Classification and symmetry properties of scaling dimensions at Anderson transitions

Ilya A. Gruzberg (University of Chicago)

Joint work with: A. Ludwig (UCSB)
A. Mirlin (Karlsruhe)
M. Zirnbauer (Köln)

References: Phys. Rev. Lett. **107**, 086403 (2011)
Phys. Rev. B **87**, 125144 (2013)

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Anderson localization

- Single electron in a random potential

$$H = -\frac{\hbar^2}{2m}\Delta + U(\mathbf{r}), \quad \overline{U(\mathbf{r})U(\mathbf{r}')} = \gamma\delta(\mathbf{r} - \mathbf{r}')$$

- Metal-insulator transition is possible upon variation of disorder strength γ

- Ensemble of disorder realizations: statistical treatment

- More complicated variants (extra symmetries)

Symmetry classification of disordered electronic systems

A. Altland, M. Zirnbauer '96

Conventional (Wigner-Dyson) classes

	T	spin rot.	chiral	p-h	symbol
GOE	+	+	−	−	AI
GUE	−	+/−	−	−	A
GSE	+	−	−	−	AII

Chiral classes

	T	spin rot.	chiral	p-h	symbol
ChOE	+	+	+	−	BDI
ChUE	−	+/−	+	−	AIII
ChSE	+	−	+	−	CII

$$H = \begin{pmatrix} 0 & t \\ t^{\dagger} & 0 \end{pmatrix}$$

Bogoliubov-de Gennes classes

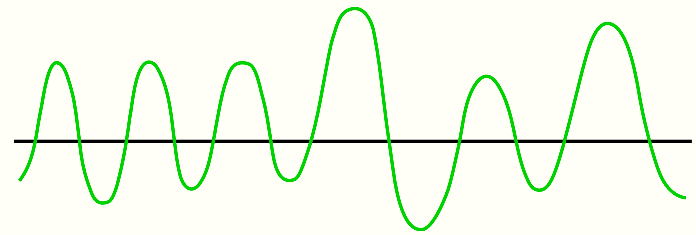
	T	spin rot.	chiral	p-h	symbol
	+	+	−	+	CI
	−	+	−	+	C
	+	−	−	+	DIII
	−	−	−	+	D

$$H = \begin{pmatrix} h & \Delta \\ -\Delta^{*} & -h^{T} \end{pmatrix}$$

Extended and localized states

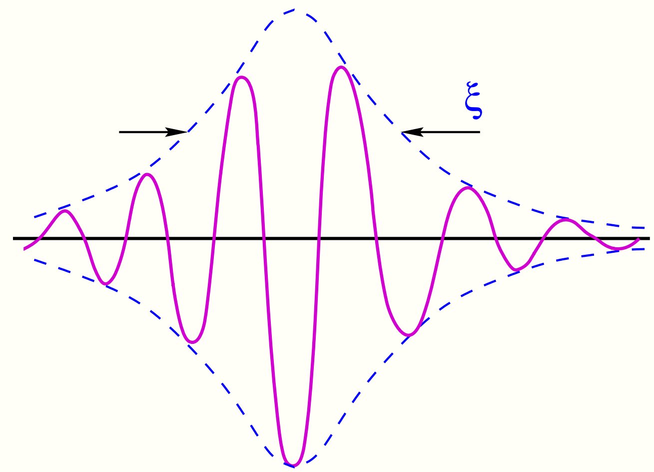
P. W. Anderson '58

- Extended, like plane waves



- Localized, with

$$|\psi(x)| \sim e^{-|x|/\xi}$$

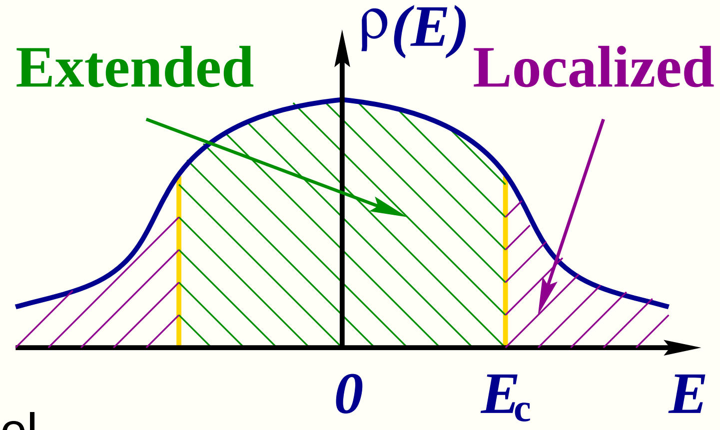


- localization length ξ

Anderson localization

• Nature of states depends on where they are in the spectrum

• Mobility edge E_c separates extended and localized states



• Anderson transition at E_c

• Field theory: supersymmetric σ-model

$$S[Q] \propto -\int d^d\boldsymbol{r}\,\mathrm{Str}[D(\nabla Q)^2 + 2i\omega\Lambda Q], \qquad Q^2 = 1$$

