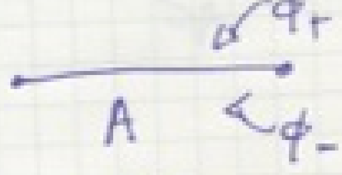
HOLOGRAPHIC ENTANGLEMENT BEYOND CLASSICAL GRAVITY

w/ T. Barrella, X. Dong, V.L. Martin

@ 1306.4682.

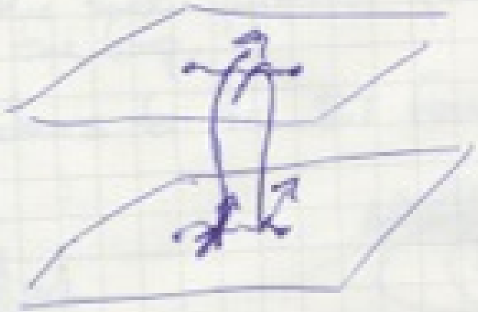
Reduced density matrix:



$$\rho[\phi_+, \phi_-] = \frac{1}{Z}\int D\phi \, e^{-S[\phi]} \, \delta(\phi(A,t^+) = \phi_+) \times \delta(\phi(A,t^-) = \phi_-)$$

Clearly then:

$$\mathrm{Tr}\,\rho^n = \frac{Z_n}{Z_1^n}$$ ← partition function on n-sheeted cover:



Rényi entropy:

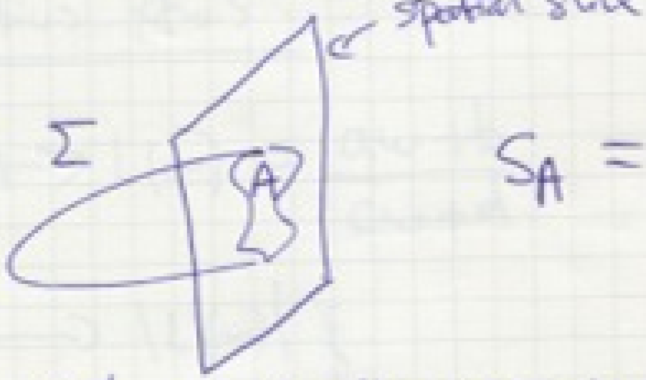
$$S_n = -\frac{1}{n-1}\log \mathrm{Tr}\,\rho^n$$

Entanglement entropy

$$S = \lim_{n\to 1} S_n = -\mathrm{tr}(\rho \log \rho)$$

S in general difficult to compute.

"outrageous" Ryu-Takayanagi proposal for theories with classical gravity duals:



$$S_A = \frac{\text{Area}(\Sigma)}{4G_N}$$

It's true! Hartman; Faulkner; Lewkowycz + Maldacena

$\underbrace{\text{Hartman; Faulkner}}_{1+1 \text{ CFTs.}}$

Connection spacetime $\leftrightarrow$ entanglement?

$\to$ Go beyond classical gravity in simplest setting, $1+1$ CFTs.

$\hookrightarrow$ directly compute the bulk partition function for the Z_n at one loop.

We will obtain one-loop corrections to the RT formula in $1+1$ dimensions.

Step I: Schottky uniformization

Einstein's equations in $2+1$ bulk: geometry is a quotient AdS_3/Γ, with $\Gamma \subset PSL(2,\mathbb{C})$

holography: $Z_{CFT}(\Sigma) = Z_{bulk}(\partial(AdS_3/\Gamma) = \Sigma)$

for us: Σ: n^{th} branched cover of $\mathbb{C}$ over region A.

therefore need to: (i) find Γ

(ii) evaluate $Z(AdS_3/\Gamma)$

AdS_3: $ds^2 = \frac{d\xi^2 + dw\,d\bar{w}}{\xi^2}$

near $\xi \to 0$ (conformal bdy) $PSL(2,\mathbb{C})$ acts via <u>Möbius xfms</u>:

$$w \mapsto L(w) = \frac{aw+b}{cw+d} \qquad ad-bc=1.$$

$$\xi \mapsto |L'(w)|\,\xi$$

$\Rightarrow$ need to find Γ s.t. $\Sigma = \mathbb{C}/\Gamma$.

