Superconformal field theory and supersymmetric sine-Gordon model with Dirichlet boundary conditions

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Outline

Introduction

supersymmetric sine-Gordon (SSG) model

$$\mathcal{S}_{\rm SSG} = \int_{-\infty}^{\infty} dt \int_0^L d\sigma \left(\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{i}{2}\bar{\Psi}\gamma^\mu \partial_\mu \Psi - \frac{m_0}{2}\cos(\beta\varphi)\bar{\Psi}\Psi + \frac{m_0^2}{2\beta^2}\cos^2(\beta\varphi) \right)$$

$$\Psi = \begin{pmatrix} \psi \\ \bar{\psi} \end{pmatrix} \qquad \gamma^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

QFT (perturbation theory from CFT)

- CFT data
- S-matrix theory
- excitation energy
- **asymptotic behaviors**

StatMech (light-cone regularization)

- CFT data
- scattering amplitudes
- excitation energy and **ground-state energy**
- **correlation functions on a finite system**

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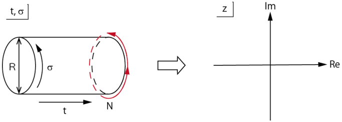
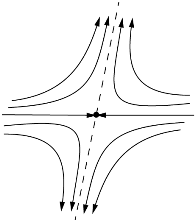
SSG model as a QFT

$$\mathcal{S}_{\rm SSG} = \int_{-\infty}^{\infty} dt \int_0^L d\sigma \left(\underbrace{\frac{1}{2}(\partial_\mu \varphi)^2}_{\text{FB}} + \underbrace{\frac{i}{2}\bar{\Psi}\gamma^\mu \partial_\mu \Psi}_{\text{FF}} - \underbrace{\frac{m_0}{2}\cos(\beta\varphi)\bar{\Psi}\Psi}_{\text{perturbation}} + \underbrace{\frac{m_0^2}{2\beta^2}\cos^2(\beta\varphi)}_{\text{negligible in UV lim.}} \right)$$

conformal mapping



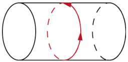
$$z = e^{i\left(\frac{2\sigma}{R} - it\right)}$$
$$\bar{z} = e^{i\left(\frac{2\sigma}{R} + it\right)}$$
$$A(t,\sigma) \to A(z,\bar{z})$$
$$A(z,\bar{z}) = \tfrac{1}{2}(\alpha(z) + \bar{\alpha}(\bar{z}))$$



RG flow

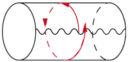
mode expansion

$$i\partial\phi(z) = \textstyle\sum_{n\in\mathbb{Z}} a_n z^{-n-1} \quad \text{(boson field)}$$
$$\psi(z) = \textstyle\sum_{n\in\mathbb{Z}+r} b_n z^{-n-1/2} \quad \text{(fermion field)}$$



$r = 1/2$
periodic b.c.

NS fermion



$r = 0$
anti-periodic b.c.

R fermion

Algebraic structure of CFT

space of states (free boson)

$$\bigoplus_{m,n\in\mathbb{Z}} \bigoplus_{p_i,q_j>0} \prod_i \bar{a}_{-p_i} \prod_j a_{-q_j} \underbrace{|m,n\rangle}_{\text{h.w.vec}}$$

$$|m,n\rangle = V_{(m,n)}(z,\bar{z}) \underbrace{|0,0\rangle}_{\text{vacuum}}$$

vertex operators

$$V_{(m,n)} := e^{:i(mR+\frac{n}{2R})\phi(z)+i(mR-\frac{n}{2R})\bar{\phi}(\bar{z}):}$$

conformal dimensions

$$\Delta^{\pm}_{\mathrm{FB}} = \frac{1}{2}\alpha_0^2 := \frac{1}{2}(mR \pm \frac{n}{2R})^2$$

energy

$$E(0) \sim -\frac{\pi}{6L}(1 - 12(\Delta_+ + \Delta_-))$$

$$\Rightarrow c = 1 \text{ CFT}$$

space of states (free fermion)

$$\bigoplus_{\hat{f}\in\mathcal{V}} \bigoplus_{p_i,q_j>0} \prod_i \bar{b}_{-p_i} \prod_j b_{-q_j} \underbrace{\hat{f}(z,\bar{z})|0\rangle}_{\text{h.w.vec}}$$