• Matrix field $Q \in G/K$, the (super)coset different for each AZ class

• $d = 0$ describes metallic grains, gives RMT results

Wave functions at Anderson transitions

- Critical wave functions are neither localized nor truly extended

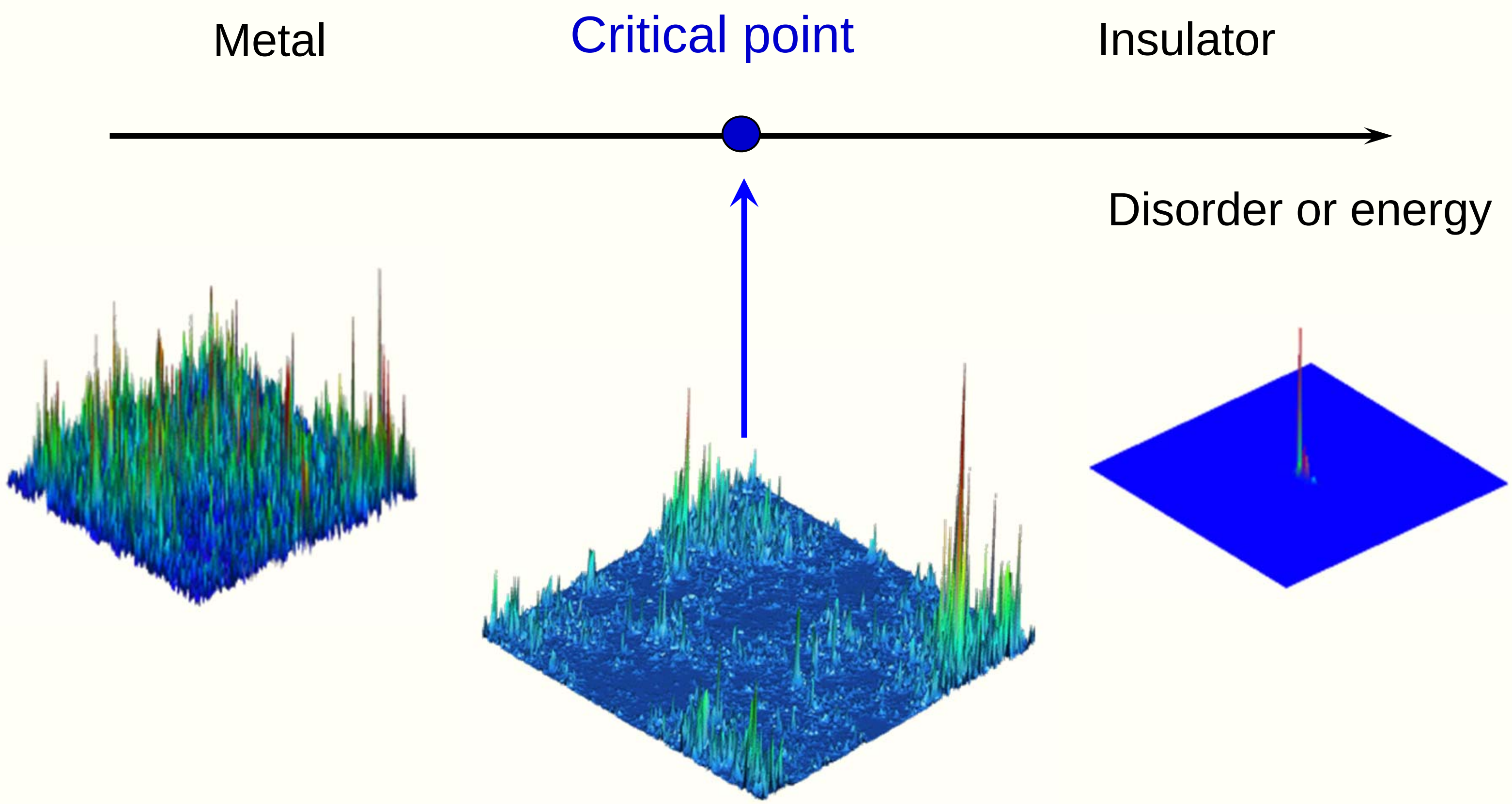
F. Wegner `80

- They are complicated statistically scale-invariant multifractals characterized by an infinite set of exponents, a multifractal spectrum

