|2}$.
Hence we can construct an associated Yangian algebra Y!

Tasks:

- Show that RTT presentation equivalent to above Drinfeld presentation. How to identify $\mathcal{T}(u)$ with the Drinfeld generators J, $\widehat{\mathrm{J}}$?

Steps:

- Consider fundamental representation where $\mathcal{T} \simeq \mathcal{R}$. Compare to above representation of J, $\widehat{\mathrm{J}}$.
- Generalise relations between $\mathcal{T}$ and J, $\widehat{\mathrm{J}}$. Derive algebra of J, $\widehat{\mathrm{J}}$ from RTT relations. Derive coalgebra of J, $\widehat{\mathrm{J}}$ from fusion relations.
- Find suitable quotient.

Along the Way:

- Watch out for differences w.r.t. conventional Yangians.

IV. Fundamental Representation

Fundamental Representation Leading Order

The R-matrix is always a representation of $\mathcal{T}$ (RTT is YBE)

$$\mathcal{T}^A{}_B(u) \simeq \mathcal{R}^{AC}{}_{BD}(u,v) E^D{}_C.$$

Expand around $u = \infty$ using Shastry's R-matrix to find

$$\mathcal{T}^A{}_B(u) \simeq \begin{pmatrix} \delta^a_b & 0 \\ 0 & \delta^\alpha_\beta U \end{pmatrix} + \mathcal{O}(u^{-1}).$$

Find that:

- Eigenvalue U of group-like central element U appears at leading order.
- Conventionally leading order is trivial for a Yangian.
- U acts as a relative factor between two graded subspaces (1,2 vs. 3,4).

Fundamental Representation Level-0

First order around $u = \infty$

$$\mathcal{T}^A{}_B(u) \simeq \begin{pmatrix} \delta^a_b & 0 \\ 0 & \delta^\alpha_\beta U \end{pmatrix} + \hbar u^{-1} \begin{pmatrix} J^a{}_b & Q^\alpha{}_b \\ U\bar{Q}^a{}_\beta & U J^\alpha{}_\beta \end{pmatrix} + \mathcal{O}(u^{-2}).$$

Recover representation of level-0 generators: $Q, \bar{Q}$ and

$$J^a{}_b = L^a{}_b + \delta^a_b (A - H), \qquad J^\alpha{}_\beta = \tilde{L}^\alpha{}_\beta + \delta^\alpha_\beta (A + H).$$

Notes:

- Lower-column generators are multiplied by U.
- Q, $\bar{Q}$ reproduce deformed representation (parameters a, b, c, d).
- An extra central generator A appears.
  Only element depending on overall factor $r(u, v)$ of $\mathcal{R}$.
- No trace of central elements C, $\bar{C}$.

Fundamental Representation Level-1

Continue expansion around $u = \infty$

$$\mathcal{T}^A{}_B(u) \simeq \ldots + \hbar u^{-2} \begin{pmatrix} \widehat{J}^a{}_b & \widehat{Q}^\alpha{}_b \\ U\widehat{\bar{Q}}^a{}_\beta & U\widehat{J}^\alpha{}_\beta \end{pmatrix} + u^{-2}\text{“}\hbar^2(J^2)^A{}_B\text{”} + \mathcal{O}(u^{-3}).$$

Recover level-1 Yangian evaluation representation:

$$\widehat{J}^A{}_B = uJ^A{}_B.$$

Notes:

- Expansion of exponent $\mathcal{T} = \exp(\ldots)$ appears.
- Some extra terms required for $\widehat{Q}, \widehat{\bar{Q}}$ involving $\bar{Q}, Q$ and U.
- $\widehat{H}$ requires extra terms involving U.
- $\mathfrak{sl}(1)$ automorphism $\widehat{B}$ (secret symmetry). [Matsumoto Moriyama Torrielli][NB Spill]
- No trace of $\widehat{C}$ and $\widehat{\bar{C}}$.