$$\mathcal{V} \in \{\underbrace{\mathbb{I},\ \psi(z)\bar{\psi}(\bar{z})}_{\text{NS sector}},\ \underbrace{\sigma(z,\bar{z})}_{\text{R sector}}\}$$

conformal dimensions

$$(\Delta^+_{\mathbb{I}}, \Delta^-_{\mathbb{I}}) = (0,0)$$

$$(\Delta^+_{\psi\bar{\psi}}, \Delta^-_{\psi\bar{\psi}}) = (\tfrac{1}{2}, \tfrac{1}{2})$$

$$(\Delta^+_{\sigma}, \Delta^-_{\sigma}) = (\tfrac{1}{16}, \tfrac{1}{16})$$

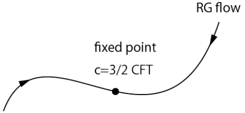
energy

$$E(0) \sim -\frac{\pi}{6L}\left(\frac{1}{2} - 12(\Delta_+ + \Delta_-)\right)$$

$$\Rightarrow c = 1/2 \text{ CFT}$$

Summary 1

- The UV lim. of the SSG model results in the $c = 3/2$ CFT.



- Hilbert space of the UV lim. of the SSG model consists of $\mathcal{H}_{\text{FB}}$ and $\mathcal{H}_{\text{FF}}$.

$$\mathcal{H}_{\text{SSG-UV}} = \mathcal{H}_{\text{FB}} \otimes \mathcal{H}_{\text{FF}}$$

- States of $\mathcal{H}_{\text{FB}}$ and $\mathcal{H}_{\text{FF}}$ are characterized by $\Delta_\pm$.

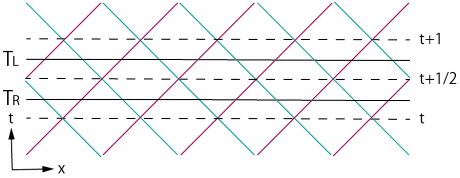
$$\begin{aligned}(\Delta_+, \Delta_-) \;&\rightarrow\; |\Delta_+, \Delta_-\rangle \quad : \text{ a unique state} \\ &\rightarrow\; E_{(\Delta_+,\Delta_-)}(0) : \text{ CFT energy}\end{aligned}$$

SSG model on a lattice

transfer matrix method

$\Downarrow$

light-cone regularization [Destri-DeVega 87]
(discretization of space-time)



$= R(x)$

$$\mathcal{T}(x) = \overrightarrow{\prod}_{j=1}^{N} R_{0j}(x-(-1)^j\Lambda)$$

$$T_{\mathrm{R}} = \mathrm{tr}_0 \mathcal{T}(\Lambda)$$

$$T_{\mathrm{L}} = \mathrm{tr}_0 \mathcal{T}(-\Lambda)$$

$$H \pm P = \frac{i\gamma}{2\pi a} \ln T_{\mathrm{R,L}}$$

- Does this discretization properly give the original theory in the scaling limit?

$$a \to 0 \quad \Lambda \to \infty \quad m_0 = \frac{2}{a} e^{-\pi\Lambda/\gamma} : \text{ fixed}$$

- Compare the CFT data derived in the original theory and that computed from the lattice system.

CFT data of the LSSG model

CFT data $\Leftarrow \lim_{m_0 L \to 0} E(L)$

periodic b.c. [Hegedus-Ravanini-Suzuki 08]

$$E(0) = \underbrace{E_{\text{bulk}}}_{\mathcal{O}(L)} + \underbrace{E_{\text{Casimir}} + E_{\text{excitation}}}_{\mathcal{O}(1/L)} \sim -\frac{\pi}{6L}\left(\frac{3}{2} - 12(\Delta_+ + \Delta_-)\right)$$

$$\Delta_\pm = \frac{1}{2}(SR \pm \frac{n}{2R})^2 : \text{ conformal dimensions}$$

$$= \Delta^\pm_{\text{FB}} + \underbrace{\Delta^\pm_{\text{FF}}}_{=0} \Rightarrow \text{NS sector}$$

- The conformal dimensions $\Delta_\pm$ only take values in the NS sector.
- The light-cone regularization realized only the NS sector of the SSG model under the periodic boundary conditions.
- How about the other boundary conditions? Can't the R sector be obtained?