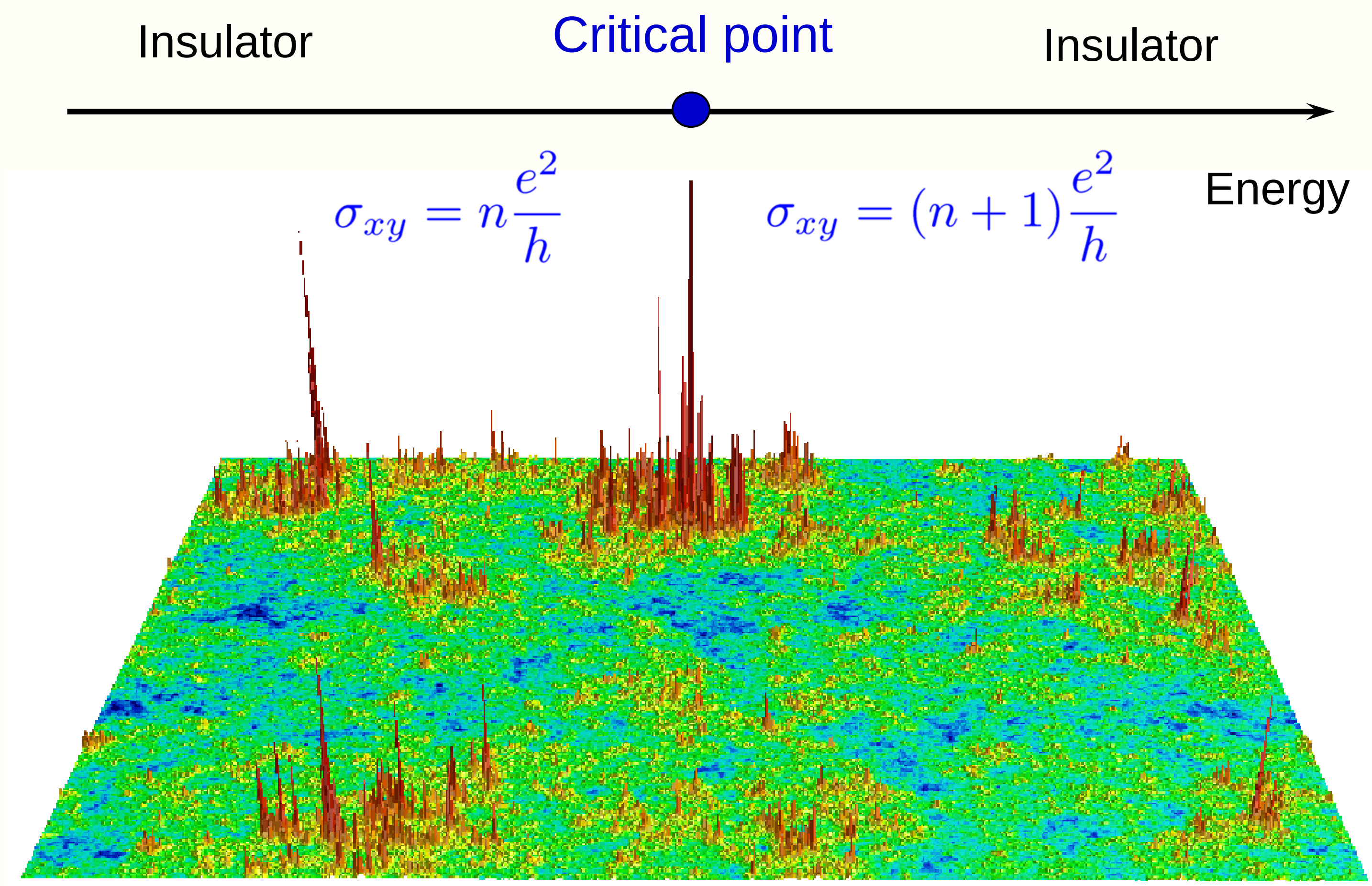
- For most Anderson transitions no analytical results are available

- Expect conformal invariance

Wave functions across Anderson transition



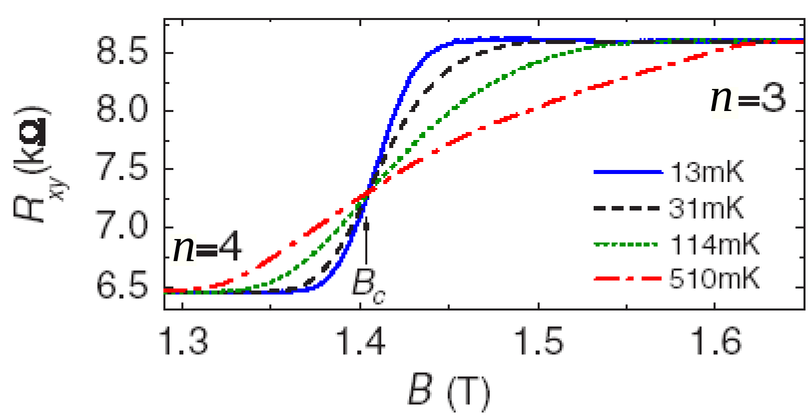
Wave function at the quantum Hall transition



