Multifractality at Anderson transitions with Coulomb interaction

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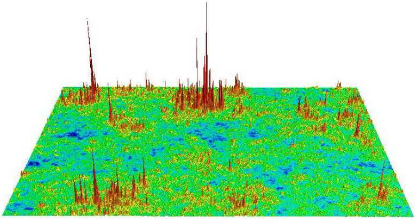
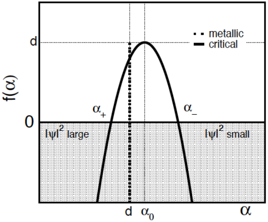
[Wegner (1980,1987); Kravtsov, Lerner (1985); Pruisken(1985); Castellani, Peliti (1986)]

$$\int_{|\boldsymbol{r}|\leqslant L} d^d\boldsymbol{r} \left\langle \left|\psi_E(\boldsymbol{r})\right|^{2q} \right\rangle_{\text{dis}} \sim L^{-\tau_q}, \quad \tau_q = \begin{cases} d(q-1), & \text{metal,} \\ d(q-1)+\Delta_q, & \text{criticality,} \\ 0, & \text{insulator} \end{cases}$$

- multifractal exponent $\Delta_q \leqslant 0$ ($\Delta_1 = 0$ due to w.f. normalization)
- Legendre transform of τ_q: $f(\alpha) = q\alpha - \tau_q, \quad \alpha = d\tau_q/dq$
- $L^{f(\alpha)}$ measures a set of points where $\left|\psi_E\right|^2 \sim L^{-\alpha}$



[adapted from Evers, Mildenberger, Mirlin]



[adapted from Evers, Mirlin (2008)]

- local density of states (LDOS)

$$\rho(E, \boldsymbol{r}) = \sum_{\alpha} \left|\psi_\alpha(\boldsymbol{r})\right|^2 \delta(E - \epsilon_\alpha)$$

where $\psi_\alpha(\boldsymbol{r})$ and ϵ_α w. f. and energy for a given disorder

- multifractality in the moments of LDOS

$$\left\langle [\rho(E, \boldsymbol{r})]^q \right\rangle_{\text{dis}} \sim L^{-\Delta_q}$$

- spatial correlations of LDOS

$$\langle \rho(E, \boldsymbol{r}) \rho(E, \boldsymbol{r} + \boldsymbol{R}) \rangle_{\text{dis}} \sim (R/L)^{\Delta_2} \qquad R \ll L$$

examples for Anderson transitions in $d = 2$:

$\Delta_2 = -0.34$ (class AII, spin-orbit coupling)
$\Delta_2 = -0.52$ (class A, integer qHe)
$\Delta_2 = -1/4$ (class C, spin qHe)

[see for a review, Evers&Mirlin (2008)]

[Altshuler, Aronov, Lee(1980), Finkelstein(1983), Castellani, DiCastro, Lee, Ma(1984)]
[Nazarov (1989), Levitov, Shytov (1997), Kamenev, Andreev (1999)]

- zero-bias anomaly in $d = 2$ $(L = \infty)$

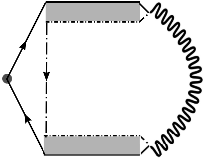
$$\langle \rho(E, \boldsymbol{r}) \rangle_{\text{dis}} \sim \exp\left(-\frac{1}{4\pi g} \ln\big(|E|\tau\big) \ln \frac{|E|}{D^2 \kappa^4 \tau} \right)$$

where

g - conductance in units e^2/h,
D - diffusion coefficient,
$\kappa = e^2 \rho_0 / \varepsilon$ - inverse static screening length

- zero-bias anomaly in $d = 2 + \epsilon$ at Anderson transitions

$$\langle \rho(E, \boldsymbol{r}) \rangle_{\text{dis}} \sim |E|^{\beta}, \qquad \beta = O(1)$$



in the absence of interaction average LDOS is non-critical ($\beta = 0$) for Wigner-Dyson classes

[see for a review, Finkelstein (1990), Kirkpatrick&Belitz (1994)]