V. Hopf Algebra

RTT Algebra

Consider now the RTT algebra

$$\mathcal{R}_{12}(u,v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)\mathcal{R}_{12}(u,v).$$

Use ansatz based on findings for fundamental representation

$$\mathcal{T}(u) = E^C{}_C \otimes \mathrm{U}^{|C|} \exp\bigl(\hbar E^B{}_A \otimes (u^{-1}\mathrm{J}^A{}_B + u^{-2}\widehat{\mathrm{J}}^A{}_B + \ldots)\bigr)$$

Expand around $u = \infty$ and write RTT algebra as commutation relations.

Notes:

- Drinfeld realisation of Yangian algebra reproduced.
- A at $\mathcal{O}(u^{-1})$ is central; appears nowhere else; project out!
- U is central; appears in many places of expansion.
- Central elements $\mathrm{C}, \bar{\mathrm{C}}, \widehat{\mathrm{C}}, \widehat{\bar{\mathrm{C}}}$ appear in algebra of Q's as functions of U.

$$\mathrm{C} = \hbar^{-1}(\mathrm{U}^2 - 1), \qquad \widehat{\mathrm{C}} = \tfrac{1}{2}(\mathrm{U}^2 + 1)\mathrm{H}, \qquad \ldots$$

Not independent elements of Y. Deformation of $\mathrm{Y}[\mathfrak{gl}(2|2)]$!

RTT Coalgebra

Next consider fusion relation

$$\Delta(\mathcal{T}^A{}_B(u)) = \mathcal{T}^A{}_C(u) \otimes \mathcal{T}^C{}_B(u).$$

Expansion yields at leading order

$$\Delta(\mathrm{U}^{|B|}\delta^A{}_B) = \delta^A{}_C \mathrm{U}^{|C|} \otimes \delta^C{}_B \mathrm{U}^{|B|} \quad \Longrightarrow \quad \Delta(\mathrm{U}) = \mathrm{U} \otimes \mathrm{U}.$$

At next order some factors of U remain

$$\Delta(\mathrm{U}^{|B|}\mathrm{J}^A{}_B) = \mathrm{U}^{|B|}\mathrm{J}^A{}_B \otimes \mathrm{U}^{|B|} + \mathrm{U}^{|A|} \otimes \mathrm{U}^{|B|}\mathrm{J}^A{}_B.$$

Remove $U^{|B|}$ from coproduct

$$\Delta(\mathrm{J}^A{}_B) = \mathrm{J}^A{}_B \otimes 1 + \mathrm{U}^{|A|-|B|} \otimes \mathrm{J}^A{}_B.$$

Prefactor of $E^C{}_C \otimes \mathrm{U}^{|C|}$ in $\mathcal{T}$ inserts U at desired places.

Secret Symmetries

A 4×4 has 16 elements. We have $\mathrm{Q}, \bar{\mathrm{Q}}, \mathrm{L}, \tilde{\mathrm{L}}, \mathrm{H}$; one is missing: B

"Secret symmetry": $\mathfrak{gl}(1)$ automorphism within $\mathfrak{gl}(2|2)$. [Matsumoto Moriyama Torrielli] [NB Spill] [de Leeuw Regelskis Torrielli]

- Never appears on r.h.s. of algebra relations.
- Shift by central elements H has no impact on Hopf algebra.
- No B at level zero! (dual picture: central element U at level -1)

Coproduct of $\widehat{\mathrm{B}}$

$$\Delta(\widehat{\mathrm{B}}) = \widehat{\mathrm{B}} \otimes 1 + 1 \otimes \widehat{\mathrm{B}} + \tfrac{1}{2}\hbar(\mathrm{Q}^{\alpha}{}_{b}\mathrm{U}^{-1} \otimes \bar{\mathrm{Q}}^{b}{}_{\alpha} + \bar{\mathrm{Q}}^{a}{}_{\beta}\mathrm{U} \otimes \mathrm{Q}^{\beta}{}_{a}).$$

Non-trivial commutators of $\widehat{\mathrm{B}}$

$$[\widehat{\mathrm{B}}, \mathrm{Q}^{\alpha}{}_{b}] = -\widehat{\mathrm{Q}}^{\alpha}{}_{b} + \varepsilon^{\alpha\gamma}\varepsilon_{bd}(\mathrm{U}^2 + 1)\bar{\mathrm{Q}}^{d}{}_{\gamma},$$
$$[\widehat{\mathrm{B}}, \bar{\mathrm{Q}}^{a}{}_{\beta}] = \widehat{\bar{\mathrm{Q}}}^{a}{}_{\beta} + \varepsilon^{ac}\varepsilon_{\beta\delta}(1 + \mathrm{U}^{-2})\mathrm{Q}^{\delta}{}_{c}.$$

All **higher-level** versions of B exist.