F. Evers

Effects of multifractality in interacting systems

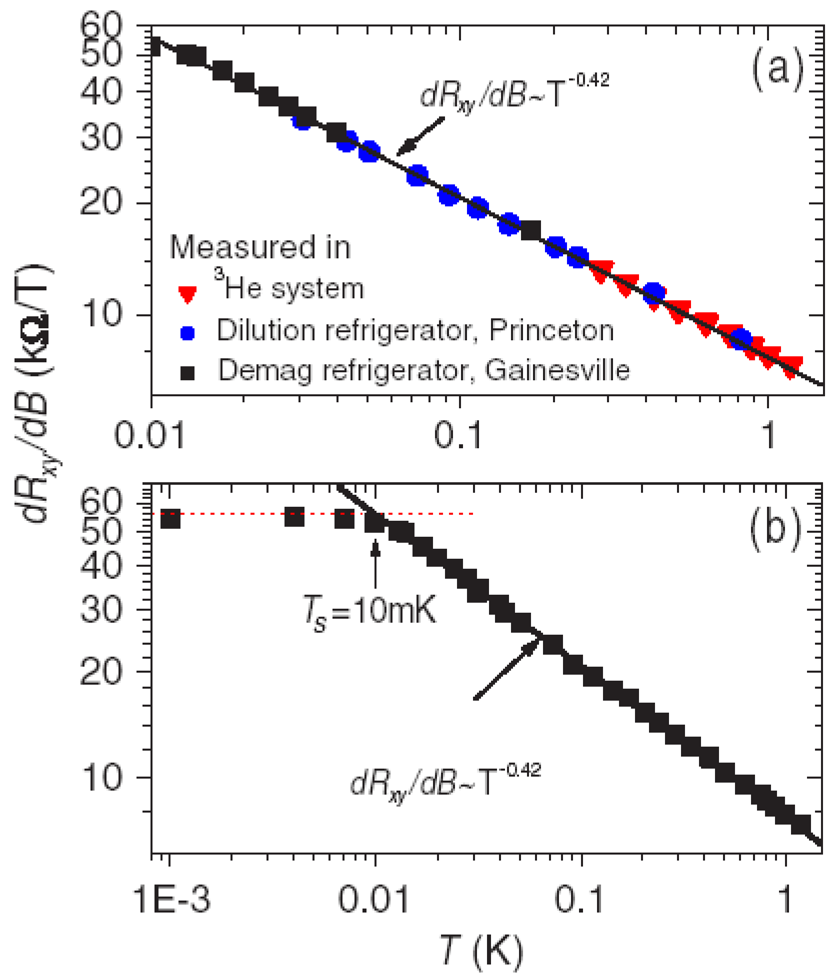
- Temperature scaling near Anderson transitions: IQH transition



W. Li et al. `09)

$$\frac{dR_{xy}}{dB} \sim T^{-\kappa}$$

$$\kappa = \frac{1}{z\nu} \approx 0.42$$



Effects of multifractality in interacting systems

- Temperature scaling at Anderson transitions cannot be explained within the single-particle picture

- It is due to interaction-induced dephasing

D-H. Lee and Z. Wang `96
Z. Wang, M. P. A. Fisher, S. M. Girvin, and J. T. Chalker `00
I. S. Burmistrov, S. Bera, F. Evers, I. V. Gornyi, and A. D. Mirlin `11

$$M_{jk} = \int d\mathbf{r}_1 d\mathbf{r}_2 K_{jk}(\mathbf{r}_1, \mathbf{r}_2) U(\mathbf{r}_1 - \mathbf{r}_2),$$

$$K_{jk}(\mathbf{r}_1, \mathbf{r}_2) = |\psi_j^2(\mathbf{r}_1)\psi_k^2(\mathbf{r}_2)| - \psi_j(\mathbf{r}_1)\psi_k(\mathbf{r}_2)\psi_j^*(\mathbf{r}_2)\psi_k^*(\mathbf{r}_1)$$

Effects of multifractality in interacting systems

M.V. Feigel'man, L. B. Ioffe, V. E. Kravtsov, and E. A. Yuzbashyan `07
I. S. Burmistrov, I. V. Gornyi, and A. D. Mirlin `12

