Free Parafermions

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Free fermions

- The fundamental system in theoretical physics
- Many properties can be computed exactly
- Keeps on keeping on
  e.g. topological classification, entanglement, quenches…
- Appear even in some non-obvious guises

For example, spin models sometimes can be mapped onto free-fermionic systems:

1d quantum transverse-field/2d classical Ising

Kauffman, Onsager; now known in its fermionic version as the ``Kitaev chain''

1d quantum XY

Jordan-Wigner; Lieb-Schultz-Mattis

2d honeycomb model

Kitaev

Such models typically remain solvable even for spatially inhomogenous couplings.

Free fermions

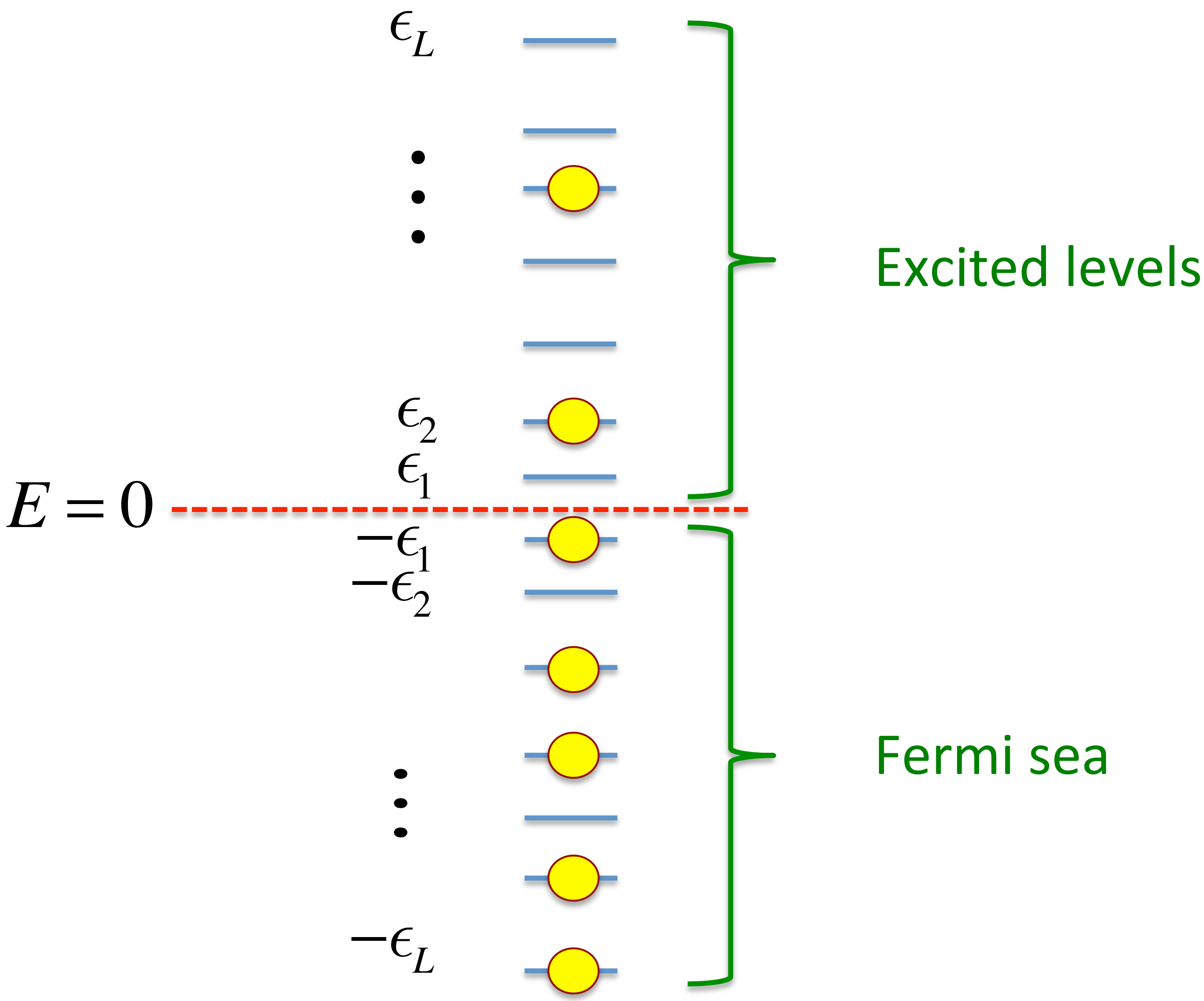
Forget statistics, forget operators, forget fields…the basic property of free fermions is that the spectrum is

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \; \ldots \; \pm \epsilon_L$$

Levels are either filled or empty.

