EXACT RESULTS IN N=4

Nikolay Gromov

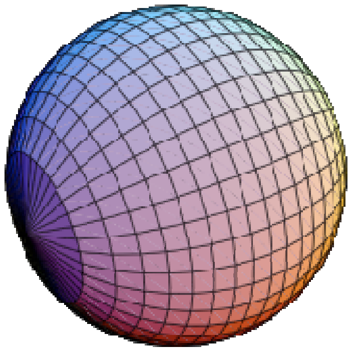
Based on works with V.Kazakov, S.Leurent, D.Volin 1305.1939
F. Levkovich-Maslyuk, G. Sizov 1305.1944

Euler Symposium on Theoretical and Mathematical Physics
July 12-17, 2013 St. Petersburg, Russia

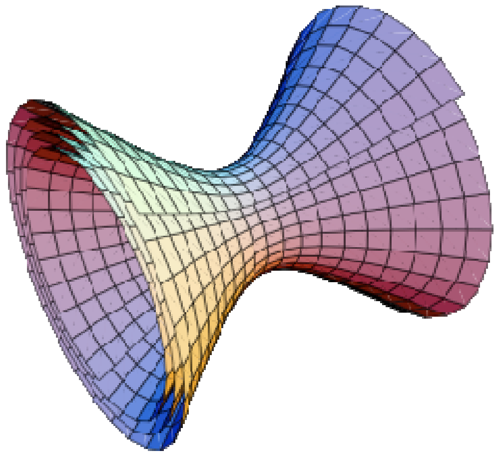
Introduction

AdS/CFT correspondence

(type IIB super) string theory in $AdS_5 x S^5$

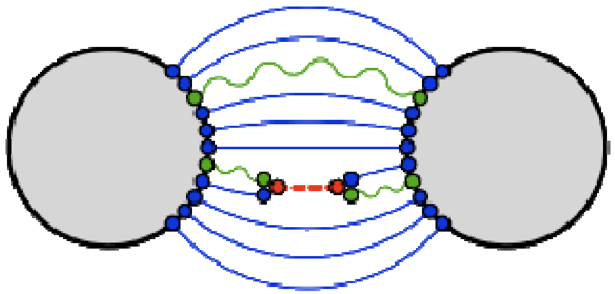


x



Maldacena

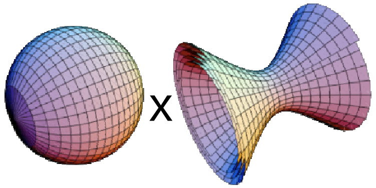
is dual to a 4 dimensional conformal field theory (N=4 SYM)

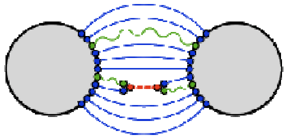


Local operators ⇔ String states

Introduction

AdS/CFT correspondence





String tension $T = \frac{\sqrt{\lambda}}{2\pi}$

String coupling $g_s = \frac{\lambda}{4\pi N}$

`t Hooft coupling $\lambda = g_{YM}^2 N$

Number of colors N

We will mainly focus on $N \to \infty$ limit

i.e. planar limit in YM, and free strings

Introduction

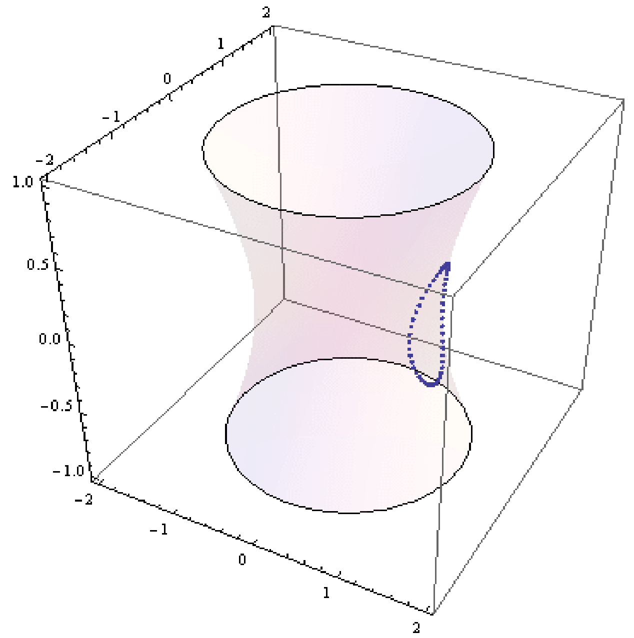
Classical integrability

$$S = \int d\tau d\sigma \sum_{a=1}^{4} \left((\partial_\tau X_a)^2 - (\partial_\sigma X_a)^2 \right)$$

The scalar fields are constrained

$$X_a^2 = 1$$

Motion of the string: $$\partial_\mu \partial^\mu X_a + (\partial_\nu X_b \partial^\nu X_b) X_a = 0$$

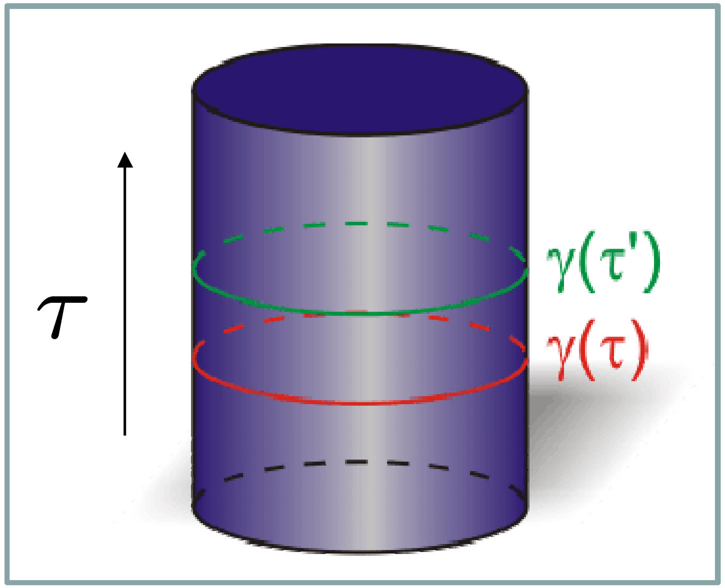


Introduction
Classical integrability

$$j_{ab,\mu} = 2X_a\partial_\mu X_b - 2X_b\partial_\mu X_a$$

$$A_\mu(z) = \frac{j_\mu + z\ \ \epsilon_{\mu\nu} j^\nu}{z^2 - 1} \qquad \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0$$

$$\Omega(z,\tau) = \mathrm{Pexp} \oint_{\gamma(\tau)} A_\sigma(z,\tau,\sigma) d\sigma$$



$$\partial_\tau \mathrm{tr}\Omega(z,\tau) = 0$$

on equations of motion

Bena, Polchinski, Roiban;
Kazakov, Marshakov, Minahan, Zarembo;