- Disordered superconductors
- Preformed localized pairs, enhanced single-particle ("parity") gap
- Enhancement of the transition temperature of a disordered close to an Anderson transition

$$T_c \sim \lambda^{d/|\Delta_2|}$$

$$H = \sum_{j\sigma} \epsilon_j c^\dagger_{j\sigma} c_{j\sigma} - \frac{\lambda}{\nu} \sum_{jk} M_{jk} c^\dagger_{j\uparrow} c^\dagger_{j\downarrow} c_{k\uparrow} c_{k\downarrow}$$

$$M_{jk} = \int d\mathbf{r}\, \psi_j^2(\mathbf{r}) \psi_k^2(\mathbf{r})$$

Multifractal wave functions

F. Wegner `80
C. Castellani, L. Peliti `86

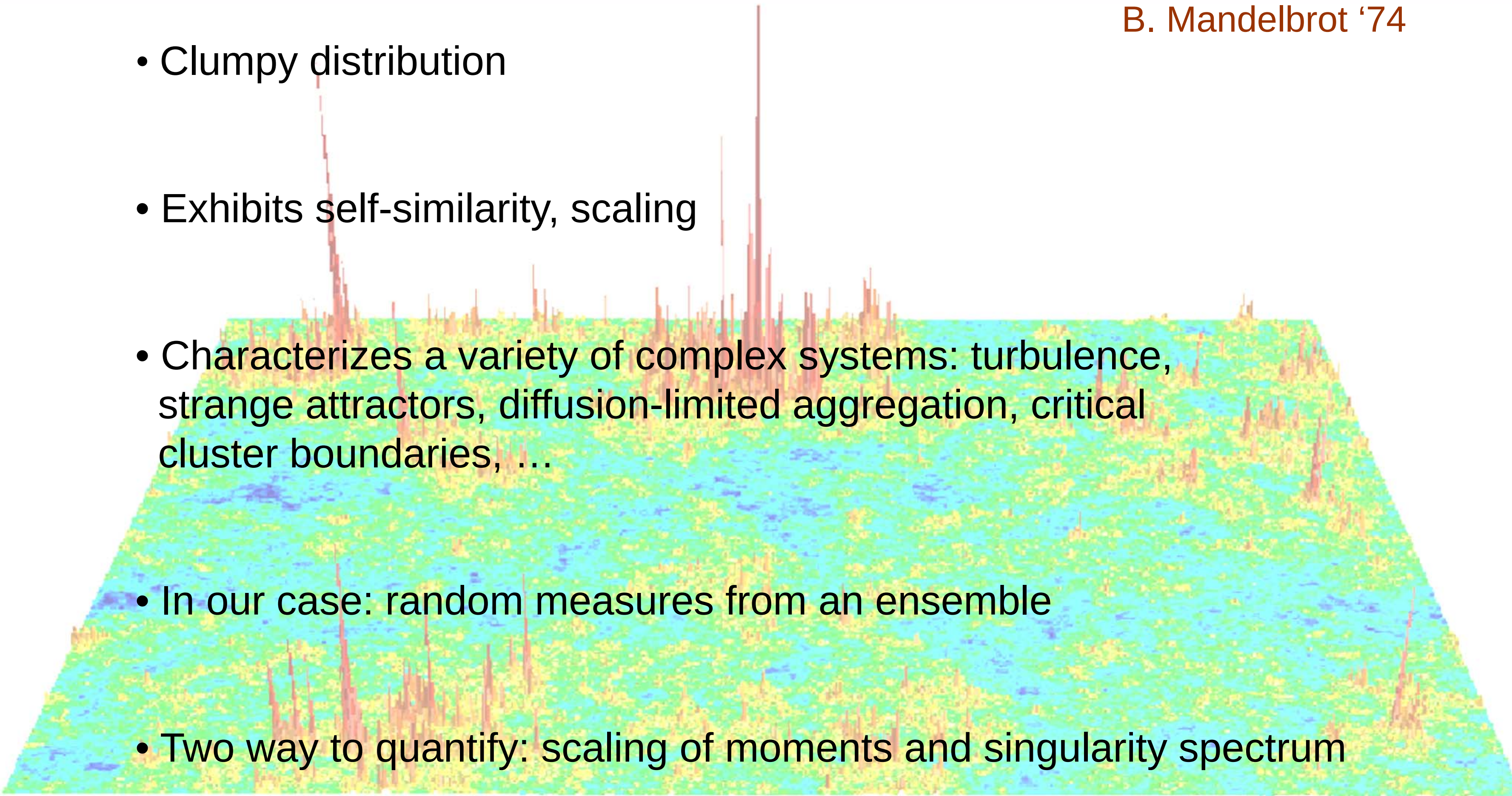
- Moments of the wave function intensity $|\psi(\boldsymbol{r})|^2$

