“Euler Symposium on Theoretical and Mathematical Physics”

Euler International Mathematical Institute, St. Petersburg, July 12-17, 2013

# Exact spectral equations in planar N=4 SYM theory


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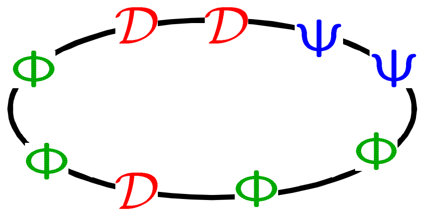
**Collaborations with**
**Gromov, Leurent, Volin**

# CFT: N=4 SYM as a superconformal 4d QFT

$$\mathcal{S}_{SYM} = \frac{1}{\lambda}\int d^4x\,\mathrm{Tr}\left(F^2 + (\mathcal{D}\Phi)^2 + \bar{\Psi}\mathcal{D}\Psi + \bar{\Psi}\Phi\Psi + [\Phi,\Phi]^2\right)$$

- 4d superconformal QFT! Global symmetry PSU(2,2|4)
- Operators in 4D (planar limit)




$$\mathcal{O}(x) = \mathrm{Tr}\,[\mathcal{D}\mathcal{D}\Psi\Psi\Phi\Phi\mathcal{D}\Psi\ldots](x)$$

$$+\text{permutations}$$

- 4D Correlators (e.g. for scalars):

scaling dimensions
structure constants
} non-trivial functions of 'tHooft coupling λ!

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle = \frac{\delta_{ij}}{|x|^{2\Delta_j(\lambda)}}$$

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}(\lambda)}{|x_{12}|^{\Delta_i+\Delta_j-\Delta_k}|x_{23}|^{\Delta_j+\Delta_k-\Delta_i}|x_{31}|^{\Delta_i+\Delta_k-\Delta_j}}$$

They describe the whole conformal theory via operator product expansion

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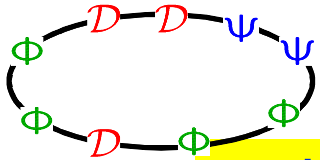
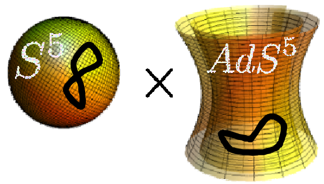
# Methods for AdS/CFT spectral problem




**Weak coupling expansion for SYM anomalous dimensions. Perturbative integrability.**

**Strong coupling from AdS-dual – classical superstring sigma model Classical integrability, algebraic curve**

**S-matrix, Asymptotic Bethe ansatz**

$$0 = \left( e^{iLp_j} \prod_{k\neq j}^{M} \hat{S}(p_j, p_k) - 1 \right)$$

$$Y_{a,s}^{+} Y_{a,s}^{-} = \frac{\left[1 + Y_{a,s+1}\right] \left[1 + Y_{a,s-1}\right]}{\left[1 + \frac{1}{Y_{a+1,s}}\right] \left[1 + \frac{1}{Y_{a-1,s}}\right]}$$

**Y-system + analyticity (exact!)**

**Thermodynamic Bethe ansatz (exact!)**

$$\log Y_{a,s} = L\,\delta_{a,0} \frac{\partial}{\partial u} \tilde{\epsilon}_a + \sum_{a',s'} K_{a,s;a',s'} * \log\left(1 + Y_{a',s'}(u)\right)$$

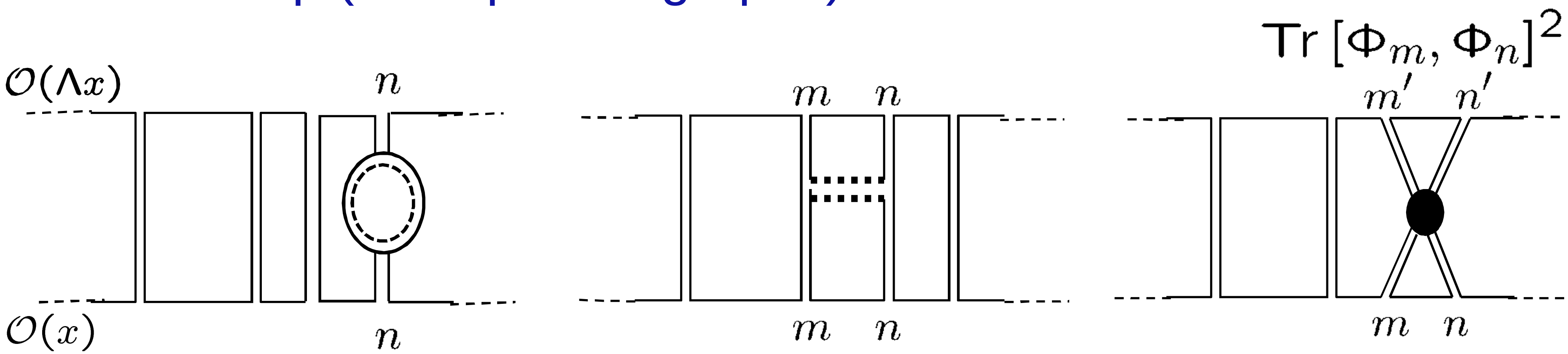
**Wronskian solution of Y-system via Baxter's Q-functions + analyticity**