The choice of a given $\pm$ is independent of the remaining choices, and does not effect the value of any ϵ_l.

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \; \dots \; \pm \epsilon_L$$



Can this be generalized?

The free-fermion approach relies on a Clifford algebra.

Integrable models provide a generalization, but the algebraic structure (Yang-Baxter etc.) is much more complicated, and you work much harder for less.

Conformal field theory is also a generalization, but applies only to Lorentz-invariant critical models.

Typically a free-fermion model has a $\mathbb{Z}_2$ symmetry: $[(-1)^F, H] = 0$ where $(-1)^F$ counts the number of fermions mod 2. In Ising this is simply symmetry under flipping all spins.

So why isn't there a $\mathbb{Z}_n$ version?

- Fradkin and Kadanoff showed long ago that 1+1d clock models with $\mathbb{Z}_n$ symmetry can be written in terms of parafermions.

- Fateev and Zamolodchikov found integrable critical self-dual lattice spin models with $\mathbb{Z}_n$ symmetry. Later they found an elegant CFT description of the continuum limit.

- Read and Rezayi constructed fractional quantum Hall wavefunctions using the CFT parafermion correlators.

But these models are definitely not free. The lattice models are not even integrable unless critical and/or chiral.

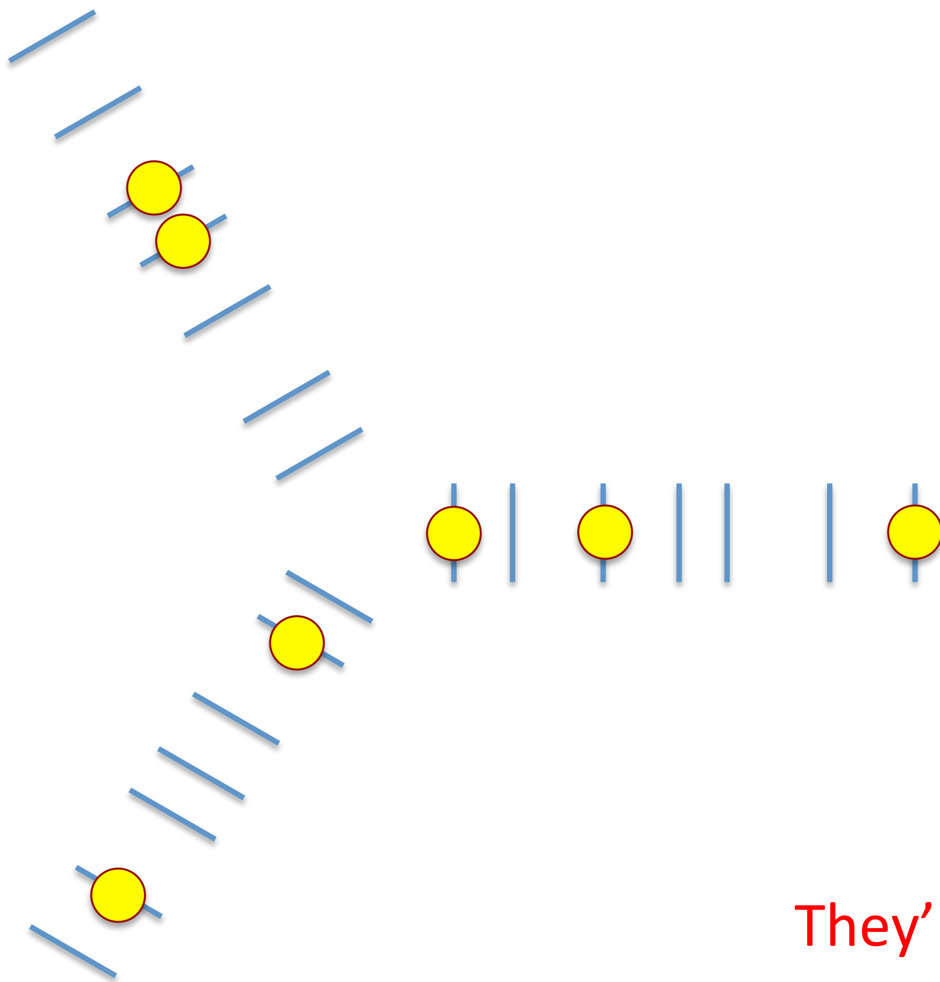
Nonetheless, Baxter found a non-Hermitian Hamiltonian with spectrum