SSG model with Dirichlet boundaries

$$\mathcal{S}_{\mathrm{SSG}} = \int_{-\infty}^{\infty} dt \int_0^L d\sigma \left(\frac{1}{2}(\partial_\mu \varphi)^2 + \frac{i}{2}\bar{\Psi}\gamma^\mu \partial_\mu \Psi - \frac{m_0}{2}\cos(\beta\varphi)\bar{\Psi}\Psi + \frac{m_0^2}{2\beta^2}\cos^2(\beta\varphi) \right)$$

$$\varphi(0,t) = \varphi_- \quad \varphi(L,t) = \varphi_+ \quad \psi(0,t) - \bar{\psi}(0,t) = 0 \quad \psi(L,t) - \bar{\psi}(L,t) = 0$$

$\Downarrow$ light-cone regularization ($H = \frac{i\gamma}{4\pi a}[\ln T_{\mathrm{R}} + \ln T_{\mathrm{L}}]$)

eigenenergy

$$E(L) = \frac{i\gamma}{4\pi a}\left[\frac{d}{dx}\ln T_2(x)\Big|_{x=\Lambda+\frac{i\gamma}{2}} - \frac{d}{dx}\ln T_2(x)\Big|_{x=\Lambda-\frac{i\gamma}{2}} \right]$$

transfer matrix

$$T_p(x) = \mathrm{tr}_0 \overleftarrow{\prod}_{j=1}^{N} R_{0j}^{(p,2)}(x - (-1)^j \Lambda)$$

$$R_{ij}^{(\ell_i,\ell_j)}(x) : \; V_i \otimes V_j \mapsto V_j \otimes V_i$$

$$V_i \equiv \mathbb{C}^{\ell_i + 1} : \; \text{irred. spin-}\ell_i/2 \text{ repr.}$$

Yang-Baxter equations

$$R_{12}^{(\ell_1,\ell_2)}(x) R_{13}^{(\ell_1,\ell_3)}(x+y) R_{23}^{(\ell_2,\ell_3)}(y)$$
$$= R_{23}^{(\ell_2,\ell_3)}(y) R_{13}^{(\ell_1,\ell_3)}(x+y) R_{12}^{(\ell_1,\ell_2)}(x)$$

- integrability
- commuting transfer matrix

diagonalization of the transfer matrix

$\Rightarrow$ Bethe ansatz equations (BAEs)

$$\frac{\phi(\theta_j + i\pi)}{\phi(\theta_j - i\pi)} = -\frac{B^{(-)}(\theta_j)}{B^{(+)}(\theta_j)} \frac{\sinh(2\theta_j - i\pi)}{\sinh(2\theta_j + i\pi)} \frac{Q(\theta_j + i\pi)}{Q(\theta_j - i\pi)}$$

$$\phi(\theta) = \sinh^N(\theta - \Theta) \sinh^N(\theta + \Theta)$$

$$B^{(\pm)}(\theta) = \sinh(\theta \pm \tfrac{i\pi H_+}{2}) \sinh(\theta \pm \tfrac{i\pi H_-}{2})$$

$$Q(\theta) = \textstyle\prod_{j=1}^{M} \sinh(\theta - \theta_j) \sinh(\theta + \theta_j) \quad (\theta_j : \text{ Bethe roots})$$

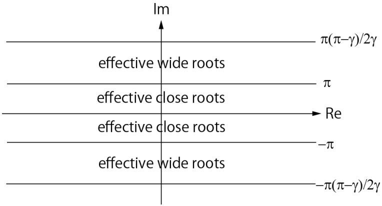
parameter relations

$$\theta = \frac{\pi x}{\gamma} \qquad \Theta = \frac{\pi \Lambda}{\gamma} \qquad \beta = 2\sqrt{\pi - 2\gamma} \qquad \varphi_\pm = \mp \frac{\gamma(H_\pm + 1)}{2\sqrt{\pi - 2\gamma}}$$