**Finite system of integral non-linear equations (FiNLIE)**

**Finite matrix Riemann-Hilbert eqs. for quantum spectral curve (P-μ)**

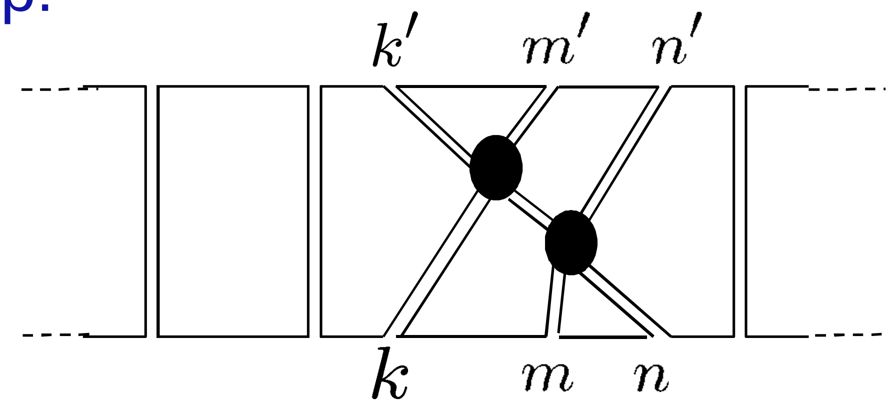
# Weak coupling calculation from SYM

- Example: O(6) sector (scalar fields): $\mathcal{O}(x) = \mathrm{Tr}\,(\Phi_{n_1}\Phi_{n_2}\cdots\Phi_{n_L})$
- Tree level: $\Delta_0 = L$ - degeneracy (for scalars)
- 1-loop (examples of graphs):




nontrivial action on R-indices.

- 2-loop:



# Dilatation operator in SYM perturbation theory

$$\lambda = N g^2_{YM}$$

Point-splitting and renormalization:

$$\mathcal{O}(x/\Lambda) = \Lambda^{\hat{D}} \mathcal{O}(x) = \Lambda^{\hat{D}^{(0)}} \left(1 + \lambda \log \Lambda \hat{D}^{(2)} + \ldots\right)$$

$$\hat{D} = \hat{D}^{(0)} + \lambda \hat{D}^{(2)} + \lambda^2 \hat{D}^{(4)} + \ldots$$

general action of dilatation operator gives a mixture:

$$\hat{D} \mathcal{O}_j(x) = \sum_j D_{ij} \mathcal{O}_j$$

Conf. dimensions are eigenvalues of ``Hamiltonian'' $\hat{D}$

$$\hat{D} \mathcal{O}_j(x) = \Delta_j \mathcal{O}_j$$

Perturbative dimensions: $\Delta = \Delta^{(0)} + \lambda \Delta^{(2)} + \lambda^2 \Delta^{(4)} + \ldots$

# Examples: su(2) and sl(2) sectors at one loop

Notations: $Z=\Phi_1+i\Phi_2,\quad X=\Phi_3+i\Phi_4,\quad Y=\Phi_5+i\Phi_6$

**su(2) operators:** $\mathrm{Tr} Z^{L-J}X^J(x)$ + permutations $\quad -Z-Z-X-Z-Z-X-Z-$

- Dilatation operator - Heisenberg Hamiltonian, integrable by Bethe ansatz!

$$\hat{D} = L + \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(1-\sigma_l\cdot\sigma_{l+1}\right) + O(\lambda^2)$$

Minahan, Zarembo
Beisert,Kristjansen,Staudacher

Solution in terms of Baxter function $Q(u)=\prod_{k=1}^{J}(u-u_k)$ obaying the Baxter eq.

$$T(u)Q(u) = \left(u+\frac{i}{2}\right)^L Q(u+i) + \left(u-\frac{i}{2}\right)^L Q(u-i)$$

where the function **T(u)** – transfer matrix eigenvalue -- is a polynomial.