$$E = \omega^{s_1}\epsilon_1 + \omega^{s_2}\epsilon_2 \pm \ \ldots \ \pm \omega^{s_L}\epsilon_L \qquad \omega = e^{2\pi i/n}$$

$$s_j = 0,1,\ldots n-1$$

A free parafermion sea?

For $\mathbb{Z}_3$:



They're exclusons!

Baxter's proof is very indirect.

In particular, he asks:

The eigenvalues therefore have the same simple structure as do direct products of L matrices, each of size N by N. For $N = 2$ this is the structure of the eigenvalues of the Ising model.[9]

For the Ising model this property follows from Kaufman's solution in terms of spinor operators [10], i.e. a Clifford algebra.[11, p.189] Whether there is some generalization of such spinor operators to handle the $\tau_2(t_q)$ model with open boundaries remains a fascinating speculation.[12]

The purpose of this talk is to display this structure, and so give a useful generalization of a Clifford algebra.

Jordan-Wigner transformation to Majorana fermions:

The Hilbert space is a chain of two-state systems $(\mathbb{C}^2)^{\otimes L}$

The fermions are written in terms of strings of spin flips:

$$\psi_{2j-1} = \sigma_j^z \prod_{k=1}^{j-1} \sigma_k^x \qquad\qquad \psi_{2j} = i\sigma_j^x \psi_{2j-1}$$

String flips all spins behind site j

$$\{\psi_a, \psi_b\} = 2\delta_{ab}$$

The 1d Ising Hamiltonian is bilinear in fermions:

$$H = -\sum_{j=1}^{L} u_{2j-1}\,\sigma_j^x - \sum_{j=1}^{L-1} u_{2j}\,\sigma_j^z \sigma_{j+1}^z$$

$$= i\sum_{a=1}^{2L-1} u_a\,\psi_a \psi_{a+1}$$

These are open boundary conditions and arbitrary couplings u_a .

$\mathbb{Z}_2$ symmetry operator flips all spins:

$$(-1)^F = \prod_{j=1}^{L} \sigma_j^x = (-1)^L \prod_{a=1}^{2L} \psi_a$$

Solving the Ising chain in one slide

Let $\Psi = \sum_{a=1}^{2L} \mu_a \psi_a$ so that $[H, \Psi] = \Psi'$ with

$$\begin{pmatrix} \mu_1' \\ \mu_2' \\ \vdots \\ \mu_{2L}' \end{pmatrix} = 2i \begin{pmatrix} 0 & u_1 & 0 & \cdots & & \\ -u_1 & 0 & u_2 & & & \\ 0 & -u_2 & 0 & & & \\ \vdots & & & & & u_{2L-1} \\ & & & & -u_{2L-1} & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{2L} \end{pmatrix}$$

because commuting bilinears in the fermions with linears gives linears.