Eigenvalues = integrals of motion

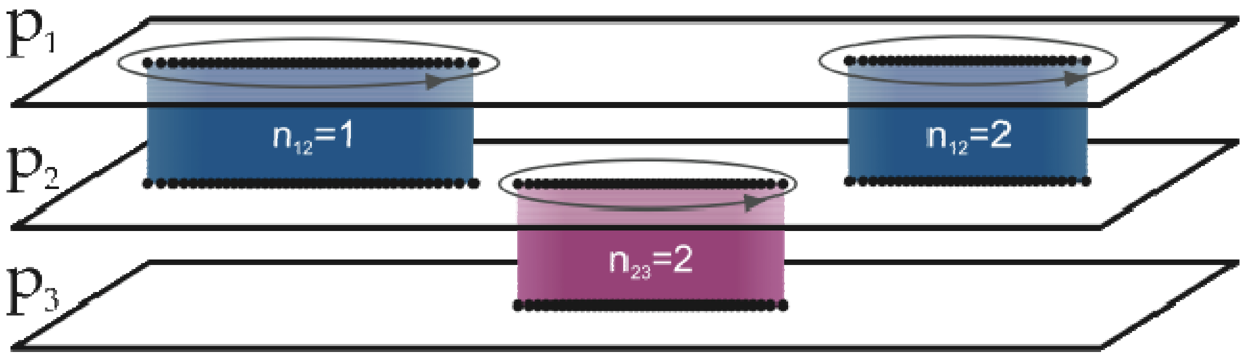
$$\Omega(x) \to (\lambda_1(x), \lambda_2(x), \lambda_3(x), \lambda_4(x) | \mu_1(x), \mu_2(x), \mu_3(x), \mu_4(x))$$

Introduction

Classical intgrability

According to Beisert, Kazakov, Sakai and Zarembo, we can map a classical string motion to an 8-sheet Riemann surface

$$\Omega(x) \to (\lambda_1(x), \lambda_2(x), \lambda_3(x), \lambda_4(x) | \mu_1(x), \mu_2(x), \mu_3(x), \mu_4(x))$$

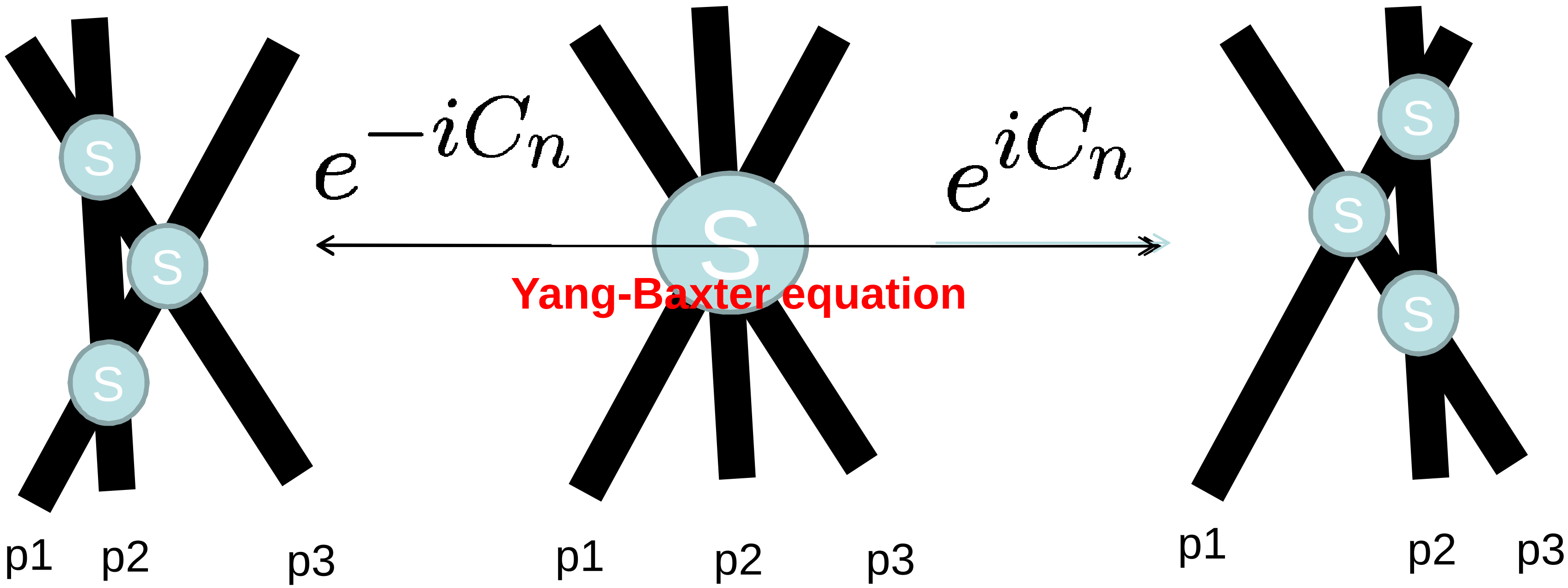


$$p_i^+ - p_j^- = 2\pi n_{ij} \ , \ \ x \in C_n^{ij}$$

$$\oint_{C_n^{ij}} p_i(z) dz = \frac{4\pi}{\sqrt{\lambda}} N_{ij}$$

Introduction

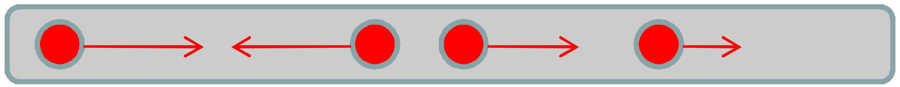
S-matrix factorization



Introduction

Asymptotic spectrum

- For spectral density we need finite volume



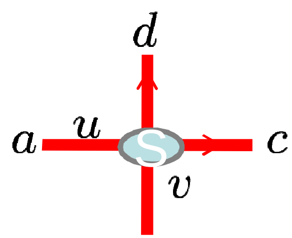
$$\Psi(x_1+L, x_2, \ldots) = e^{ip_1 L} S(p_1, p_2) \ldots S(p_1, p_n) \Psi(x_1, x_2, \ldots)$$