[Gruzberg, Read, Ludwig (1999), Beamond, Cardy, Chalker (2002)]
[Mirlin, Evers, Mildenberger (2003)]

- the average LDOS ($L=\infty$)

$$\langle \rho(E,\boldsymbol{r}) \rangle \sim |E-E_c|^{\beta}, \qquad \beta = 1/7, \qquad E_c = 0$$

- multifractality in the moments of LDOS

$$\langle [\rho(E,\boldsymbol{r})]^q \rangle_{\rm dis} \sim \langle [\rho(E,\boldsymbol{r})] \rangle_{\rm dis}^q \, [\xi(E)]^{-\Delta_q} \sim |E-E_c|^{\beta q + \nu \Delta_q},$$

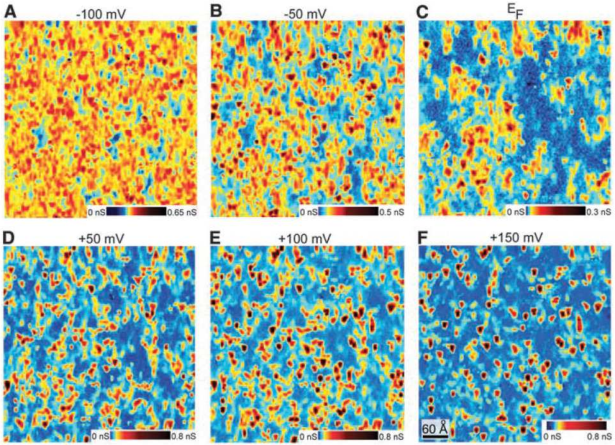
where $\xi(E) \sim |E-E_c|^{-\nu}$, $\nu = 4/7$, $\Delta_2 = -1/4$, $\Delta_3 = -3/4$

- spatial correlations of LDOS

$$\langle \rho(E,\boldsymbol{r}) \rho(E,\boldsymbol{r}+\boldsymbol{R}) \rangle_{\rm dis} \sim \langle [\rho(E,\boldsymbol{r})] \rangle_{\rm dis}^2 \, (R/\xi(E))^{\Delta_2} \qquad R \ll \xi(E)$$

[Richardella et al. (2010)]

- differential conductance over an area of 500 Å× 500 Å in $Ga_{1-x}Mn_xAs$ with $x = 1.5\%$

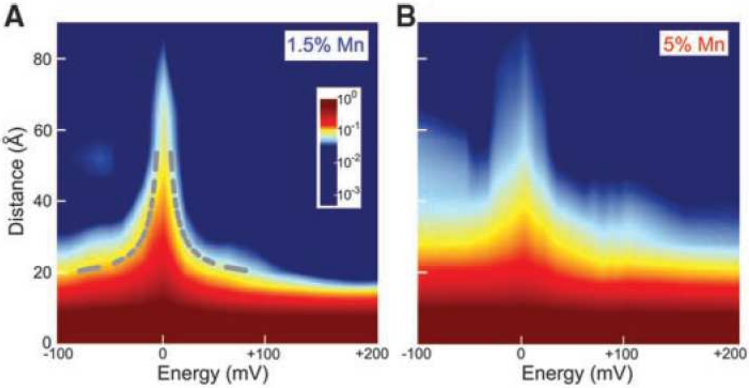


mean-free path $l = 10$ Å

[Richardella et al. (2010)]

- the autocorrelation function of LDOS

$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle - \langle \rho(E,\boldsymbol{r})\rangle^2}{\langle \rho^2(E,\boldsymbol{r})\rangle^2 - \langle \rho(E,\boldsymbol{r})\rangle^2}$$



How Coulomb interaction affects mesoscopic fluctuations of the local density of states?