- Solutions of BAEs are called roots.
- each configuration of roots $\Rightarrow$ each state
- other important objects:
  $h_j \in \mathbb{R};\ T_2(h_j) = 0\ (j = 1, \ldots, N_H)$: holes in the distribution of 2-string roots
  $h_j^{(1)} \in \mathbb{R};\ T_1(h_j^{(1)}) = 0\ (j = 1, \ldots, N_1)$: type-1 holes in the distribution of real roots

classification of Bethe roots

(effective roots $\widetilde{\theta}_j = \theta_j - \frac{i\pi}{2}\,\mathrm{sign}(\mathrm{Im}\theta_j)$)



counting equations

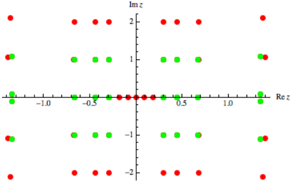
$$N_H - 2N_S = M_C + 2M_W + 2S + \lfloor -\frac{3\omega}{2\pi} + \frac{\delta_a}{2} - \lfloor \frac{\omega}{2\pi} + \frac{1+\delta_a}{2} \rfloor \rfloor$$
$$+ \frac{1}{2}\{\mathrm{sign}(1-H_+) + \mathrm{sign}(1-H_-) + \mathrm{sign}(1+H_+) + \mathrm{sign}(1+H_-)\}$$
$$N_1 - 2N_S^R = S - M_R + M_{C>\gamma} + M_W + \lfloor -\frac{\omega}{\pi} + \frac{1}{2} \rfloor + \frac{1}{2}\{\mathrm{sign}(H_+) + \mathrm{sign}(H_-)\}$$

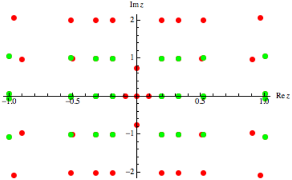
difficult to solve BAEs ⇒ nonlinear integral equations

[Klümper-Batchelor-Pearce 91, Destri-DeVega 95, Suzuki 99]

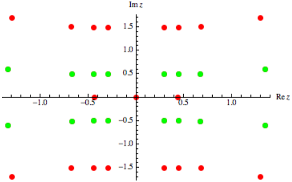
distribution of effective roots/holes ($\Lambda = \gamma = 0,\ N = M = 8$)

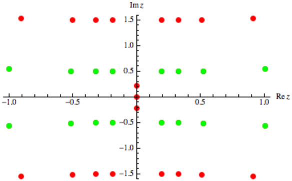
zeros of $\mathrm{B}(\theta)$





zeros of $\mathrm{Y}(\theta - \frac{i\pi}{2})$





$(H_+, H_-) = (1.5, 2.2)$ $(H_+, H_-) = (1.5, -0.5)$

auxiliary functions

$$\mathrm{B}(\theta) = 1 + \mathrm{b}(\theta) \qquad \mathrm{Y}(\theta) = 1 + \mathrm{y}(\theta)$$

$$\frac{1}{i}\ln \mathrm{b}(\theta_j) = 2\pi I_j \qquad \frac{1}{i}\ln \mathrm{y}(h_j^{(1)} - \tfrac{i\pi}{2}) = 2\pi I_j^{(1)} \qquad (I_j, I_j^{(1)} \in \mathbb{Z} + \tfrac{1}{2})$$

nonlinear integral equations (NLIEs)

$$\ln \mathrm{b}(\theta) = (\mathrm{G} * \ln \mathrm{B})(\theta) - (\mathrm{G}^{[+2\epsilon]} * \ln \bar{\mathrm{B}})(\theta) + (\mathrm{G}_K^{[\frac{\pi}{2}+\epsilon]} * \ln \mathrm{Y})(\theta) + 2im_0 L \sinh\theta$$
$$+ i\sum_j c_j\{\mathrm{g}_{(j)}(\theta - \theta_j) + \mathrm{g}_{(j)}(\theta + \theta_j)\} + i\mathrm{D}_{\mathrm{B}}(\theta) + i\pi C_b$$

$$\ln \mathrm{y}(\theta) = (\mathrm{G}_K^{[\frac{\pi}{2}-\epsilon]} + \ln \mathrm{B})(\theta) + (\mathrm{G}_K^{[-\frac{\pi}{2}+\epsilon]} * \ln \bar{\mathrm{B}})(\theta)$$
$$+ i\sum_j c_j\{\widetilde{\mathrm{g}}_{(j)}^{(1)}(\theta - \theta_j) + \widetilde{\mathrm{g}}_{(j)}^{(1)}(\theta + \theta_j)\} + i\mathrm{D}_{\mathrm{SB}}(\theta) + i\pi C_y$$