- Scaling with the system size L

$$P_q = \int d^d\boldsymbol{r} |\psi(\boldsymbol{r})|^{2q},$$

$$\overline{P_q} = L^d \overline{|\psi(r)|^{2q}} \sim \begin{cases} L^0, & \text{insulator} \\ L^{-\tau_q}, & \text{critical} \\ L^{-d(q-1)}, & \text{metal} \end{cases}$$

Multifractal measures



B. Mandelbrot '74

- Clumpy distribution
- Exhibits self-similarity, scaling
- Characterizes a variety of complex systems: turbulence, strange attractors, diffusion-limited aggregation, critical cluster boundaries, …
- In our case: random measures from an ensemble
- Two way to quantify: scaling of moments and singularity spectrum

Multifractal moments and spectrum

- Probability measure with support in a cube of size L
- Divide the cube into N boxes B_i of size a, $N = \left(\frac{L}{a}\right)^d$
- Measure of each box $p_i = \int_{B_i} d\mu(\mathbf{r})$
- (Complex) moments of the measure scale with L/a

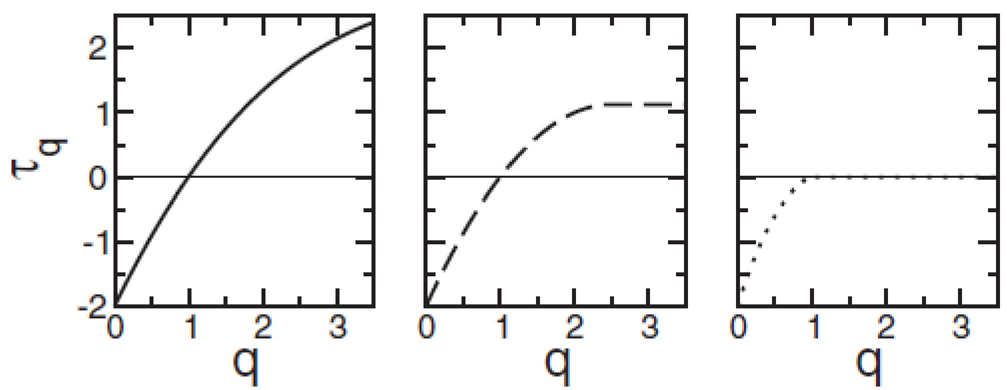
$$P_q = \sum_{i=1}^{N} p_i^q \sim \left(\frac{L}{a}\right)^{-\tau_q}$$

- Multifractal spectrum τ_q
- For random measures: distinguish mean and typical MF spectra

General properties of multifractal spectra

From the definition of τ_q it follows that for real q

- τ_q is non-decreasing: $\tau_q' \geq 0$
- τ_q is convex: $\tau_q'' \leq 0$
- $\tau_0 = -d$ (dimension of the support)
- $\tau_1 = 0$ (normalization of the measure)



Multifractal spectrum of a metal and an insulator

- Extreme cases

  - Uniform measure

$$p_i = \frac{1}{N} = \left(\frac{L}{a}\right)^{-d}, \quad P_q = N p_i^q = \left(\frac{L}{a}\right)^{-d(q-1)}$$

MF spectrum is linear: $\tau_q = d(q-1)$

  - Measure localized in volume $\xi^d, \quad a < \xi < L$

$(\xi/a)^d$ boxes filled with $p_i = (\xi/a)^{-d}$

$P_q = \left(\frac{\xi}{a}\right)^{-d(q-1)}$ is independent of $L \Rightarrow \tau_q = 0$

Solvable case: Dirac fermion in a random gauge field

- Dirac Hamiltonian in 2D $H = v\sigma_\mu(i\partial_\mu - A_\mu)$
- Hodge decomposition of gauge field $A_\mu(\mathbf{r}) = \epsilon_{\mu\nu}\partial_\nu\phi(\mathbf{r}) + \partial_\mu\chi(\mathbf{r})$
- Exact zero energy wave functions

$$\psi(\mathbf{r}) = \mathcal{N}^{-1/2}e^{i\xi(\mathbf{r})}e^{\phi(\mathbf{r})}, \quad \mathcal{N} = \int_{L^2} e^{2\phi(\mathbf{r})}d^2\mathbf{r}$$

- Disorder $\overline{A_\mu(\mathbf{r})A_\nu(\mathbf{r}')} = 2\pi\gamma_A\delta_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}')$

$$\mathcal{P}[\phi] \propto \exp\left(-\frac{1}{4\pi\gamma_A}\int(\nabla\phi)^2d^2\mathbf{r}\right)$$