- free electrons in $d = 2 + \epsilon$-dimensions

$$H_0 = \int d^d\boldsymbol{r}\,\overline{\psi}_\sigma(\boldsymbol{r})\left[-\frac{\nabla^2}{2m}\right]\psi_\sigma(\boldsymbol{r})$$

- scattering off white-noise random potential

$$H_{\rm dis} = \int d^d\boldsymbol{r}\,\overline{\psi}_\sigma(\boldsymbol{r})\,V(\boldsymbol{r})\psi_\sigma(\boldsymbol{r}), \qquad \langle V(\boldsymbol{r})\,V(0)\rangle = \frac{1}{2\pi\rho_0\tau}\delta(\boldsymbol{r})$$

- scattering off magnetic impurities

$$H_{\rm mag} = \int d^d\boldsymbol{r}\,\overline{\psi}_\sigma(\boldsymbol{r})\,\boldsymbol{U}(\boldsymbol{r})\boldsymbol{\sigma}_{\sigma\sigma'}\psi_{\sigma'}(\boldsymbol{r}), \qquad \langle U_a(\boldsymbol{r})\,U_b(0)\rangle = \frac{\delta_{ab}}{6\pi\rho_0\tau_s}\delta(\boldsymbol{r})$$

- Coulomb interaction:

$$H_{\rm int} = \frac{1}{2}\int d^d\boldsymbol{r_1}d^d\boldsymbol{r_2}\,\frac{e^2}{\varepsilon|\boldsymbol{r_1}-\boldsymbol{r_2}|}\,\overline{\psi}_\sigma(\boldsymbol{r_1})\psi_\sigma(\boldsymbol{r_1})\overline{\psi}_{\sigma'}(\boldsymbol{r_2})\psi_{\sigma'}(\boldsymbol{r_2})$$

$H_0 + H_{\rm dis} + H_{\rm mag}$ do not preserve time-reversal and spin-rotational symmetry $\Rightarrow$ class A

- disorder-averaged moments of the LDOS

$$\left\langle \left[\rho(E,\boldsymbol{r})\right]^q \right\rangle_{\text{dis}} = \left\langle \left[-\frac{1}{\pi}\,\mathrm{Im}\,\mathcal{G}^R(E,\boldsymbol{r},\boldsymbol{r})\right]^q \right\rangle_{\text{dis}}$$

where the retarded single-particle Green function

$$\mathcal{G}^R(\boldsymbol{r}t;\boldsymbol{r'}t') = -i\theta(t-t')\left\langle \left\{\overline{\psi}(\boldsymbol{r}t),\psi(\boldsymbol{r'}t')\right\}\right\rangle$$

- assumptions

$$\mu \gg \frac{1}{\tau}, \frac{1}{\tau_s} \gg T, |E|$$

where

μ – chemical potential
τ – mean-free time for scattering off potential impurities
τ_s – mean-free time for scattering off magnetic impurities
T – temperature
E – energy measured from the chemical potential

in what follows we consider $T = 0$

[Finkelstein(1983)]

- action for the nonlinear sigma-model

$$\mathcal{S} = -\frac{g}{4}\int d^d\boldsymbol{r}\,\mathrm{tr}(\nabla Q)^2 + 4\pi Tz\int d^d\boldsymbol{r}\,\mathrm{tr}\,\eta(Q-\Lambda)$$
$$-\pi T\Gamma_s\sum_{\alpha,n}\int d\boldsymbol{r}\,\mathrm{tr}\,I^\alpha_n Q\,\mathrm{tr}\,I^\alpha_{-n}Q$$

where the matrix (in Matsubara and replica spaces) field Q satisfies

$$Q^2(\boldsymbol{r}) = 1, \qquad \mathrm{tr}\,Q(\boldsymbol{r}) = 0, \qquad Q^\dagger(\boldsymbol{r}) = Q(\boldsymbol{r}),$$

g – conductivity in units e^2/h,
$\Gamma_s = -z$ – interaction amplitude in the singlet channel,

and matrices

$$\Lambda^{\alpha\beta}_{nm} = \mathrm{sgn}\,n\,\delta_{nm}\delta^{\alpha\beta}, \qquad \eta^{\alpha\beta}_{nm} = n\,\delta_{nm}\delta^{\alpha\beta}, \qquad (I^\gamma_k)^{\alpha\beta}_{nm} = \delta_{n-m,k}\delta^{\alpha\beta}\delta^{\alpha\gamma}$$

symmetry class “MI(LR)"as denoted in Kirkpatrick&Belitz (1994)