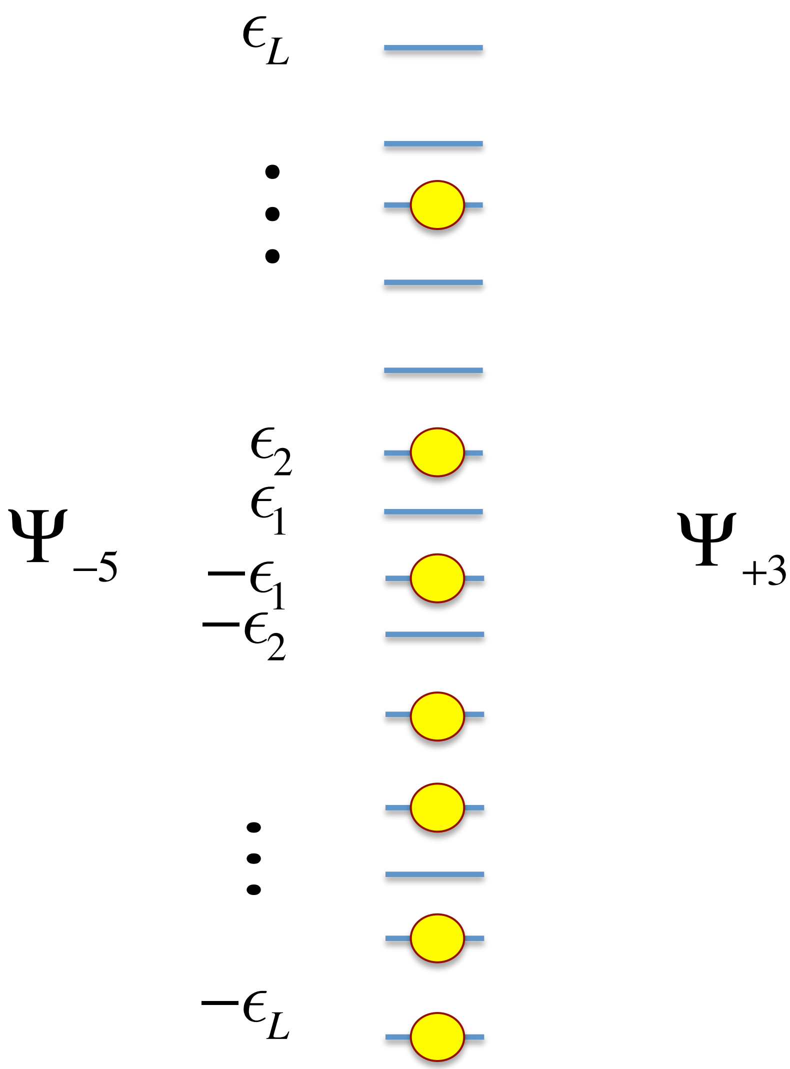
Diagonalizing this matrix gives $[H, \Psi_{\pm k}] = \pm 2\epsilon_k \Psi_k$

2L raising/lowering operators obey the Clifford algebra $\{\Psi_k, \Psi_l\} = 2\delta_{k,-l}$

Because $(\Psi_k + \Psi_{-k})^2 = 2$ no state is annihilated by both. Consistency requires

$$E = \pm\epsilon_1 \pm \epsilon_2 \pm \ \ldots \ \pm \epsilon_L$$

$$[H, \Psi_{\pm k}] = \pm 2\epsilon_k \Psi_k$$



On to n-state models

For 3 states, i.e. a Hilbert space of $(\mathbb{C}^3)^{\otimes L}$:

flip is now "shift"

Spin up or down is now "clock"

$$\tau = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 \\ 0 & 0 & e^{-2\pi i/3} \end{pmatrix}$$

$$\tau^3 = \sigma^3 = 1, \quad \tau^2 = \tau^\dagger, \quad \sigma^2 = \sigma^\dagger$$

$$\tau\sigma = e^{2\pi i/3}\sigma\tau$$

Parafermions from the Fradkin-Kadanoff transformation:

In a 2d classical theory, they're the product of order and disorder operators. In the quantum chain with $\omega = e^{2\pi i/n}$

$$\psi_{2j-1} = \sigma_j \prod_{k=1}^{j-1} \tau_k \qquad \psi_{2j} = \omega^{(n-1)/2}\ \tau_j \psi_{2j-1}$$

$$\psi_a^n = 1, \qquad \psi_a^{n-1} = \psi_a^\dagger$$

Instead of anticommutators:

$$\psi_a \psi_b = \omega \psi_b \psi_a$$

for $a < b$

Baxter's Hamiltonian

shift — clock interaction

$$H = -\sum_{j=1}^{L} u_{2j-1}\,\tau_j - \sum_{j=1}^{L-1} u_{2j}\,\sigma_j^{\dagger}\sigma_{j+1}$$

I did not forget the h.c. – the Hamiltonian is not Hermitian.

This is the Hamiltonian limit of the $\tau_2(q)$ model.