- $\phi(\mathbf{r})$ is GFF with $\overline{\phi(\mathbf{r})\phi(\mathbf{r}')} = \gamma_A \ln\frac{L}{|\mathbf{r}-\mathbf{r}'|}$
- $|\psi(\mathbf{r})|^2 \propto e^{2\phi(\mathbf{r})}$ is a Liouville measure

Dirac fermion in a random gauge field: MF spectrum

Ludwig et al. `94

- Weak disorder $\gamma_a < 1$, parabolic spectrum with termination point

$$\tau_q = \begin{cases} 2(q-1)(1-\gamma_A q), & q \leq q_c = \dfrac{1+\gamma_A}{2\gamma_A} \\ \dfrac{(1-\gamma_A)^2}{2\gamma_A}, & q \geq q_c \end{cases}$$

Chamon et al.`96
Castillo et al. `97
Carpentier and LeDoussal `01

- Freezing transition at $\gamma_a = 1$

- Strong disorder $\gamma_a \geq 1$, parabolic spectrum with freezing

$$\tau_q = \begin{cases} -2(1-\gamma_A^{1/2} q)^2, & q \leq q_c = {\gamma_A}^{-1/2} \\ 0, & q \geq q_c \end{cases}$$

Multifractality and field theory

- Anomalous exponents: $\tau_q = d(q-1) + \Delta_q$ F. Wegner '80
- Δ_q related to scaling dimensions x_q of operators $\mathcal{O}_q$ in a σ model

$$\Delta_q = x_q - q x_1$$

- For WD classes $x_1 = 0$. In $d = 2 + \epsilon$
  - Orthogonal and unitary classes

$$\Delta_q^{(O)} = q(1-q)\epsilon + \frac{\zeta(3)}{4} q(1-q)[q(1-q)-1]\epsilon^4 + O(\epsilon^5)$$

$$\Delta_q^{(U)} = q(1-q)(\epsilon/2)^{1/2} - \frac{3\zeta(3)}{8} q^2(1-q)^2\epsilon^{3/2} + O(\epsilon^{5/2})$$

- Notice the symmetry $\Delta_q = \Delta_{1-q}$

Symmetry of MF spectra for WD classes

$$\mathcal{P}(\rho) = \rho^{-3}\mathcal{P}(\rho^{-1}), \quad \overline{\rho^q} = \overline{\rho^{1-q}}$$

A. Mirlin, Y. Fyodorov `94

- ρ is a normalized local density of states
- Exact within the sigma models in the WD classes
- At criticality it follows that $x_q = x_{1-q} \quad (\Delta_q = \Delta_{1-q})$
- Due to universality this is exact for any critical system in the WD classes

- Our goals:
  - to understand the origin of the symmetry
  - to extend it to other AZ symmetry classes
  - to extend to subleading scaling operators and identify the corresponding physical observables

Results

- General relation for dimensions of scaling operators describing MF moments of an observable at criticality

$$x_q = x_{q_* - q}$$

follows from global conformal invariance

- q_* is left unspecified, but it is as universal as critical exponents

- For Anderson localization in the WD classes we show that

$$\mathcal{P}(\rho) = \rho^{-3}\mathcal{P}(\rho^{-1}), \quad \overline{\rho^q} = \overline{\rho^{1-q}} \quad \Rightarrow \quad \Delta_q = \Delta_{1-q}$$

are consequences of the Weyl group symmetry of the sigma model

Global conformal invariance

- $x_q = \tau_q - d(q-1) + qx_\nu$
- From the properties of τ_q it follows that $x_0 = 0$ and $x_q < 0$ for sufficiently large q
- Also $x'_0 \geq 0 \quad \Rightarrow$ there is a $q_* \geq 0$ such that $x_{q_*} = 0$!
- OPE $\mathcal{O}_p(\mathbf{r}_1)\mathcal{O}_q(\mathbf{r}_2) \sim |\mathbf{r}_1 - \mathbf{r}_2|^{x_{p+q}-x_p-x_q}\mathcal{O}_{p+q}\left(\frac{\mathbf{r}_1+\mathbf{r}_2}{2}\right)$
- For $p = q_* - q$ the operator on the right has zero dimension

$$\Rightarrow \quad x_q = x_{q_*-q}$$

- q_* is left unspecified, but it is as universal as critical exponents

Random matrix theory and q_*

- q_* is determined by symmetries only
- Can find it in the zero-dimensional sigma model, or RMT
- Find moments of LDOS in a RMT:

$$\overline{\rho^q} \propto \int d\epsilon \left(\frac{\delta}{\epsilon^2 + \delta^2} \right)^q P(\epsilon), \qquad P(\epsilon) \sim |\epsilon|^m$$