- Level-2 version explicitly constructed.
- Coproduct consistent and compatible with classical limit (unusual U's).

VI. Quotient

Crossing Relation

RTT realisation of algebra Y constructed. Can we quotient out an ideal?
R-matrix $\mathcal{R}$ satisfies crossing equation. Can demand the same for $\mathcal{T}$

$$\varepsilon^{AC}\varepsilon_{BD}\mathrm{S}[\mathcal{T}^{D}{}_{C}(u')] = \mathcal{T}(u)^{A}{}_{B}.$$

Crossing involves a transformation of spectral parameter $u \to u'$:

- u' denotes a different sheet for the representation parameters.
- $u' = \infty$ is different from the point $u = \infty$.

Notes: for the fundamental representation $\mathcal{T} \simeq \mathcal{R}$

- L and $\tilde{L}$ are mapped to themselves by ε^2.
- H is mapped to $-H$ by ε^2; different expansion point $u' = \infty$.
- Q and $\bar{\mathrm{Q}}$ are interchanged by ε^2.
  $\mathrm{Q}(u)$ and $\bar{\mathrm{Q}}(u)$ as two sheets of $\mathcal{T}(u)$.

Conclusion. $\mathcal{T}(u)$ contains two copies of the same algebra. Divide out!

Spectral Parameter Plane

How does $\mathcal{T}^A{}_B(u)$ depend on u as complex variable?

- Reflects behaviour of fundamental representation.
- Four quadratic branch points ($u = \pm 2 \pm \frac{1}{2}\hbar$): genus 1 surface.
- Complex structure τ_1 of torus for fundamental representation.

Puzzle. Why special reference to fundamental representation?

- Higher representations have a different complex structure τ_n.
  Branch points at different points $u = \pm 2 \pm \frac{1}{2}n\hbar$.

Resolution.

- Y is spanned by polynomials in $\mathcal{T}^A{}_B(u)$.
- Some branch points cancel in $\mathcal{T}^A{}_B(u + \frac{1}{2}\hbar)\mathcal{T}^C{}_D(u - \frac{1}{2}\hbar)$.
  Recover structure of complex structure of higher representations.
- $\mathcal{T}^A{}_B(u)$ was defined using the fundamental representation. Choice!
- $\mathcal{T}^A{}_B(u \pm \frac{1}{2}\hbar)\mathcal{T}^C{}_D(u \mp \frac{1}{2}\hbar)$ from RTT using symmetric representation.
- $\mathcal{T}^A{}_B(u)\mathcal{T}^C{}_D(u')$ corresponds to non-evaluation representation RTT.

VII. Conclusions

Conclusions

Reviewed:

- Shown R-matrix for Hubbard chain and AdS/CFT worldsheet.
- Described Yangian algebra via Drinfeld realisation.
- RTT realisation method based on concrete R-matrix.

New Results:

- Constructed RTT realisation.
- Compared to Drinfeld realisation.
- Explained some peculiar features.

Outlook.

- Relate to larger algebra $\mathfrak{h} = \mathfrak{sl}(2) \ltimes \mathfrak{psl}(2|2) \ltimes \mathfrak{gl}(1)^3$. [NB de Leeuw, Hecht in progress]
- Construct Yangian double.
- Construct RTT realisation for quantum affine algebra.
- Construct/compare Drinfeld II realisation & universal R-matrix. [Spill Torrielli]
- Explain deviation from difference form.
  Quantum-deformed Lorenz boost operator.