kernels

$$\mathrm{G}(\theta) = \int_{-\infty}^{\infty} \frac{dk}{2\gamma} e^{-ik\theta} \frac{\sinh(\frac{\pi}{\gamma} - 3)\frac{\pi k}{2}}{2\cosh\frac{\pi k}{2}\sinh(\frac{\pi}{\gamma} - 2)\frac{\pi k}{2}}$$

$$\mathrm{G}_K(\theta) = \int_{-\infty}^{\infty} \frac{dk}{2\gamma} e^{-ik\theta} \frac{1}{e^{\frac{\pi k}{2}} + e^{-\frac{\pi k}{2}}}$$

particle source terms

$$c_j = \begin{cases} +1 & \text{for holes} \\ -1 & \text{otherwise} \end{cases}$$

$$\mathrm{g}_{(j)}(\theta) = \begin{cases} \mathrm{g}_{\mathrm{II}}(\theta) := \mathrm{g}(\theta) + \mathrm{g}(\theta - i\pi\,\mathrm{sign}(\mathrm{Im}\theta)) & \text{for wide roots} \\ \mathrm{g}(\theta + i\epsilon) + \mathrm{g}(\theta - i\epsilon) & \text{for specials} \\ \mathrm{g}_K(\theta) & \text{for type-1 holes} \\ \mathrm{g}(\theta) & \text{otherwise} \end{cases}$$

$$\widetilde{\mathrm{g}}^{(1)}_{(j)}(\theta) = \lim_{\epsilon \to +0} \mathrm{g}^{(1)}_{(j)}(\theta + \tfrac{i\pi}{2} - i\epsilon)$$

$$\mathrm{g}^{(1)}_{(j)}(\theta) = \begin{cases} (\mathrm{g}_K)_{\mathrm{II}}(\theta) = 0 & \text{for wide roots} \\ \mathrm{g}_K(\theta + i\epsilon) + \mathrm{g}_K(\theta - i\epsilon) & \text{for specials} \\ \mathrm{g}_K(\theta) & \text{otherwise} \end{cases}$$

$$\mathrm{g}(\theta) = 2\gamma \int_0^\theta d\theta' \mathrm{G}(\theta') \qquad \mathrm{g}_K(\theta) = 2\gamma \int_0^\theta d\theta' \mathrm{G}_K(\theta')$$

boundary terms

$$\mathrm{D_B}(\theta) = \mathsf{F}(\theta; H_+) + \mathsf{F}(\theta; H_-) + \mathsf{J}(\theta)$$

$$\mathsf{F}(\theta; H) = \begin{cases} \mathrm{sign}(H) \int_0^\theta d\theta' \int_{-\infty}^{\infty} \frac{dk}{2\gamma} e^{-ik\theta} \frac{\sinh(\frac{\pi}{\gamma} - |H|)\frac{\pi k}{2}}{2\cosh\frac{\pi k}{2} \sinh(\frac{\pi}{\gamma} - 2)\frac{\pi k}{2}} & |\mathrm{Im}\theta| > \frac{\pi(1-|H|)}{2} \\ \int_0^\theta d\theta' \int_{-\infty}^{\infty} \frac{dk}{2\gamma} e^{-ik\theta} \frac{\sinh(-\frac{\pi}{\gamma} - H + 2)\frac{\pi k}{2}}{2\cosh\frac{\pi k}{2} \sinh(\frac{\pi}{\gamma} - 2)\frac{\pi k}{2}} & |\mathrm{Im}\theta| < \frac{\pi(1-|H|)}{2} \end{cases}$$

$$\mathsf{J}(\theta) = \int_0^\theta d\theta' \int_{-\infty}^{\infty} \frac{dk}{2\gamma} e^{-ik\theta} \frac{\cosh\frac{\pi k}{4} \sinh(\frac{\pi}{\gamma} - 2)\frac{\pi k}{4}}{\cosh\frac{\pi k}{2} \sinh(\frac{\pi}{\gamma} - 2)\frac{\pi k}{4}}$$