Bazhanov and Stroganov

``Higher'' commuting Hamiltonians

$$[H^{(j)}, H] = 0$$

Let $h_a \equiv u_a \psi_a^2 \psi_{a+1}$ so $H = \sum_{a=1}^{2L-1} u_a \psi_a^2 \psi_{a+1} = \sum_{a=1}^{2L-1} h_a$

$$H^{(2)} = \sum_{a=1}^{2L-3} \sum_{b=a+2}^{2L-1} h_a h_b$$

$$H^{(3)} = \sum_{a=1}^{2L-5} \sum_{b=a+2}^{2L-3} \sum_{c=b+2}^{2L-1} h_a h_b h_c$$ etc.

Note ``exclusion'' rule! Only one h_a for every 2 adjacent sites

For n-state model: n-2 h_a allowed on 2 adjacent sites

To find the energies and generalized Clifford algebra, we need the raising/lowering operators.

What worked so well for the fermions doesn't seem to work here:

Not linear in the parafermions!

For example: $$[H, \psi_1] \propto \psi_2$$

$$[H, \psi_2] \sim \psi_3 + \psi_1^2 \psi_2^2$$

Ignoring constants

It starts to look nasty very quickly.

But staring at this long enough, a pattern emerges.

Find that only surviving terms are of the form

$$h_{b_1} h_{b_2} \ldots h_{b_l} \psi_a$$

with $$|b_i - b_j| = 2, 3, \ldots$$

The same exclusion rule!

So repeatedly commuting with H doesn't generate all 3^{2L} operators.

In fact…

Let $v_0 \equiv \psi_1$

$$v_1 \equiv [H, v_0]$$

$$v_2 \equiv [H, v_1]$$

$$v_{3j} \equiv [H, v_{3j-1}] - u_{2j-1}^3 v_{3j-3} (\bar{\varpi} - 1)$$

$$v_{3j+1} \equiv [H, v_{3j}] - u_{2j}^3 v_{3j-2} (\bar{\varpi} - 1)$$

J=1,2,3...

$$v_{3j+2} \equiv [H, v_{3j+1}]$$

Then $v_{3L} = 0$

Can then find a set of ``rotating'' operators

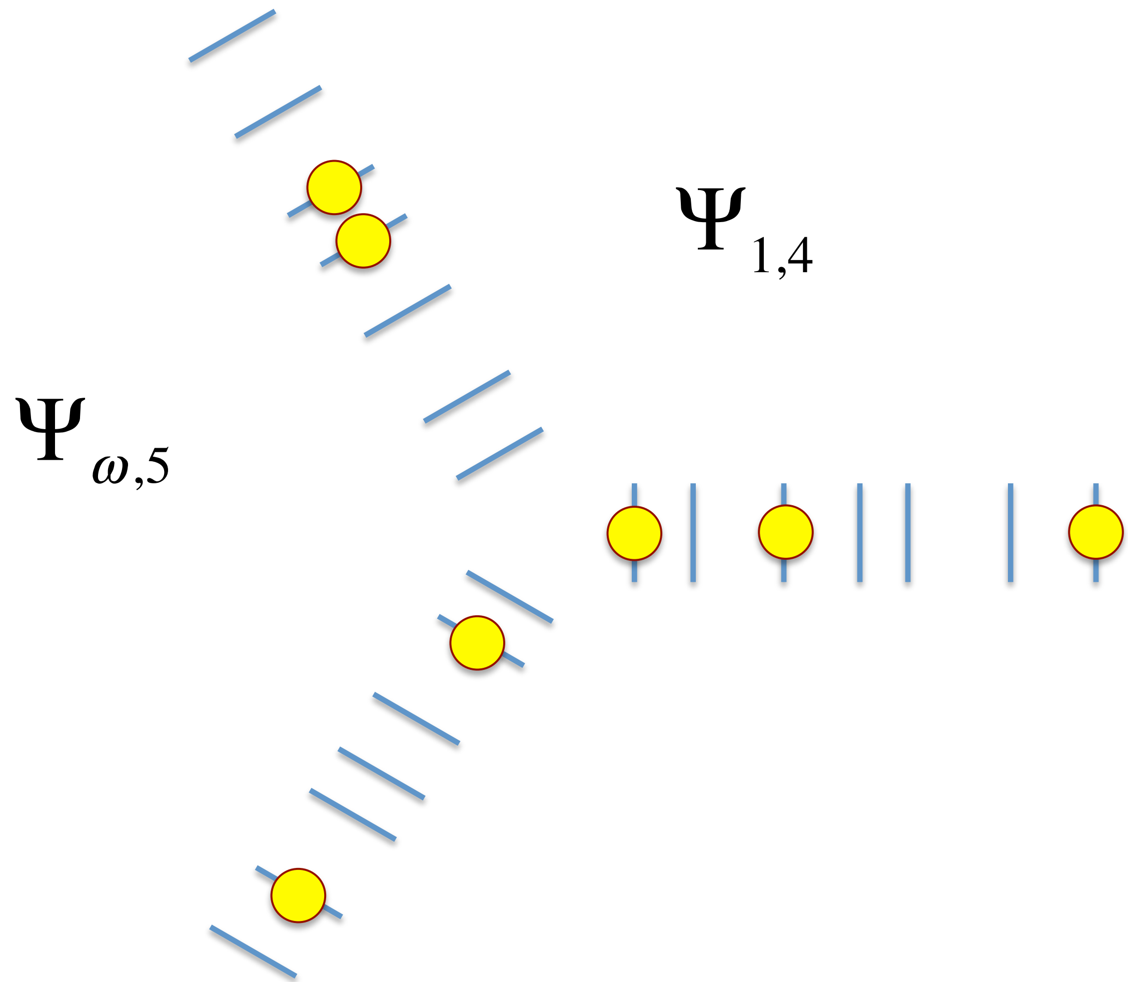
$$\Psi_{1,k}\,, \Psi_{\omega,k}\,, \Psi_{\bar{\omega},k}$$