- disorder-averaged LDOS

$$\Big\langle \rho(E,\boldsymbol{r}) \Big\rangle_{\rm dis} = \rho_0 \, \mathrm{Re} \, \langle P_1^R(E) \rangle_{\mathcal{S}}, \qquad P_1(i\varepsilon_n) = Q_{nn}^{\alpha\alpha}(\boldsymbol{r})$$

where $\varepsilon_n = \pi T(2n+1)$ is fermionic Matsubara frequencies

- disorder-averaged 2d moment of LDOS

$$\Big\langle \rho(E,\boldsymbol{r})\rho(E',\boldsymbol{r}) \Big\rangle_{\rm dis} = (\rho_0^2/2)\, \mathrm{Re} \Big\langle P_2^{RR}(E,E') - P_2^{RA}(E,E') \Big\rangle_{\mathcal{S}}$$

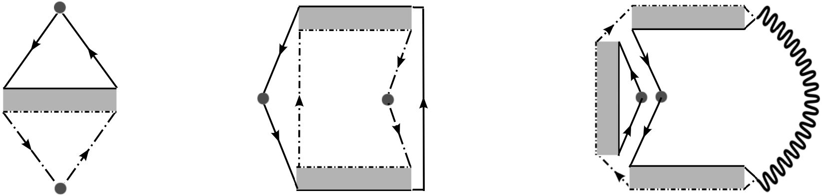
$$P_2(i\varepsilon_n, i\varepsilon_m) = Q_{nn}^{\alpha_1\alpha_1}(\boldsymbol{r})\, Q_{mm}^{\alpha_2\alpha_2}(\boldsymbol{r}) - Q_{nm}^{\alpha_1\alpha_2}(\boldsymbol{r})\, Q_{mn}^{\alpha_2\alpha_1}(\boldsymbol{r})$$

$\ldots$

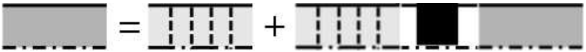
- P_q are eigenoperators of renormalization group transformations

renormalization of P_q by means of perturbation theory around $Q = \Lambda$

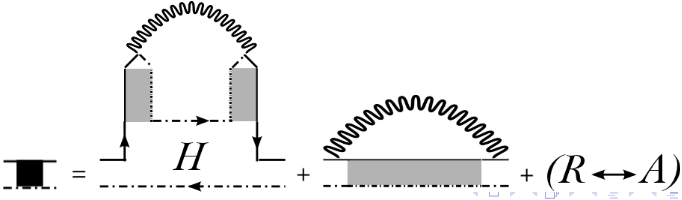
- one- and two-loop contributions to the 2d moment of LDOS



- diffuson with self-energy due to interaction



- diffuson self-energy due to interaction



[Abrahams, Anderson, Licciardello, Ramakrishnan (1979), Wegner (1980)]
[Efetov, Larkin, Khmelnitsky (1980)]

- RG equation for conductivity $g = (e^2/h)L^{2-d}/(\pi t)$:

$$-\frac{dt}{d\ln L} = \beta(t)$$

- fixed point t_* and localization/correlation length exponent ν:

$$\beta(t_*) = 0, \qquad \xi \propto \left|t - t_*\right|^{-\nu}, \qquad \nu = -1/\beta'(t_*)$$

- dynamical exponent z:

$$L_\omega \sim \omega^{-1/z}$$

in general, there is another dynamical exponent z_T, $L_\phi \sim T^{-1/z_T}$

[Hikami (1983), Bernreuther&Wegner (1986)]
[Castellani, DiCastro, Lee, Ma (1984), Finkelstein(1984)]
[Baranov, Pruisken, Škorić (1999), Baranov, Burmistrov, Pruisken (2002)]