- From periodicity of the wave function

$$e^{ip_i L} = \prod_{j=1}^{M} S(p_i, p_j)$$

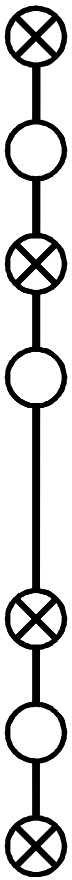
SU(2|2) invariant tensor with 4 fundamental indexes

$$S_{ab}^{cd}(u, v)$$



Introduction

Asymptotic spectrum



Beisert, Staudacher; Beisert, Hernandez, Lopez; Beisert,Eden,Staudacher

$$1 = \prod_{j=1}^{K_2} \frac{u_{1,k}-u_{2,j}+\frac{i}{2}}{u_{1,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{1-1/x_{1,k}x^+_{4,j}}{1-1/x_{1,k}x^-_{4,j}},$$

$$1 = \prod_{j\neq k}^{K_2} \frac{u_{2,k}-u_{2,j}-i}{u_{2,k}-u_{2,j}+i} \prod_{j=1}^{K_3} \frac{u_{2,k}-u_{3,j}+\frac{i}{2}}{u_{2,k}-u_{3,j}-\frac{i}{2}} \prod_{j=1}^{K_1} \frac{u_{2,k}-u_{1,j}+\frac{i}{2}}{u_{2,k}-u_{1,j}-\frac{i}{2}},$$

$$1 = \prod_{j=1}^{K_2} \frac{u_{3,k}-u_{2,j}+\frac{i}{2}}{u_{3,k}-u_{2,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k}-x^+_{4,j}}{x_{3,k}-x^-_{4,j}},$$

$$1 = \left(\frac{x^-_{4,k}}{x^+_{4,k}}\right)^L \prod_{j\neq k}^{K_4} \frac{u_{4,k}-u_{4,j}+i}{u_{4,k}-u_{4,j}-i} \prod_{j}^{K_4} \left(\frac{1-1/x^+_{4,k}x^-_{4,j}}{1-1/x^-_{4,k}x^+_{4,j}}\right) \sigma^2(x_{4,k},x_{4,j})$$

$$\times \prod_{j=1}^{K_1} \frac{1-1/x^-_{4,k}x_{1,j}}{1-1/x^+_{4,k}x_{1,j}} \prod_{j=1}^{K_3} \frac{x^-_{4,k}-x_{3,j}}{x^+_{4,k}-x_{3,j}} \prod_{j=1}^{K_5} \frac{x^-_{4,k}-x_{5,j}}{x^+_{4,k}-x_{5,j}} \prod_{j=1}^{K_7} \frac{1-1/x^-_{4,k}x_{7,j}}{1-1/x^+_{4,k}x_{7,j}},$$

$$1 = \prod_{j=1}^{K_6} \frac{u_{5,k}-u_{6,j}+\frac{i}{2}}{u_{5,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{5,k}-x^+_{4,j}}{x_{5,k}-x^-_{4,j}},$$

$$1 = \prod_{j\neq k}^{K_6} \frac{u_{6,k}-u_{6,j}-i}{u_{6,k}-u_{6,j}+i} \prod_{j=1}^{K_5} \frac{u_{6,k}-u_{5,j}+\frac{i}{2}}{u_{6,k}-u_{5,j}-\frac{i}{2}} \prod_{j=1}^{K_7} \frac{u_{6,k}-u_{7,j}+\frac{i}{2}}{u_{6,k}-u_{7,j}-\frac{i}{2}},$$

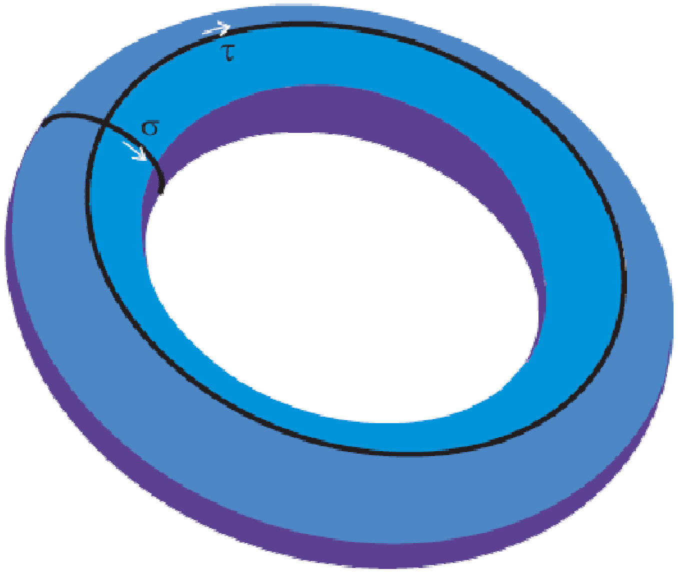
$$1 = \prod_{j=1}^{K_6} \frac{u_{7,k}-u_{6,j}+\frac{i}{2}}{u_{7,k}-u_{6,j}-\frac{i}{2}} \prod_{j=1}^{K_4} \frac{1-1/x_{7,k}x^+_{4,j}}{1-1/x_{7,k}x^-_{4,j}}.$$

$$x + \frac{1}{x} = \frac{u}{g} \;, \quad x^{\pm} + \frac{1}{x^{\pm}} = \frac{u \pm i/2}{g} \qquad E = \sum_k \epsilon_k = \sum_k 2gi \left(\frac{1}{x^+_{4,k}} - \frac{1}{x^-_{4,k}}\right)$$

Derivation of Y-system

Zamolodchikov's trick

…,Matsubara, Zamolodchikov,…



$$Z(\tau,\sigma) = Z(\sigma,\tau)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\sum e^{-E_n(L)R} \qquad \sum e^{-E_n(R)L}$$