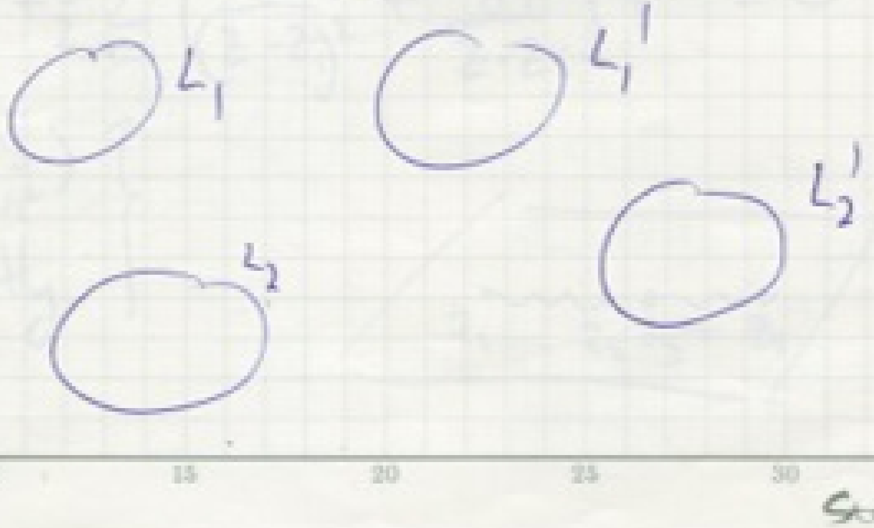
with Γ subgp of Möbius xfms.

$\Rightarrow$ called <u>Schottky uniformization</u>.

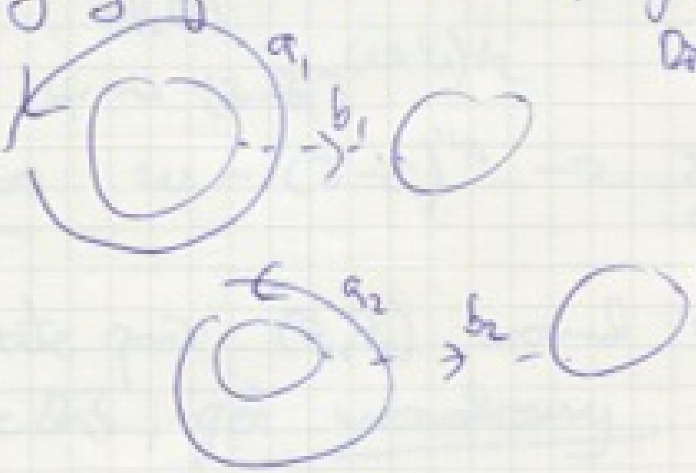
Why reasonable?

Möbius maps circles to circles:

~~Sup~~ The Schottky gp will be freely generated by g elements of $PSL(2,\mathbb{C})$. Consider $2g$ disjoint circles s.t. $C_i' = L_i(C_i)$



under $\Sigma = \mathbb{C}/\Gamma$, circles give g generators of $\pi_1(\Sigma)$. Paths connecting circles give remaining g generators of genus g Riemann surface.



<u>How to find Γ?</u>

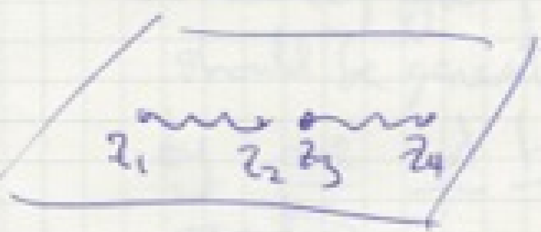
- Given multi-valued coordinate z on branched cover, want to find ω s.t.
  1. single valued
  2. Γ acts by Möbius xfm on ω.

- Simplest nontrivial cases:
  - two intervals on plane.
  - one interval on torus. (CFT on circle, finite T)

- Define:

$$\psi''(z) + \frac{1}{2}\sum_{i=1}^{4}\left(\frac{\Delta}{(z-z_i)^2} + \frac{\gamma_i}{z-z_i}\right)\psi = 0$$

$\Delta = \frac{1}{2}\left(1 - \frac{1}{n^2}\right)$
γ_i fixed shortly.