$$\mathrm{D_{SB}}(\theta) = \mathsf{F}_y(\theta; H_+) + \mathsf{F}_y(\theta; H_-) + \lim_{\epsilon \to +0} 2\mathsf{g}_K(\theta + \tfrac{i\pi}{2} - i\epsilon)$$

$$\mathsf{F}_y(\theta; H) = \begin{cases} 0 & |\mathrm{Im}\theta| > \frac{\pi(1-|H|)}{2} \\ \widetilde{\mathsf{g}}_K(\theta - \frac{i\pi(1-H)}{2}) + \widetilde{\mathsf{g}}_K(\theta + \frac{i\pi(1-H)}{2}) & |\mathrm{Im}\theta| < \frac{\pi(1-|H|)}{2} \end{cases}$$

IR limit

the large volume limit $m_0 L \to \infty \Rightarrow \ln \mathrm{B} = \ln \bar{\mathrm{B}} = 0$

NLIEs

$$\ln \mathrm{b}(\theta) = (\mathrm{G}_K^{[\frac{\pi}{2}+\epsilon]} * \ln \mathrm{Y})(\theta) + 2im_0 L \sinh\theta + i\sum_j c_j\{\mathrm{g}_{(j)}(\theta-\theta_j) + \mathrm{g}_{(j)}(\theta+\theta_j)\} + i\mathrm{D}_\mathrm{B}(\theta) + i\pi C_b$$

$$\ln \mathrm{y}(\theta) = i\sum_j c_j\{\widetilde{\mathrm{g}}_{(j)}^{(1)}(\theta-\theta_j) + \widetilde{\mathrm{g}}_{(j)}^{(1)}(\theta+\theta_j)\} + i\mathrm{D}_\mathrm{SB}(\theta) + i\pi C_y$$

- quantization condition for two-particle state
  ⇒ scattering amplitude [Hegedus-Ravanini-Suzuki 08] (for periodic case)
- quantization condition for one-particle state
  ⇒ reflection amplitude [Ahn-Nepomechie-Suzuki 07] (partially)

$$e^{2im_0 L \sinh h_1} R(h_1; H_+) R(h_1; H_-) = 1 \qquad R = R_\mathrm{SG} \otimes R_\mathrm{RSOS}$$

$$e^{2im_0 L \sinh h_1 + i\mathrm{g}(2h_1) + i\mathrm{D}_B(h_1) + (\mathrm{G}_K^{[\frac{\pi}{2}+\epsilon]} * \ln \mathrm{Y})(h_1) + i\pi C_b} = -1$$

relations between QFT and NLIEs

$$R_\mathrm{SG}(h_1; H_+) R_\mathrm{SG}(h_1; H_-) = e^{i\mathrm{g}(2h_1) + i\mathrm{D}_B(h_1)}$$

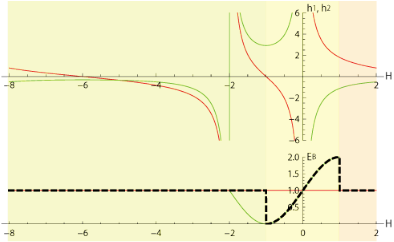
$$R_\mathrm{RSOS}(h_1; H_+) R_\mathrm{RSOS}(h_1; H_-) = -e^{(\mathrm{G}_K^{[\frac{\pi}{2}+\epsilon]} * \ln \mathrm{Y})(h_1) + i\pi C_b}$$

boundary-dependence of reflection amplitudes

- H_+: $H_+ > 1$ focus on the $(-)$-boundary
- H_-: $\frac{\pi}{\gamma} > H_- > -\frac{\pi}{\gamma}$ periodicity of NLIEs

	$R_{\rm SG}(\theta)$	$R_{\rm RSOS}(\theta)$
(i) $H_- > 1$	$R_+^+(\theta)$	$R_{\rm NS}(\theta)$
(ii) $1 > H_- > 0$	$R_+^+(\theta)e^{ig(\theta-i\nu_0)+ig(\theta+i\nu_0)}$	$R_{\rm NS}(\theta)e^{i\sigma(\theta-i\nu_0)+i\sigma(\theta+i\nu_0)}$
(iii) $0 > H_- > -1$	$R_+^+(\theta)$	$R_{\rm R}(\theta)$
(iv) $-1 > H_- > -2$	$R_+^+(\theta)e^{ig(\theta-i\nu_0)+ig(\theta+i\nu_0)}$	$R_{\rm R}(\theta)e^{i\sigma(\theta-i\nu_0)+i\sigma(\theta+i\nu_0)}$
(v) $-2 > H_-$	$R_-^-(\theta)$	$R_{\rm NS}(\theta)$