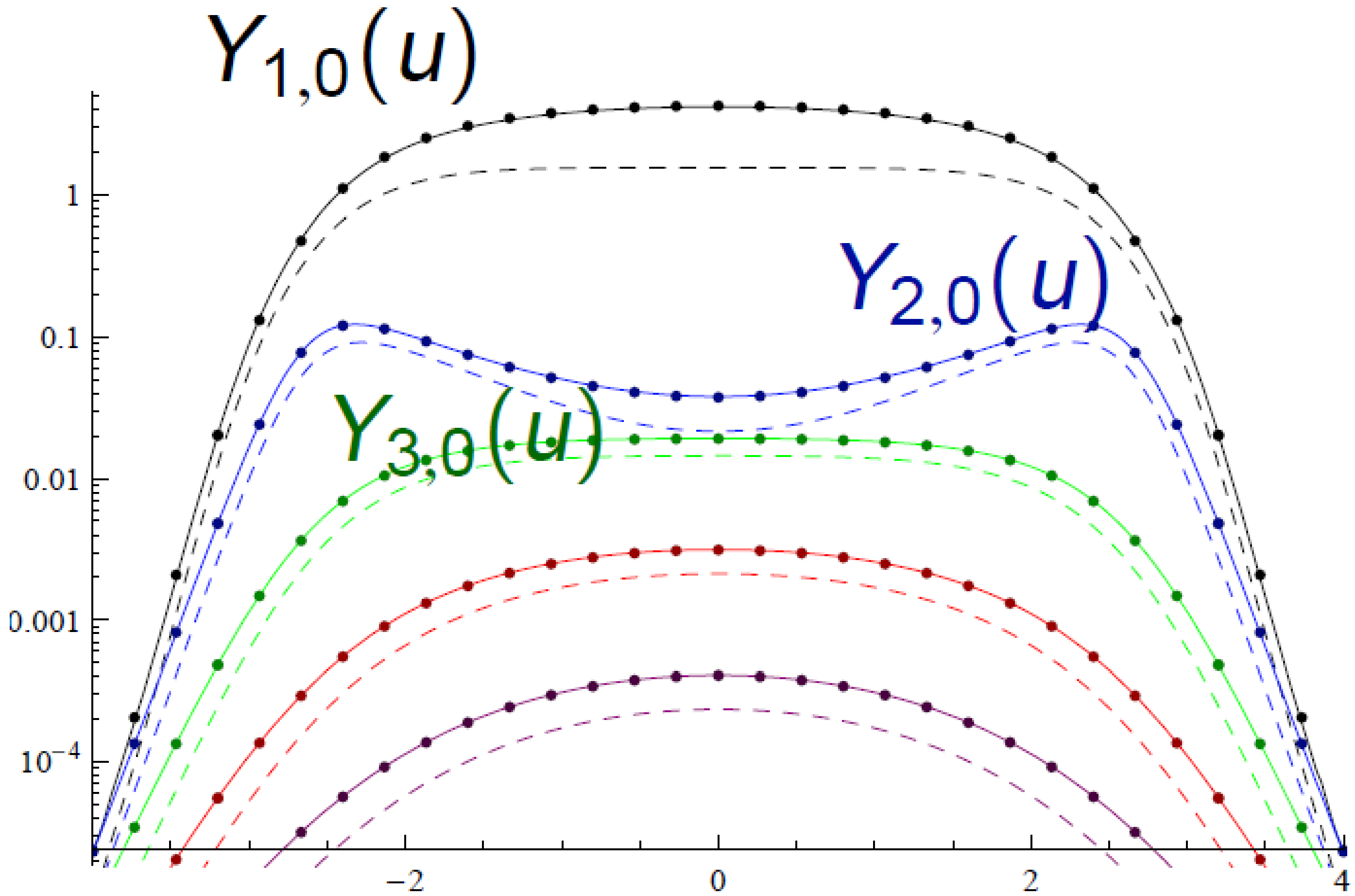
$$\downarrow$$

$$e^{-E_0(L)R}$$

I.e. from the asymptotical spectrum (infinite R) we can compute the Ground state energy for ANY finite volume!

$$E_0(L) = -\lim_{R\to\infty} \frac{\log \sum e^{-E_n(R)L}}{R}$$

Maxwell-like distributions



Introduction

Some 2D Integrable models

$$\log Y_{\otimes} = +K_{m-1} * \log \frac{1+1/Y_{\circ_m}}{1+Y_{\triangle_m}} + \mathcal{R}^{(01)}_{1m} * \log(1+Y_{\bullet_m}) + \left[\log \frac{R^{(+)}}{R^{(-)}}\right] + \log(-1)$$

$$\log Y_{\oplus} = -K_{m-1} * \log \frac{1+1/Y_{\circ_m}}{1+Y_{\triangle_m}} - \mathcal{B}^{(01)}_{1m} * \log(1+Y_{\bullet_m}) - \left[\log \frac{B^{(+)}}{B^{(-)}}\right] - \log(-1)$$

$$\log Y_{\triangle_n} = -K_{n-1,m-1} * \log(1+Y_{\triangle_m}) - K_{n-1} * \log \frac{1+Y_{\otimes}}{1+1/Y_{\oplus}} + \left(\mathcal{R}^{(01)}_{nm} + \mathcal{B}^{(01)}_{n-2,m}\right) * \log(1+Y_{\bullet_m})$$

$$+ \left[\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \frac{R^{(+)}(u+ik)}{R^{(-)}(u+ik)} + \sum_{k=-\frac{n-3}{2}}^{\frac{n-3}{2}} \log \frac{B^{(+)}(u+ik)}{B^{(-)}(u+ik)}\right]$$

$$\log Y_{\circ_n} = K_{n-1,m-1} * \log(1+1/Y_{\circ_m}) + K_{n-1} * \log \frac{1+Y_{\otimes}}{1+1/Y_{\oplus}}$$

$$\log Y_{\bullet_n} = L \log \frac{x^{[-n]}}{x^{[+n]}} + \left(2\mathcal{S}_{nm} - \mathcal{R}^{(11)}_{nm} + \mathcal{B}^{(11)}_{nm}\right) * \log(1+Y_{\bullet_m}) + \left[\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} i\Phi(u+ik)\right]$$

$$+ 2\left(\mathcal{R}^{(10)}_{n1} * \log(1+Y_{\otimes}) - \mathcal{B}^{(10)}_{n1} * \log(1+1/Y_{\oplus}) + \left(\mathcal{R}^{(10)}_{nm} + \mathcal{B}^{(10)}_{n,m-2}\right) * \log(1+Y_{\triangle_m})\right)$$