Anomalous dimensions: $\Delta - L = \frac{\lambda}{8\pi^2}\,\partial_u \log\frac{Q(u+\frac{i}{2})}{Q(u-\frac{i}{2})}\Big|_{u=0} + \mathcal{O}(\lambda^2)$

with trace cyclicity condition $Q\left(\frac{i}{2}\right) = Q\left(-\frac{i}{2}\right)$

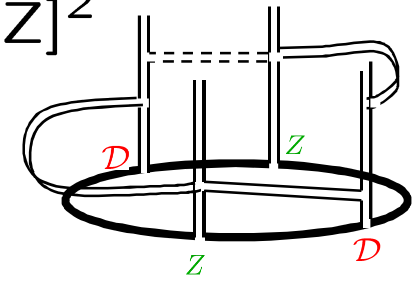
**sl(2) operators:** $\mathrm{Tr} Z\nabla^S Z^{L-1}(x)$ + permutations

Baxter relation slightly changes…

# Perturbative Konishi: Y-system versus Feynman graphs

- Integrability allows to sum exactly enormous number of Feynman diagrams of N=4 SYM
- ABA/S-matrix approach not enough due to wrapping. Luscher finite size corrections (7 loops). For >7 loops – Y-system needed!

$\mathrm{Tr}\ [\mathcal{D}, Z]^2$




$$
\begin{aligned}
\Delta_{\text{Konishi}} = \quad & 4 + 12\,g^2 - 48\,g^4 + 336\,g^6 + 96\,g^8(-26 + 6\,\zeta_3 - 15\,\zeta_5) \\
& -96\,g^{10}(-158 - 72\,\zeta_3 + 54\,\zeta_3^2 + 90\,\zeta_5 - 315\,\zeta_7) \\
& -48\,g^{12}(160 + 5472\,\zeta_3 - 3240\,\zeta_3\zeta_5 + 432\,\zeta_3^2 - 2340\,\zeta_5 - 1575\,\zeta_7 + 10206\,\zeta_9) \\
& +48\,g^{14}(-44480 + 108960\,\zeta_3 + 8568\,\zeta_3\zeta_5 - 40320\,\zeta_3\zeta_7 - 8784\,\zeta_3^2 + 2592\,\zeta_3^3 \\
& \qquad -4776\,\zeta_5 - 20700\,\zeta_5^2 - 26145\,\zeta_7 - 17406\,\zeta_9 + 152460\,\zeta_{11}) \\
& +48\,g^{16}(1133504 + 263736\,\zeta_2\zeta_9 - 1739520\,\zeta_3 - 90720\,\zeta_3\zeta_5 - 129780\,\zeta_3\zeta_7 \\
& \qquad +78408\,\zeta_3\zeta_8 + 483840\,\zeta_3\zeta_9 + 165312\,\zeta_3^2 - 82080\,\zeta_3^2\zeta_5 + 41472\,\zeta_3^3 \\
& \qquad +178200\,\zeta_4\zeta_7 - 409968\,\zeta_5 + 121176\,\zeta_5\zeta_6 + 463680\,\zeta_5\zeta_7 + 49680\,\zeta_5^2 \\
& \qquad +455598\,\zeta_7 + 194328\,\zeta_9 - 555291\,\zeta_{11} - 2208492\,\zeta_{13} - 14256\,\zeta_{1.2.8})
\end{aligned}
$$

**Bajnok,Janik**
**Leurent,Serban,Volin**
**Bajnok,Janik,Lukowski**
**Lukowski,Rej,**
**Velizhanin,Orlova**
**Leurent, Volin**

**Leurent, Volin (8 loops from FiNLIE)**

$\lambda = 16\pi^2 g^2$

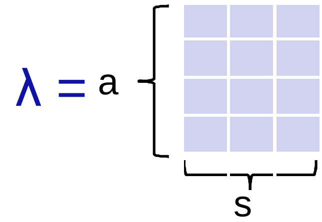
- Confirmed up to 5 loops by direct graph calculus (6 loops promised)

**Fiamberti,Santambrogio,Sieg,Zanon**
**Velizhanin**
**Eden,Heslop,Korchemsky,Smirnov,Sokatchev**

# Discrete Hirota eq.: T-system and Y-system

- Consider generalizations to other algebras, such as GL(K|M)