- no interaction

$$\beta(t) = \epsilon t - \frac{1}{2}t^3 - \frac{3}{8}t^5 + O(t^6)$$

critical resistance

$$t_*^{(\mathrm{n})} = \sqrt{2\epsilon}\left(1 - \frac{3\epsilon}{4}\right) + O(\epsilon^{5/2})$$

critical exponents

$$\nu_{\mathrm{n}} = \epsilon/2 - 3/4 + O(\epsilon)$$
$$z_{\mathrm{n}} = d = 2+\epsilon$$
$$\beta_{\mathrm{n}} = 0$$

- Coulomb interaction

$$\beta(t) = \epsilon t - 2t^2 - 4At^3 + O(t^4)$$

$$t_* = \frac{\epsilon}{2}(1 - A\epsilon) + O(\epsilon^3)$$

$$\nu = 1/\epsilon - A + O(\epsilon)$$
$$z = 2 + \epsilon/2 + B\epsilon^2 + O(\epsilon^3)$$
$$\beta = 1/2 + O(\epsilon)$$

where $B = (2A - \pi^2/6 - 3)/4 \approx -0.34$ and

$$A = \frac{139}{96} + \frac{(\pi^2-18)^2}{192} + \frac{19}{32}\zeta(3) + \left(1 + \frac{\pi^2}{48}\right)\ln^2 2 - \left(44 - \frac{\pi^2}{2} + 7\zeta(3)\right)\frac{\ln 2}{16} + \mathcal{G} - \frac{1}{48}\ln^4 2 - \frac{1}{2}\mathrm{li}_4\left(\frac{1}{2}\right) \approx 1.64$$

- scaling of the moments of LDOS

$$\left\langle [\rho(E, \boldsymbol{r})]^q \right\rangle \sim \langle \rho(E) \rangle^q \mathcal{L}^{-\Delta_q} \sim \mathcal{L}^{-\beta z q - \Delta_q} \qquad \mathcal{L} = \min\{L_E, L, \xi\}$$

where multifractal exponents and anomalous dimensions are

$$\Delta_q = \zeta_q(t_*) = \frac{q(1-q)\epsilon}{4}\left[1 + \left(1 - A - \frac{\pi^2}{12}\right)\epsilon\right] + O(\epsilon^3)$$

$$\zeta_q(t) = \frac{q(1-q)t}{2}\left[1 + \left(2 - \frac{\pi^2}{6}\right)t\right] + O(t^3)$$

cf. anomalous dimensions for non-interacting electrons

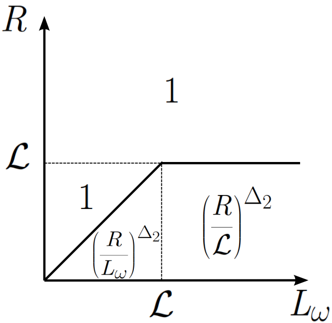
$$\Delta_q^{(\mathrm{n})} = \zeta_q^{(\mathrm{n})}(t_*^{(\mathrm{n})}) = q(1-q)\left(\frac{\epsilon}{2}\right)^{1/2} - \frac{3\zeta(3)}{32} q^2 (q-1)^2 \epsilon^2 + O(\epsilon^{5/2})$$

$$\zeta_q^{(\mathrm{n})}(t) = \frac{q(1-q)t}{2}\left(1 + \frac{3t^2}{8} + \frac{3\zeta(3)}{16} q(q-1)t^3\right) + O(t^5),$$

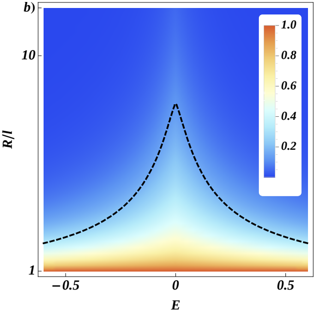
[Höf&Wegner (1986), Wegner (1987)]

- spatial and energy correlations of LDOS ($\mathcal{L} = \min\{L_E, L, \xi\}$)