Eqn has 2 independent sols: ψ_1, ψ_2.

Define:

$$w = \frac{\psi_1(z)}{\psi_2(z)}.$$

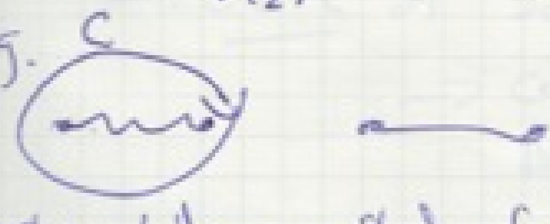
near z_i: $\psi \sim (z-z_i)^{(1 \pm 1/n)/2}$

$\Rightarrow w \sim (z-z_i)^{1/n}$ → single valued on n-th cover.

If we integrate pair (ψ_1, ψ_2) around contour C enclosing some $B.P.$, get monodromy:

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \mapsto M(C)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad M(C) = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in PSL(2,$$

e.g.



This tells us that for w must be identified:

$$w \sim \frac{aw+b}{cw+d} \rightarrow \text{quotient } \mathbb{C}/\Gamma\,!$$

Final step: too many quotients! $\qquad g = n-1$

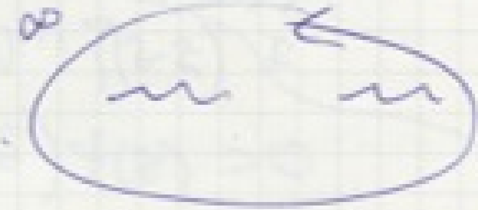
e.g. $n=2$:

→ two nontrivial cycles

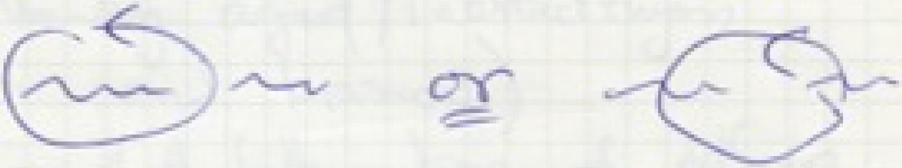
⇒ Schottky group should be generated by one not two generators.

choose accessory parameters γ_i such that:

(I) make monodromy at ∞
→ fixes 3 of 4 γ_is.



(II) one of the two cycles has trivial monodromy:



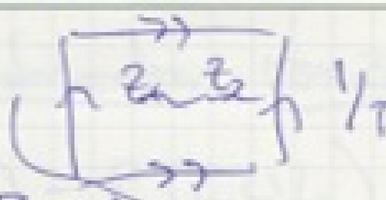
→ choice of Schottky uniformization
→ corresponds to different bulk AdS_3/Γ → lowest action dominates.
→ can have phase transitions

Having trivialized all but one cycle, have generators of Schottky gp:

$$\left.\begin{aligned} L_1 &= M_2 M_1 \\ L_i &= M_2^{i-1} L_1 M_2^{-(i-1)} \qquad i = 2, n-1 \end{aligned}\right\} \begin{array}{l} n-1 = g \\ \text{generators.} \end{array}$$

[Assumed $\mathbb{Z}_n$ symmetric uniformization]

Similar for torus



$$\psi''(z) + \frac{1}{2}\sum_{i=1}^{2}\Big(\Delta\, \wp(z-z_i) + \gamma(-1)^{i+1}\zeta(z-z_i) + \delta\Big)\psi(z) = 0$$