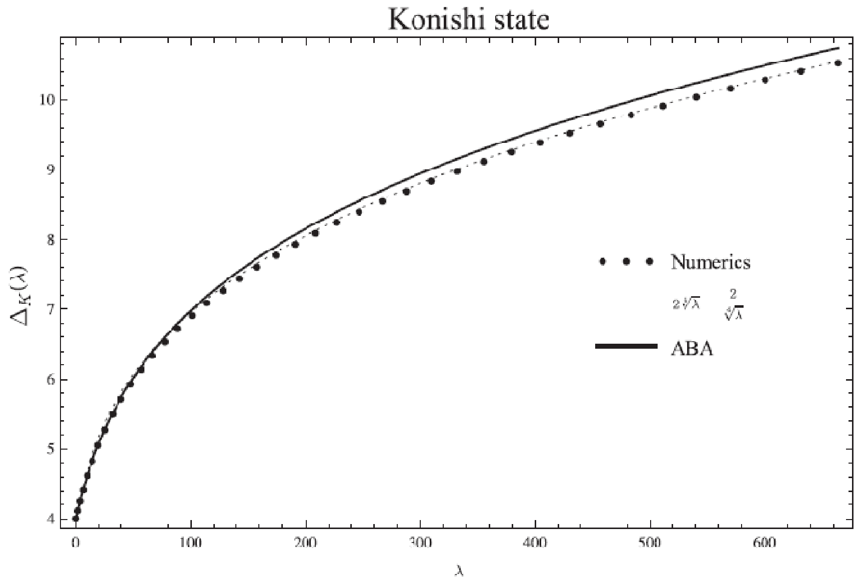
Bombardelli, Fioravanti, Tateo
N.G., Kazakov, Vieira
Arutynov, Frolov

Numerical solution of Y-system

Simplest (Konishi) operator

$$\mathcal{O} = tr(ZZWW) - tr(ZWZW)$$

N.G., Kazakov, Vieira '09



Fit:

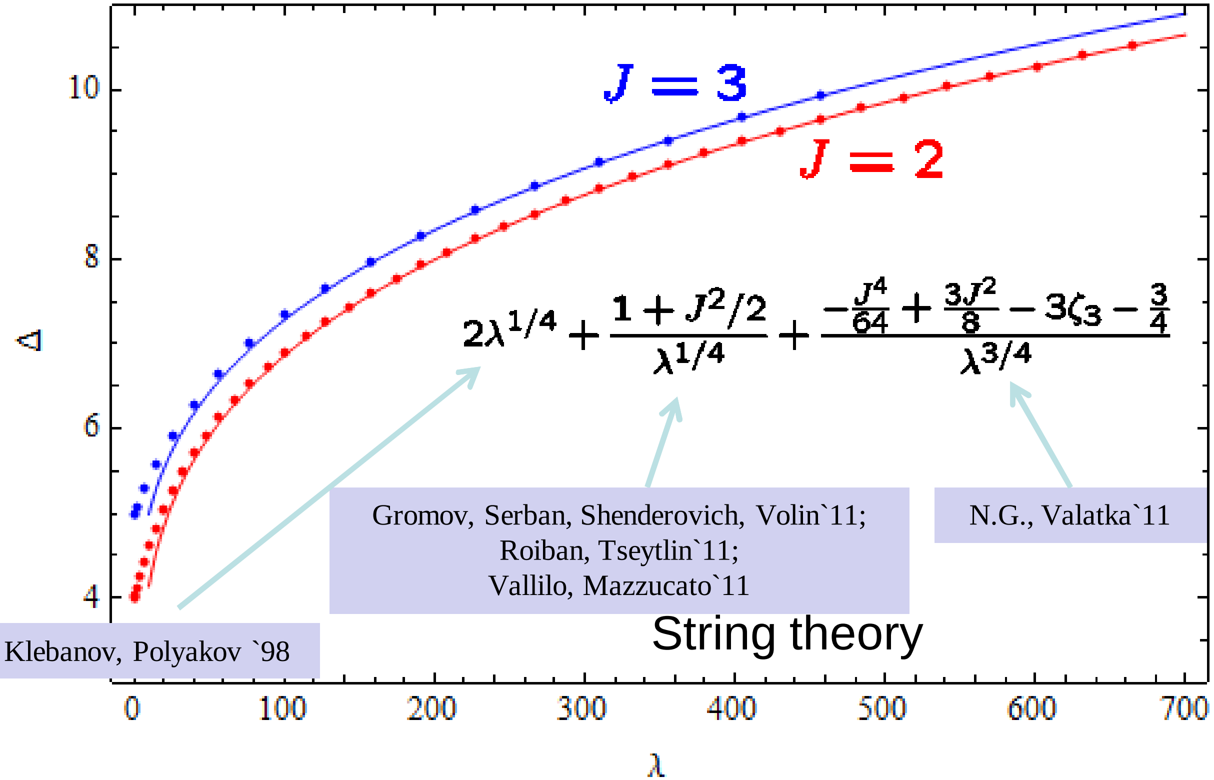
$$\Delta_K = 2\lambda^{1/4}\left(1.0002 + \frac{0.994}{\lambda^{1/2}} - \frac{1.30}{\lambda} + \frac{3.1}{\lambda^{3/2}} + \dots\right)$$

Numerical solution of Y-system

Simplest (Konishi) operator

N.G., Kazakov, Vieira ‘09

Numerics



Gubser, Klebanov, Polyakov `98

Gromov, Serban, Shenderovich, Volin`11;
Roiban, Tseytlin`11;
Vallilo, Mazzucato`11

N.G., Valatka`11

String theory

Agrees with weak coupling gauge theory up to 5 loop!

$$4 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6} + \frac{3(-26 + 6\zeta_3 - 15\zeta_5)\lambda^4}{2048\pi^8} + \frac{3(158 + 72\zeta_3 - 54\zeta_3^2 - 90\zeta_5 + 315\zeta_7)\lambda^5}{32768\pi^{10}} + O[\lambda]^6$$