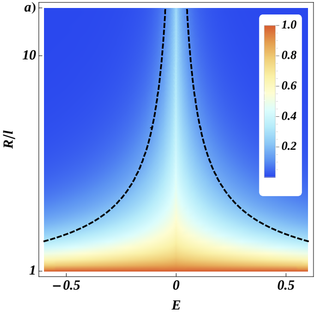
$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E+\omega,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle \rho(E)\rangle\langle \rho(E+\omega)\rangle} \sim \begin{cases} (R/L_\omega)^{\Delta_2}, & R \ll L_\omega \ll \mathcal{L} \\ 1, & L_\omega \ll R, \mathcal{L} \\ (R/\mathcal{L})^{\Delta_2}, & R \ll \mathcal{L} \ll L_\omega \\ 1, & \mathcal{L} \ll R, L_\omega \end{cases}$$



$$\frac{\langle\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle\rangle}{\langle\langle \rho^2(E,\boldsymbol{r})\rangle\rangle} \sim \begin{cases} (R/\mathcal{L})^{-\Delta_2}, & R \ll \mathcal{L} = \min\{|E|^{-1/z}, (\mu-\mu_c)^{-1/\nu}\} \\ 0, & \mathcal{L} \ll R \end{cases}$$



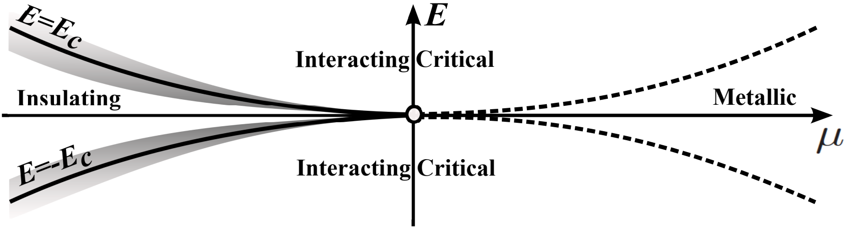
metallic phase $t < t_*$ $(\mu > \mu_c)$



critical phase $t = t_*$ $(\mu = \mu_c)$

qualitatively in agreement with the experiment

- phase diagram near interacting critical point $\mu = \mu_c$ $(t = t_*)$



- mobility edge for single particle excitations $E = \pm E_c$,

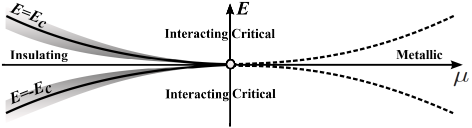
$$E_c \sim (t - t_*)^{\nu z} \sim (\mu_c - \mu)^{\nu z}$$

- divergent localization and dephasing lengths

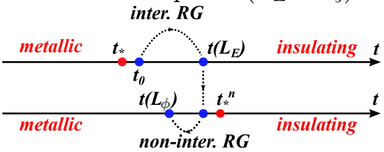
$$\xi(E) \sim \left||E| - E_c\right|^{-\nu_{\rm n}}, \qquad L_\phi(E) \sim \begin{cases} \left(|E| - E_c\right)^{-z_{\rm n}}, & |E| > E_c \\ \infty, & |E| < E_c \end{cases}$$

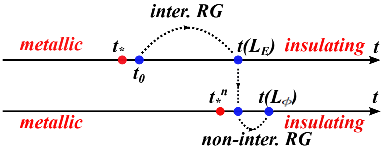
where $z_{\rm n} = d^2/[d + \Delta_2^{({\rm n})}] = 2 + \sqrt{2\epsilon} + \ldots$

- phase diagram near interacting critical point $\mu = \mu_c$ $(t = t_*)$



- two-step RG $(L_E \ll \xi)$:





$t(L_E) < t_*^n$ $\qquad$ $t(L_E) > t_*^n$

- mobility edge for single particle excitations

$$t(L_E) = t_*^n \qquad \Longrightarrow \qquad E_c \sim (t - t_*)^{\nu z} \sim (\mu_c - \mu)^{\nu z}$$

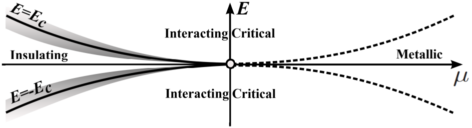
we use the condition $z\nu > 1$

- interacting criticality $|E| \gg E_c$ ($L_E \ll \xi$)