Faulkner (following Zograf/Takhtadzhyan and Krasnov) showed that on-shell bulk action of AdS_3/Γ satisfied:

$$\frac{\partial S_E}{\partial z_i} = -\frac{cn}{6}\gamma_i$$

→ central charge

→ found by trivializing monodromy

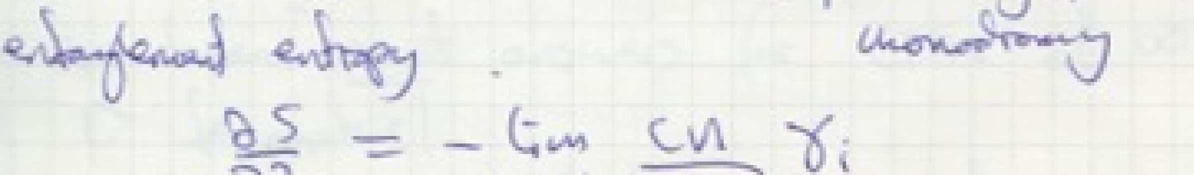
⇒ entanglement entropy

$$\frac{\partial S}{\partial z_i} = -\lim_{n\to 1}\frac{cn}{6(n-1)}\gamma_i$$

by solving for γ_i analytically as $n \to 1$, rederived the RT answer for two intervals on plane.

Interesting feature (known from RT)

small separation: large separation



phase transition → change of dominant uniformization.

In well-separated phase, mutual information vanishes:

$$I_n(L_1,L_2) \equiv S_n(L_1) + S_n(L_2) - S_n(L_1 \cup L_2)$$

However, general results (Cardy-Calabrese), this vanishing cannot be exact.

↳ will recover leading mutual information in 1-loop correction.

Similarly in torus: (ws):

- above HP: $S = \frac{c}{6} \log\left(\sin^2 \frac{\pi\Delta\phi}{2} \right) \quad (1)$
- below HP: $S = \frac{c}{6} \log \sin^2 \frac{\pi\Delta\phi}{R} \quad (2)$

These are known exact answers for CFTs at ∞ and 0 temperature!

→ (1) is missing finite size corrections.

(2) is missing mixed nature of density matrix $S(L) \neq S(R \setminus L)$.

→ find these at 1-loop.

Grandi-Ndoreg-Yin derived a nice formula for functional determinants on AdS_3/Γ:

$$\log Z|_{\text{one-loop}} = -\sum_{\gamma} \sum_{n=1}^{\infty} \log |1 - q_\gamma^n|$$

$\gamma \in \mathcal{P}$

for metric fluctuations.

"virasoro descendants of ground state" on torus

γ has eigenvalues: $q_\gamma^{\pm 1/2}$, $|q_\gamma| < 1$.

conjugacy classes of <u>primitive</u> elements of Γ (i.e. γ s.t. $\gamma \neq \gamma_g^n$, $n > 1$)

<u>strategy</u>:
- find Γ for all n (Schottky uniformize)
- generate $\mathcal{P}$ for all n $\longrightarrow$ primitive words.
- find eigenvalues of words
- do sum
- analytically continue to $n \to 1$.

Cross ratio $x = \dfrac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}$

Mutual information only depends on this. (4 pt. fn of twist operators) small x expansion known from OPEs:

$$g_n = -\underset{\substack{\uparrow \\ \text{multiplicity}}}{N} \frac{n}{2(n-1)} \left(\frac{x}{4n^2}\right)^{2h} \sum_{k=1}^{n-1} \frac{1}{\left(\sin \frac{\pi k}{n}\right)^{4h}}$$

dimension.

$$\to \quad g = -N \left(\frac{x}{4}\right)^{2h} \frac{\sqrt{\pi}}{4} \frac{\Gamma(2h+1)}{\Gamma(2h+\frac{3}{2})}$$

[Cardy - Calabrese]

We managed to implement the strategy exactly in an expansion in x.

<u>One key step</u>: To each order in x, only a finite number of words contribute to the sum.

<u>leading order</u>: <u>CDWs</u> "consecutively decreasing words".

$$\gamma_{k,m} = L_{k+m} L_{k+m-1} \cdots L_{m+1}$$

$\to$ can find eigenvalues explicitly

<u>higher orders</u>: products of <u>CDWs</u>.

$$\boxed{g_{\text{Ising}} = -\Big(\underbrace{\frac{x^4}{630} + \frac{2x^5}{693} + \frac{15x^6}{4004} + \frac{x^7}{234}}_{\text{only CDWs}} + \frac{167x^8}{36936} + O(x^9)\Big)}$$

matches CC result.