Fieamberti, Fantambrogio, Sieg, Zanon `08
Eden, Heslop, Korchemsky, Smirnov, Sokatchev `12

Bajnok, Janik, Lukowski `08
Bajnok, Hegedus, Janik, Lukowski `09
Arutyunov, Frolov, Suzuki `10

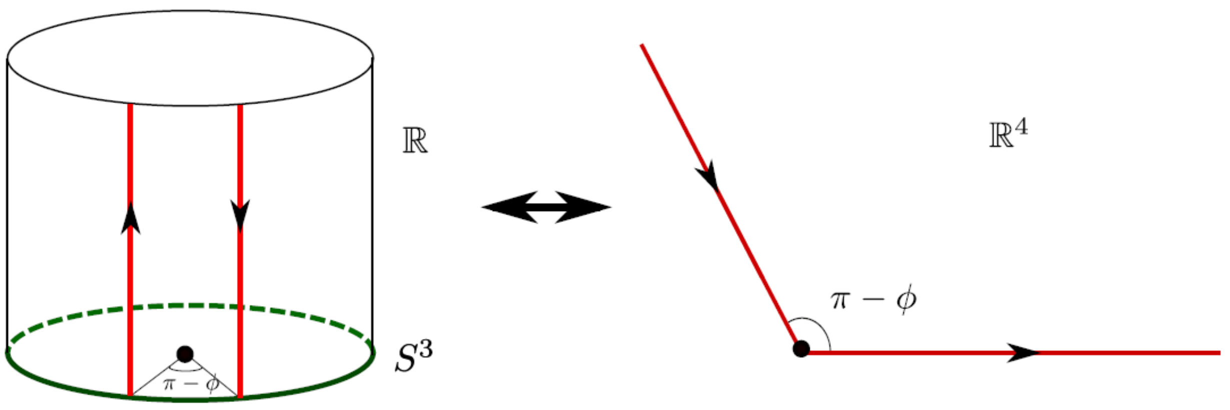
Introduction

Some 2D Integrable models

Bombardelli, Fioravanti, Tateo
N.G., Kazakov, Vieira
Arutynov, Frolov

$$
\begin{aligned}
\log Y_{\otimes} &= +K_{m-1} * \log \frac{1+1/Y_{\bigcirc_m}}{1+Y_{\triangle_m}} + \mathcal{R}^{(01)}_{1m} * \log(1+Y_{\bullet_m}) + \left[\log \frac{R^{(+)}}{R^{(-)}}\right] + \log(-1) \\
\log Y_{\oplus} &= -K_{m-1} * \log \frac{1+1/Y_{\bigcirc_m}}{1+Y_{\triangle_m}} - \mathcal{B}^{(01)}_{1m} * \log(1+Y_{\bullet_m}) - \left[\log \frac{B^{(+)}}{B^{(-)}}\right] - \log(-1) \\
\log Y_{\triangle_n} &= -K_{n-1,m-1} * \log(1+Y_{\triangle_m}) - K_{n-1} * \log \frac{1+Y_{\otimes}}{1+1/Y_{\oplus}} + \left(\mathcal{R}^{(01)}_{nm} + \mathcal{B}^{(01)}_{n-2,m}\right) * \log(1+Y_{\bullet_m}) \\
&+ \left[\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} \log \frac{R^{(+)}(u+ik)}{R^{(-)}(u+ik)} + \sum_{k=-\frac{n-3}{2}}^{\frac{n-3}{2}} \log \frac{B^{(+)}(u+ik)}{B^{(-)}(u+ik)}\right] \\
\log Y_{\bigcirc_n} &= K_{n-1,m-1} * \log(1+1/Y_{\bigcirc_m}) + K_{n-1} * \log \frac{1+Y_{\otimes}}{1+1/Y_{\oplus}} \\
\log Y_{\bullet_n} &= L \log \frac{x^{[-n]}}{x^{[+n]}} + \left(2\mathcal{S}_{nm} - \mathcal{R}^{(11)}_{nm} + \mathcal{B}^{(11)}_{nm}\right) * \log(1+Y_{\bullet_m}) + \left[\sum_{k=-\frac{n-1}{2}}^{\frac{n-1}{2}} i\Phi(u+ik)\right] \\
&+ 2\left(\mathcal{R}^{(10)}_{n1} * \log(1+Y_{\otimes}) - \mathcal{B}^{(10)}_{n1} * \log(1+1/Y_{\oplus}) + \left(\mathcal{R}^{(10)}_{nm} + \mathcal{B}^{(10)}_{n,m-2}\right) * \log(1+Y_{\triangle_m})\right)
\end{aligned}
$$

Example: Wilson line with cusp



$\langle W \rangle = e^{-TV}$

$\langle W \rangle \sim \left(\frac{\Lambda_{IR}}{\Lambda_{UV}} \right)^{\Gamma_{\text{cusp}}}$