$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle \rho(E)\rangle^2} \sim \begin{cases} (R/L_E)^{\Delta_2}, & R \ll L_E \sim |E|^{-1/z} \\ 1, & L_E \ll R \ll \xi \sim E_c^{-1/z} \end{cases}$$

- deep below the mobility edge $|E| \ll E_c$ ($L_E \gg \xi$)

$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle \rho(E)\rangle^2} \sim (R/\xi)^{\Delta_2} (L/\xi)^{d}, \quad R \ll \xi \ll L$$



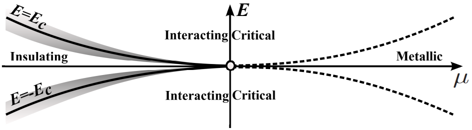
- non-interacting criticality $\big||E|-E_c\big| \ll E_c$:

  - above the mobility edge $|E| > E_c$

$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle\rho(E)\rangle^2} \sim \begin{cases} (R/\xi)^{\Delta_2}\,(\xi/L_\phi(E))^{\Delta_2^{(\mathrm{n})}}\,, & R \ll \xi \\ (R/L_\phi(E))^{\Delta_2^{(\mathrm{n})}}\,, & \xi \ll R \ll L_\phi(E) \end{cases}$$

  - below the mobility edge $|E| < E_c$

$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle\rho(E)\rangle^2} \sim \begin{cases} (R/\xi)^{\Delta_2}\,(\xi/\xi(E))^{\Delta_2^{(\mathrm{n})}}\,(L/\xi(E))^{d}\,, & R \ll \xi \\ (R/\xi(E))^{\Delta_2^{(\mathrm{n})}}\,(L/\xi(E))^{d}\,, & \xi \ll R \ll \xi(E) \end{cases}$$



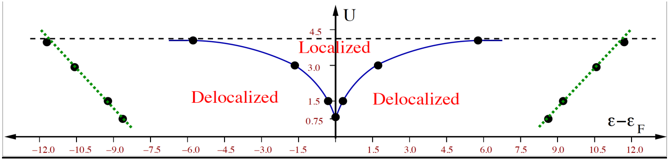
where $\xi(E) \sim \big||E|-E_c\big|^{-\nu_\mathrm{n}}$, $L_\phi(E) \sim \big(|E|-E_c\big)^{-1/z_\mathrm{n}}$. We use that $z_\mathrm{n}\nu_\mathrm{n} > 1$

[Amini, Kravtsov, Müller (2013)]

- LDOS correlation function made from Hartree-Fock w.f.

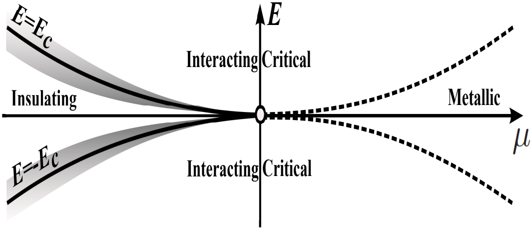
$$\frac{\langle \rho_{HF}(E, \boldsymbol{r}) \rho_{HF}(E, \boldsymbol{r}+\boldsymbol{R}) \rangle}{\langle \rho_{HF}^2(E, \boldsymbol{r}) \rangle^2}$$

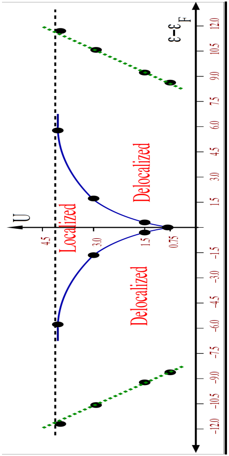
- extracted phase diagram interaction vs energy



Spatial correlations of LDOS / comparison with numerics on Hartree-Fock w.f.

- phase diagram interaction vs energy





[Amini, Kravtsov, Müller (2013)]

rough Hartree-Fock approximation produces qualitatively similar phase diagram

- multifractality of LDOS does exist in the interacting disordered electron systems
- in the case of Coulomb interaction the multifractal exponents and corresponding anomalous dimensions are different from the non-interacting case