- T-function $T_{a,s}(u)$ for rectangular representations




satisfies full quantum Hirota equation:

$$T^{+}_{a,s}T^{-}_{a,s} = T_{a,s-1}\,T_{a,s+1} + T_{a+1,s}\,T_{a-1,s}$$

notations:

$$f^{[\pm a]} := f(u \pm ia/2)$$

$$f^{\pm} := f(u \pm i/2)$$

- Y-system

$$Y^{+}_{a,s}\,Y^{-}_{a,s} = \frac{\left[1+Y_{a,s+1}\right]\,\left[1+Y_{a,s-1}\right]}{\left[1+\frac{1}{Y_{a+1,s}}\right]\,\left[1+\frac{1}{Y_{a-1,s}}\right]}$$

$$Y_{a,s} = \frac{T_{a,s+1}\,T_{a,s-1}}{T_{a+1,s}\,T_{a-1,s}}$$

Discrete "classical" integrable dynamics!
We will use it to "solve" Y-system in terms of finite number of Q-functions

Krichever,Lipan, Wiegmann,Zabrodin
Gromov,V.K.,Leurent,Volin

# Wronskian solutions of Hirota equation

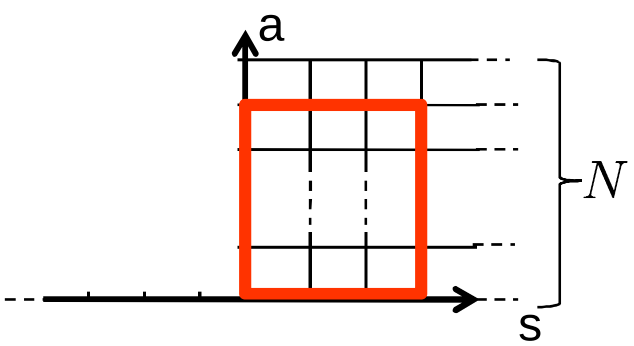
- Example: solution of Hirota equation in a band of width N in terms of differential forms with 2N coefficient functions $Q_j(u),\ \tilde{Q}_j(u)$
  Solution combines dynamics of gl(N) representations and quantum fusion:

$$Q \equiv Q_{(1)} := \sum_{j=1}^{N} Q_j(u)\xi^j\,, \qquad \{\xi_i,\xi_j\}=0, \qquad \xi_1\wedge\xi_2\wedge\cdots\wedge\xi_N = 1$$

- $l$-form encodes all Q-functions with $l$ indices:

$$Q_{(l)} \equiv D^{-l}\left(DQ_{(1)}D\right)^{\wedge l} D^{-l}\,, \qquad D = e^{\frac{i}{2}\partial_u}$$

$Q_{i_1 i_2 \ldots i_l}$ is coeff. of $\xi_{i_1}\wedge\xi_{i_2}\wedge\ldots\wedge\xi_{i_l}$




- E.g. for gl(2) : $Q_{(2)} = 2\left(Q_1^+Q_2^- - Q_1^-Q_2^+\right)\ \xi_1\wedge\xi_2$

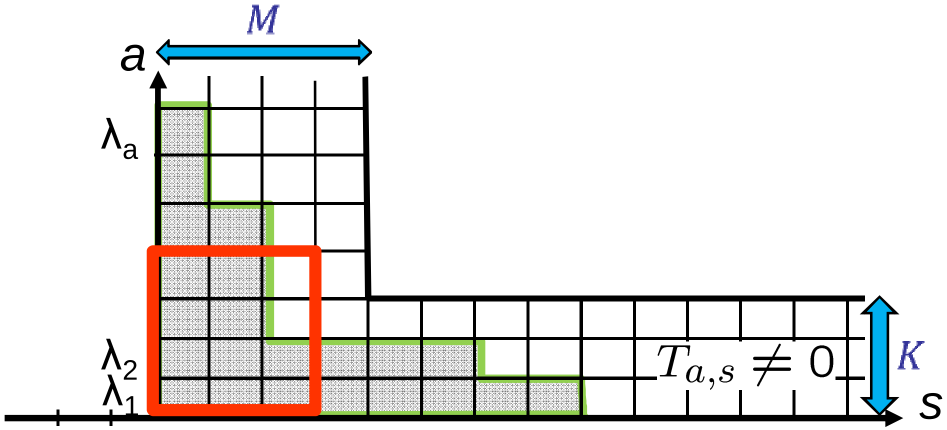
- Solution of Hirota equation in a strip (via arbitrary $Q$- and $\tilde{Q}$-forms):