- $\theta = i\nu_0 = \frac{i\pi(1-H_-)}{2}$: boundary bound state
- two-string state = excitation of boundary-trapped particle
- mapping onto a spin chain $\Rightarrow \mathcal{H}_{\rm B} = h_1 S^z + h_2 (S^z)^2$



boundary energy

UV limit

$\{\theta_j\} \Rightarrow \{\theta_j^+, \theta_j^0\}$

- $\theta_j^+ = \hat{\theta}_j - \ln(m_0 L)$: go to infinity
- θ_j^0: remain finite
- $\mathrm{f}^+(\hat{\theta}) := \mathrm{f}(\theta - \ln(m_0 L))$

plateau equations (NLIEs in the UV lim.)

$$\ln \mathrm{b}^+(\hat{\theta}) = (\mathrm{G} * \ln \mathrm{B}^+)(\hat{\theta}) - (\mathrm{G}^{[+2\epsilon]} * \ln \bar{\mathrm{B}}^+)(\hat{\theta}) + (\mathrm{G}_K^{[-\frac{\pi}{2}+\epsilon]} * \ln \mathrm{Y}^+)(\hat{\theta}) + ie^{\hat{\theta}} + i\sum_j c_j \mathrm{g}_{(j)}(\hat{\theta} - \hat{\theta}_j) + i\pi \hat{C}_b$$

$$\ln \mathrm{y}^+(\hat{\theta}) = (\mathrm{G}_K^{[\frac{\pi}{2}-\epsilon]} * \ln \mathrm{B}^+)(\hat{\theta}) + (\mathrm{G}_K^{[-\frac{\pi}{2}+\epsilon]} * \ln \bar{\mathrm{B}}^+)(\hat{\theta}) + i\sum_j c_j \mathrm{g}_{(j)}^{(1)}(\hat{\theta} - \hat{\theta}_j) + i\pi \hat{C}_y$$

$\hat{C}_b, \hat{C}_y \in \{0, 1\}$

simplified counting equations

$$N_H - 2N_S = 2\hat{S} + M_C + 2M_W$$

$$N_1 - 2N_S^R = \hat{S} - M_R + M_{C>\gamma} + M_W + \delta$$

algebraic relations

$$T_1(\theta - \tfrac{i\pi}{2})T_1(\theta + \tfrac{i\pi}{2}) = f(\theta) + T_0(\theta)T_2(\theta)$$
$$y(\theta) = T_0(\theta)T_2(\theta)/f(\theta)$$

energy formula

$$E(L) = \frac{i}{4a}\left[\frac{d}{d\theta}\ln T_2(\theta)\Big|_{\theta=\Theta+\frac{i\pi}{2}} - \frac{d}{d\theta}\ln T_2(\theta)\Big|_{\theta=\Theta-\frac{i\pi}{2}}\right]$$
$$E(0) \sim E_{\text{Casimir}}(0) + E_{\text{excitation}}(0)$$
$$E_{\text{Casimir}}(0) = \frac{1}{2\pi L}\mathrm{Im}\int_{-\infty}^{\infty} d\hat{\theta}\, e^{\hat{\theta}} \ln \bar{\mathrm{B}}^{+}(\hat{\theta})$$
$$E_{\text{exctitation}}(0) = \frac{1}{2L}\sum_{j=1}^{N_H^+} e^{\hat{h}_j} - \frac{1}{2L}\sum_{j=1}^{M_C^+} e^{\hat{c}_j}$$

dilogarithm function

$$L_+(x) = \frac{1}{2}\int_0^x dy \left\{\frac{1}{y}\ln(1+y) - \frac{1}{1+y}\ln y\right\}$$