obeying $$[H, \Psi_{\omega^s,k}] = (\omega^{s+1} - \omega^s)\,\epsilon_k\, \Psi_{\omega^s,k}$$

The ϵ_k are positive and real: they are the positive eigenvalues of

$$\begin{pmatrix} 0 & u_1^{3/2} & 0 & \dots & \\ u_1^{3/2} & 0 & u_2^{3/2} & & \\ 0 & u_2^{3/2} & 0 & & \\ \vdots & & & & u_{2L-1}^{3/2} \\ & & & u_{2L-1}^{3/2} & 0 \end{pmatrix}$$

$$[H, \Psi_{\omega^s,k}] = (\omega^{s+1} - \omega^s)\epsilon_k \Psi_{\omega^s,k}$$



These rotating operators satisfy the generalized Clifford algebra

$$\Psi^2_{\omega^s,k} = 0 \qquad \Psi_{\omega^{s-1},k}\Psi_{\omega^s,k} = 0$$

$$\Psi_{\omega^s,k}\Psi_{\omega^{s'},k'} \propto \Psi_{\omega^{s'},k'}\Psi_{\omega^s,k} \qquad k \neq k'$$

$$(\Psi_{1,k} + \Psi_{\omega,k} + \Psi_{\bar{\omega},k})^3 \propto 1$$

Conjecture that first three can be subsumed in

$$(\epsilon_{k'}\omega^{s'} - \epsilon_k\omega^s)\Psi_{\omega^s,k}\Psi_{\omega^{s'},k'} = (\epsilon_k\omega^s - \epsilon_{k'}\omega^{s'+1})\Psi_{\omega^{s'},k'}\Psi_{\omega^s,k}$$

This algebra is independent of the Hamiltonian we used – these operators can be defined for any $\mathbb{Z}_3$-invariant spin system.

Future directions

- Take copies and fill pairs of levels to make a parafermion sea with real energy? Is this the chiral part of a CFT?
- Zero modes! Topological order!
  Mong, Clarke, Lindner, Alicea, et al
- Solvable models of interacting parafermions?
- Redo for 2d classical models, any interesting geometric problems?
  Does the Pfaffian generalize to a Read-Rezayian?
- Use to build a (presumably gapless) 2d wavefunction?
- Closed boundary conditions? Full $\tau_2(q)$ model?
- The Clifford algebra plays a major role in mathematics and e.g. in the classification of topological systems. Is there a $\mathbb{Z}_n$ version?