- $q_* = m_l + 1 = \begin{cases} 1, & \text{WD classes,} \\ 2, & \text{class CI,} \\ 3, & \text{class C.} \end{cases}$

- More systematic derivation using sigma model and Wyle symmetry

Results

- We generalize these relations to other symmetry classes

$$\overline{\rho^q} = \overline{\rho^{q_*-q}} \qquad \Rightarrow \qquad x_q = x_{q_*-q}$$

- Symmetry points for all AZ classes

AZ class	A	AI	AII	AIII	BDI	CII	C	CI	D	DIII
Symmetry point q^*	1	1	1	1^*	$1/2^*$	2^*	3	2	0^*	0^*

- Sigma model Weyl symmetry can be straightforwardly applied only to WD classes and BdG classes with spin rotation symmetry (C and CI)

- Other classes are more subtle

- Classes D and DIII: domain walls and absence of localization

General scaling operators

- More complicated combinations of wave functions

$$A_p(\mathbf{r}_1, \ldots, \mathbf{r}_p) = \left| \mathrm{Det} \begin{pmatrix} \psi_1(\mathbf{r}_1) & \cdots & \psi_1(\mathbf{r}_p) \\ \vdots & \ddots & \vdots \\ \psi_p(\mathbf{r}_1) & \cdots & \psi_p(\mathbf{r}_p) \end{pmatrix} \right|^2$$

- General correlators $K_\lambda = \overline{A_1^{q_1 - q_2} A_2^{q_2 - q_3} \ldots A_{n-1}^{q_{n-1} - q_n} A_n^{q_n}}$
- Scaling dimensions x_λ
- $\lambda = (q_1, q_2, \ldots, q_n)$ is a highest weight of G with complex q_i
- Weyl group W symmetry implies the general symmetry relation

$$x_\lambda = x_{w(\lambda)}, \quad \forall w \in W$$

- This includes sign changes and permutations of (shifted) q_i

$$q_j \to -c_j - q_j, \quad q_i \to q_j + \frac{c_j - c_i}{2}$$

General scaling operators: examples

- Weight $\lambda = (q_1, q_2)$ leads to eight operators with the same x_λ

$$\begin{aligned}&(q_1,q_2),\quad (1-q_1,q_2),\quad (q_1,3-q_2),\quad (1-q_1,3-q_2),\\&(2-q_2,2-q_1),\quad (-1+q_2,2-q_1),\\&(2-q_2,1+q_1),\quad (-1+q_2,1+q_1)\end{aligned}$$

- For example, $\lambda = (0,0)$ (identity operator) has $x_\lambda = 0$,

so all operators with weights

$$\begin{aligned}&(0,0),\quad (1,0),\quad (0,3),\quad (1,3),\\&(2,2),\quad (-1,2),\quad (2,1),\quad (-1,1)\end{aligned}$$

have $x_\lambda = 0$

- One of these is $\overline{A_2^2}$ that appeared in the study of dephasing near the IQH transition

Composite operators

- Identify exact scaling gradientless operators

$$\tilde{Q}_{AA,bb} = \frac{1}{2}(Q_{RR} - Q_{AA} + Q_{RA} - Q_{AR})_{bb}$$

$$d_m = \mathrm{Det}\left[\tilde{Q}^{(m)}\right], \qquad \tilde{Q}^{(m)} = \begin{pmatrix} \tilde{Q}_{11} & \cdots & \tilde{Q}_{1m} \\ \vdots & \ddots & \vdots \\ \tilde{Q}_{m1} & \cdots & \tilde{Q}_{mm} \end{pmatrix}.$$

$$\varphi_{(q_1,\ldots,q_n)} = d_1^{q_1-q_2} d_2^{q_2-q_3} \ldots d_{n-1}^{q_{n-1}-q_n} d_n^{q_n}$$

- Arbitrary complex q_i
- Iwasawa decomposition for $G = KAN$

Composite operators

- Iwasawa decomposition for $G = KAN$ leads to

$$\tilde{Q} = \begin{pmatrix} 1 & \dots & * & * \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & * \\ 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-2x_1} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & e^{-2x_{n-1}} & 0 \\ 0 & \dots & 0 & e^{-2x_n} \end{pmatrix} \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ * & \dots & 1 & 0 \\ * & \dots & * & 1 \end{pmatrix}$$

$$\varphi_{(q_1,\dots,q_n)} = \exp\Big(-2 \sum_{j=1}^{n} q_j x_j \Big)$$

- $\lambda = (q_1, \dots, q_n)$ is a highest weight

- Young tableaux for positive integer q_i

- Connection with Wegner's classification, numerous checks

Open questions

- Chiral classes and classes D and DIII with domain walls
- Connection with transport observables (transmission eigenvalues)
- Interacting systems