$$
\begin{aligned}
E(0) \sim \frac{1}{4\pi L} \{ & L_+(\mathrm{b}^+(\infty)) - L_+(\mathrm{b}^+(-\infty)) + L_+(\bar{\mathrm{b}}^+(\infty)) - L_+(\bar{\mathrm{b}}^+(-\infty)) \\
& + L_+(\mathrm{y}^+(\infty)) - L_+(\mathrm{y}^+(-\infty))\} \\
& + \frac{i}{8\pi L}\left[\{e^{\hat{\theta}} + \sum_j c_j \mathrm{g}_{(j)}(\hat{\theta} - \hat{\theta}_j) + \hat{C}_b\}(\ln \mathrm{B}^+ - \ln \bar{\mathrm{B}}^+)\right]_{-\infty}^{\infty} \\
& + \frac{i}{8\pi L}\left[\{\sum_j c_j \mathrm{g}^{(1)}_{(j)}(\hat{\theta} - \hat{\theta}_j) + \hat{C}_y\} \ln \mathrm{Y}^+\right]_{-\infty}^{\infty} \\
& + \frac{\pi}{L}(I_{N_H^+} - 2I_{N_S^+} - I_{M_C^+} - I_{M_W^+} - I_{M_{SC}^+}) \\
& - \frac{\pi}{2L}\{\hat{C}_b(N_H^+ - 2N_S^+ - M_C^+ - 2M_W^+ - 2M_{SC}^+) + \hat{C}_y N_1^+\}
\end{aligned}
$$

asymptotic values

$$
\mathrm{b}^+(\infty) = \bar{\mathrm{b}}^+(\infty) = 0 \qquad \mathrm{y}^+(\infty) = (-1)^{\delta_y}
$$

$$
\mathrm{b}^+(-\infty) = e^{3i\rho^+} 2\cos(\rho_+) \quad \bar{\mathrm{b}}^+(-\infty) = e^{-3i\rho_+} 2\cos(\rho_+) \quad \mathrm{y}_+(-\infty) = \frac{\sin(3\rho_+)}{\sin(\rho_+)}
$$

$$
\rho_+ = -(\pi - 3\gamma)(2\hat{S}^0) + (\pi - 2\gamma)(N_1^0 + \mathrm{D}_\mathrm{B}(\infty) + C_b)
$$

conformal dimensions

$$E(0) \sim -\frac{\pi}{24L}\left(\frac{3}{2} - 24\Delta_0\right)$$

$$\Delta_0 = \frac{1}{2}\left(\frac{\varphi_+ - \varphi_-}{\sqrt{\pi}} + mR + \frac{n}{R}\right)^2 + \frac{\delta_y}{16}$$

$$m = \tfrac{1}{4}\{\mathrm{sign}(1 - H_+) + \mathrm{sign}(1 - H_-) + \mathrm{sign}(1 + H_+) + \mathrm{sign}(1 + H_-)\} - \hat{S}$$

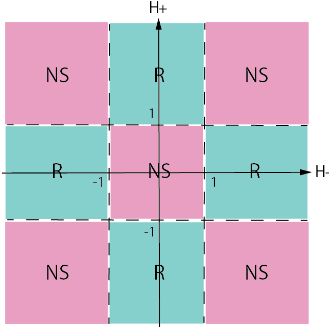
$$n = \tfrac{1}{2}\{\mathrm{sign}(1 - |H_+|) + \mathrm{sign}(1 - |H_-|) + \tfrac{1}{2} + 3\hat{S}_0 + N_1^0 + C_b\} = 0$$

- $\delta_y = 0 \Rightarrow$ NS sector
- $\delta_y = 1 \Rightarrow$ R sector

parity condition

$$(\delta_y - 2\hat{S}^+)_{\mathrm{mod}2} = 0 \qquad \hat{S}, \hat{S}^+ \in \mathbb{Z}$$

- light-cone regularization
  $\Rightarrow$ different sectors depending on boundary parameters



Summary 2

- Known result: light-cone regularization only realizes the NS sector of SSG model [Hegedus-Ravanini-Suzuki 08]
- Our result: the R sector is also obtained from light-cone regularization.
- We compared the reflection amplitudes and conformal dimensions derived both from the NLIEs and the perturbation of CFT.
- The physical interpretation was given in the relation with the corresponding spin chain.
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Thank you for listening!