Conformal invariance $\Rightarrow V = \Gamma_{\text{cusp}}$

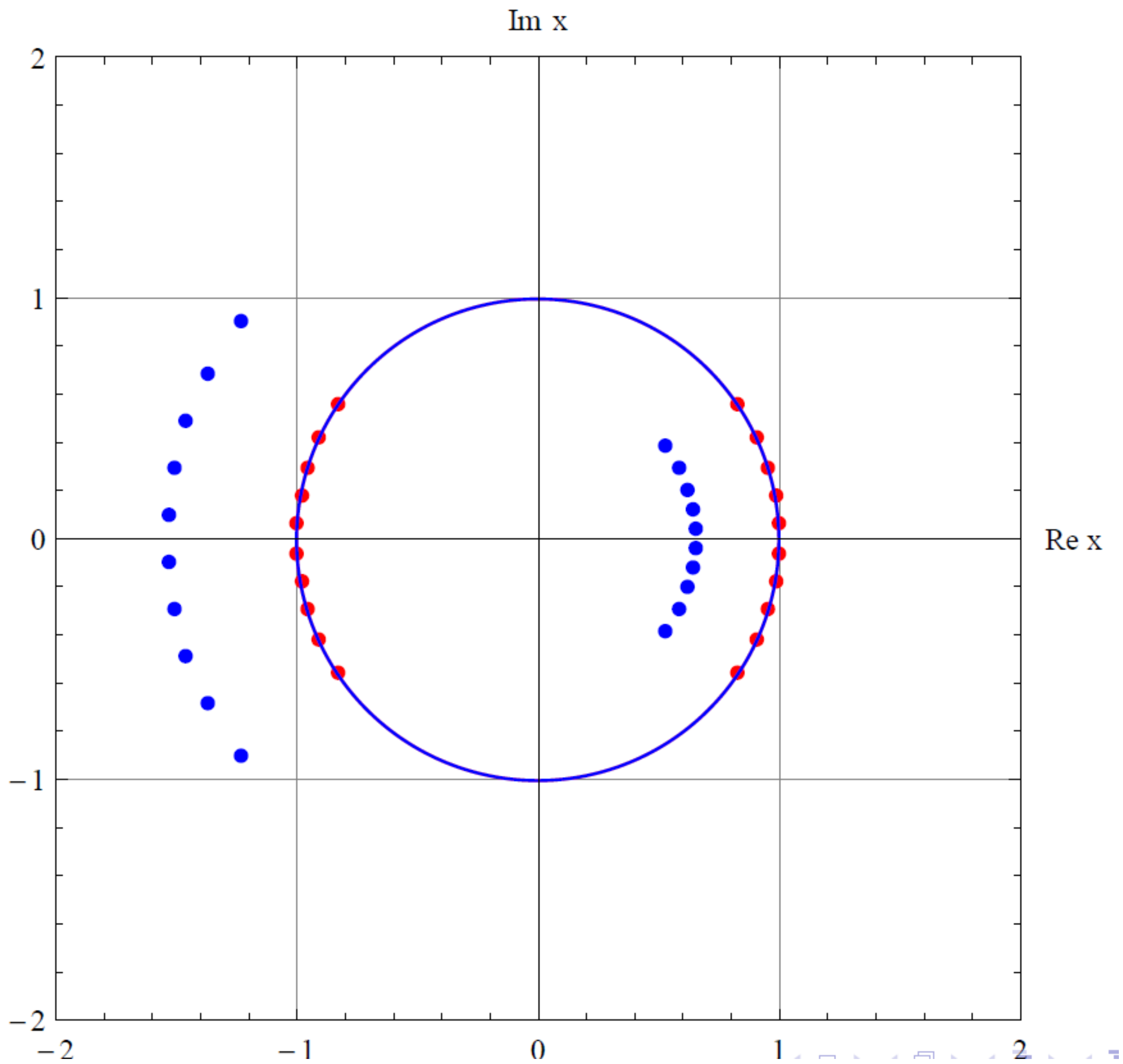
$$W = \text{Tr}\, \mathcal{P} \exp \int dt \left[iA \cdot \dot{x}_q + \vec{\Phi} \cdot \vec{n}\, |\dot{x}_q| \right]$$

- Cusp angle ϕ
- Angle θ between the couplings to scalars on two rays
- R-charge L of a local operator inserted at the cusp
- 't Hooft coupling λ

For $\theta^2 - \phi^2 = 0$ this observable is protected

Cusp anomalous dimension is related to a variety of physical quantities, as

- IR divergences of scattering amplitudes, $i\phi$ is a boost angle for massive particles and $i\phi \to \infty$ for massless.
- Bremsstrahlung function — radiation of a moving particle ($\phi \to 0$)
- The quark-antiquark potential in the flat space ($\phi \to \pi$)



Im x
Re x
2
1
0
−1
−2
−2
−1
0
1
2

Cusp anomalous dimension for arbitrary L, finite $\theta \approx \phi$ and any value of 't Hooft coupling

$$\Gamma_L(\lambda) = \frac{\phi - \theta}{4} \partial_\theta \log \frac{\det \mathcal{M}_{2L+1}}{\det \mathcal{M}_{2L-1}}$$

$$\mathcal{M}_N = \begin{pmatrix} I_1^\theta & I_0^\theta & \cdots & I_{2-N}^\theta & I_{1-N}^\theta \\ I_2^\theta & I_1^\theta & \cdots & I_{3-N}^\theta & I_{2-N}^\theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_N^\theta & I_{N-1}^\theta & \cdots & I_1^\theta & I_0^\theta \\ I_{N+1}^\theta & I_N^\theta & \cdots & I_2^\theta & I_1^\theta \end{pmatrix}$$

$$I_n^\theta = \tfrac{1}{2} I_n \left(\sqrt{\lambda} \sqrt{1 - \tfrac{\theta^2}{\pi^2}} \right) \left[\left(\sqrt{\tfrac{\pi+\theta}{\pi-\theta}} \right)^n - (-1)^n \left(\sqrt{\tfrac{\pi-\theta}{\pi+\theta}} \right)^n \right] .$$

Using

$$I_n^\theta = \frac{1}{2\pi i}\oint \frac{dx}{x^{n+1}}\sinh(2\pi g\,(x+1/x))\,e^{2g\theta(x-1/x)}$$

for every element of

$$\mathcal{M}_N = \begin{pmatrix} I_1^\theta & I_0^\theta & \cdots & I_{2-N}^\theta & I_{1-N}^\theta \\ I_2^\theta & I_1^\theta & \cdots & I_{3-N}^\theta & I_{2-N}^\theta \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I_N^\theta & I_{N-1}^\theta & \cdots & I_1^\theta & I_0^\theta \\ I_{N+1}^\theta & I_N^\theta & \cdots & I_2^\theta & I_1^\theta \end{pmatrix}$$

we obtain

$$\det\mathcal{M}_N = \oint\prod_{i=1}^{N+1}\frac{dx_i}{2\pi i x_i^{N+2}}\frac{\Delta^2(x_i)}{(N+1)!}\sinh\left[2\pi g\,\left(x_i+\frac{1}{x_i}\right)\right]e^{2g\theta\left(x_i-\frac{1}{x_i}\right)}$$

In the case of $L \sim g \to \infty$ limit of $\det \mathcal{M}_{2L+1}$ the saddle-point equation is

$$-\theta \frac{x_j^2+1}{x_j^2-1} + \frac{L}{g}\frac{x_j}{x_j^2-1} - \frac{1}{g}\frac{x_j^2}{x_j^2-1}\sum_{i\neq j}^{2L+1}\frac{1}{x_j-x_i} = \pi\,\mathrm{sgn}(\mathrm{Re}(x_j)).$$

Introduce the quantum quasimomentum $p(x)$

$$p(x) = -\theta\frac{x^2+1}{x^2-1} + \frac{L}{g}\frac{x}{x^2-1} - \frac{2L}{g}\frac{x^2}{x^2-1}\,G_L^{cl}(x),$$

where

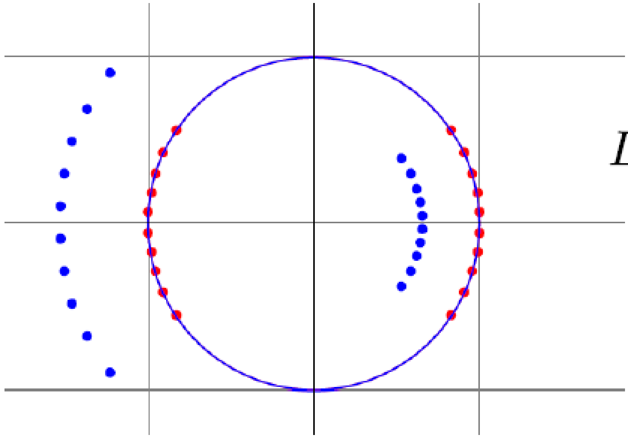
$$G_L^{cl}(x) = \frac{1}{2L}\sum_{k=1}^{2L+1}\frac{1}{x-x_k}.$$

The saddle-point equation then is

$$\frac{1}{2}\left(p(x_i+i\epsilon)+p(x_i-i\epsilon)\right) = \pi\,\mathrm{sgn}(\mathrm{Re}(x_i)).$$

As $L \to \infty$, the roots aggregate into two cuts and $p(x)$ becomes a classical algebraic curve with two cuts.

Roots of the Baxter polynomial

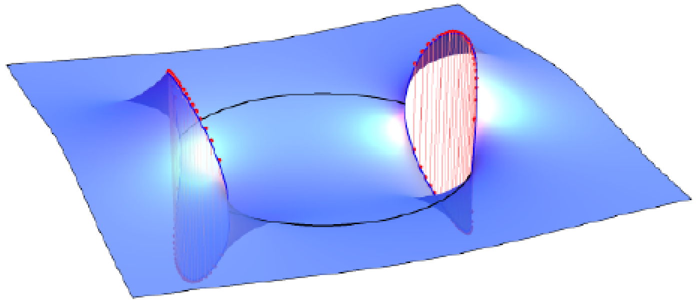


$L \sim \sqrt{\lambda} \gg 1$

$\Longrightarrow$

Classical limit

The algebraic curve



$$p(x) = \pi - 4\,i\,\mathbb{E}(a^2 \sin^2(\phi))\,\mathbb{F}_1 + 4\,i\,\mathbb{K}(a^2 \sin^2(\phi))\,\mathbb{F}_2 - a\left(\frac{x + \frac{1}{r}e^{-i\phi}}{x + re^{i\phi}}\right)\left(\frac{2\,r\,e^{i\phi}}{x^2 - 1}\right) y(x)\,\mathbb{K}(a^2 \sin^2(\phi)),$$

[Valatka&Sizov, to appear]

where

$$\mathbb{F}_1 = \mathbb{F}\left(\sin^{-1}\sqrt{a\left(\frac{x - \frac{1}{r}e^{-i\phi}}{x + re^{i\phi}}\right)\left(\frac{2\,r\,e^{2i\phi}}{e^{2i\phi} - 1}\right)}\,\middle|\,a^2 \sin^2(\phi)\right),$$

$$\mathbb{F}_2 = \mathbb{E}\left(\sin^{-1}\sqrt{a\left(\frac{x - \frac{1}{r}e^{-i\phi}}{x + re^{i\phi}}\right)\left(\frac{2\,r\,e^{2i\phi}}{e^{2i\phi} - 1}\right)}\,\middle|\,a^2 \sin^2(\phi)\right),$$

Slope function

$$\mathrm{tr}\; D^S Z^J$$

At weak coupling (one-loop) the anomalous dimension for any integer S is given by

$$\left(\frac{u_k - i/2}{u_k + i/2}\right)^J = -\prod_{j=1}^{S} \frac{u_k - u_j + i}{u_k - u_j - i} \;\; , \;\; \gamma = 2g^2 \frac{1}{u_k^2 + 1/4}$$

For J=2 one can solve it explicitly:

$$Q_S(u) \equiv \prod_j^S (u - u_j)$$

$$Q_2 = \frac{1}{4}(1 - 12u^2) \;\; , \;\; Q_4 = \frac{1}{192}(560u^4 - 520u^2 + 27) \;\; , \;\; \ldots$$

$$\gamma_{S=2}^{1-\mathrm{loop}} = 12g^2 \;\; , \;\; \gamma_{S=4}^{1-\mathrm{loop}} = \frac{44g^2}{3} \;\; , \;\; \gamma_{S=6}^{1-\mathrm{loop}} = \frac{50g^2}{3} \;\; , \;\; \ldots$$

Easy to guess the general form:

$$\gamma_S^{1-\mathrm{loop}} = 8g^2 H_S$$

Has a simple pole at S=-1 as predicted by BFKL

For us important question is what is the analytical continuation of Q

$$Q_S(u) = {}_3F_2[-S, S+1, 1/2 - iu; 1, 1; 1]$$

Good to positive integer S, but is obviously symmetric S -> -1-S. So cannot give a singularity at S=-1

$$Q(u) = \frac{e^{\frac{i\pi S}{2}}\Gamma^3(S+1)}{2\Gamma(2S+1)} \frac{1/\cosh(\pi u)}{\sin(\pi S/2)} \left(Q_S(-u) \sin\left(\frac{1}{2}\pi(S-2iu)\right) + Q_S(u) \sin\left(\frac{1}{2}\pi(S+2iu)\right)\right)$$

The correct combination has an asymptotic

$$u^S + A(S)\frac{\sinh(2\pi u)}{u^{S+1}}$$

Conclusions

- Can iterate and compute more orders in the near BPS expansion
- Solve Pμ in different regimes – strong coupling systematic expansion, BFKL
- More observables can be studied (available on the market already)
- Full string theory as a matrix integral (multi-matrix integral)?
- Correlation functions? Relation to Bubble ansatz