$$T_{a,s} = Q_{(a)}^{[s]} \wedge \tilde{Q}_{(N-a)}^{[-s]}$$

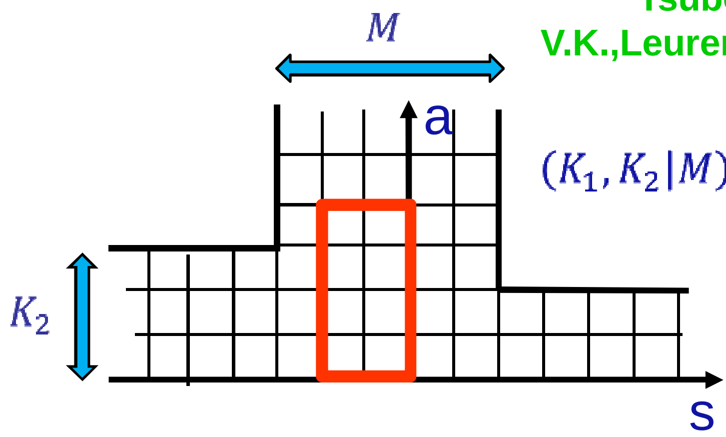
- For su(N) spin chain (half-strip) we impose:

$$\tilde{Q}(u) = Q^{[N]}, \qquad \tilde{Q}_{(0)} = Q_{(0)} = 1$$

# Solution of Hirota in $(K|M)$ fat hook and $(K_1,K_2|M)$ T-hook




Tsuboi
V.K.,Leurent,Volin




- Bosonic and fermionic 1-(sub)forms (all $\xi's$ anticomute):

$$B := Q_{j|\emptyset}\,\xi^j\,, \quad j = 1,\dots K; \qquad F := Q_{\emptyset|\hat{j}}\,\xi^{\hat{j}}\,, \quad \hat{j} = 1,\dots M;$$

- Wronskian solution for the $(K|M)$ fat hook:

$$T_{a,s} = B^{[-M+s]}_{(K-a)} \wedge (B+F)^{[-K-s]}_{(M+a)}, \qquad s-M \geq a-K$$

$$T_{a,s} = F^{[-K+a]}_{(M-s)} \wedge (B+F)^{[-M-a]}_{(K+s)}, \qquad s-M \leq a-K$$

- Similar Wronskian solution exists in $(K_1,K_2|M)$-hook
- We need the hook (2,2|4) for our superconformal group

# Wronskian solution of u(2,2|4) T-system in T-hook

Gromov,V.K.,Tsuboi
Gromov,Tsuboi,V.K.,Leurent
Tsuboi

$$\mathsf{T}_{1,s} = +\mathsf{Q}_1^{[s]}\mathsf{Q}_{\bar 1}^{[-s]} - \mathsf{Q}_2^{[s]}\mathsf{Q}_{\bar 2}^{[-s]} \;,\quad s \geq +1$$

$$\mathsf{T}_{2,s} = +\mathsf{Q}_{12}^{[s]}\mathsf{Q}_{\overline{12}}^{[-s]} \;,\quad s \geq +2$$

$$\mathsf{T}_{a,+2} = +\mathsf{Q}_{12}^{[a]}\mathsf{Q}_{\overline{12}}^{[-a]} \;,\quad a \geq 2$$

$$\mathsf{T}_{a,+1} = (-1)^{a+1}\left(\mathsf{Q}_{12\hat 1}^{[a]}\mathsf{Q}_{\overline{12\hat 1}}^{[-a]} - \mathsf{Q}_{12\hat 2}^{[a]}\mathsf{Q}_{\overline{12\hat 2}}^{[-a]} + \mathsf{Q}_{12\hat 3}^{[a]}\mathsf{Q}_{\overline{12\hat 3}}^{[-a]} - \mathsf{Q}_{12\hat 4}^{[a]}\mathsf{Q}_{\overline{12\hat 4}}^{[-a]}\right) \;,\quad a \geq 1$$

$$\mathsf{T}_{a,0} = +\mathsf{Q}_{12\hat 1\hat 2}^{[a]}\mathsf{Q}_{\overline{12\hat 1\hat 2}}^{[-a]} - \mathsf{Q}_{12\hat 1\hat 3}^{[a]}\mathsf{Q}_{\overline{12\hat 1\hat 3}}^{[-a]} + \mathsf{Q}_{12\hat 1\hat 4}^{[a]}\mathsf{Q}_{\overline{12\hat 1\hat 4}}^{[-a]} + \mathsf{Q}_{12\hat 2\hat 3}^{[a]}\mathsf{Q}_{\overline{12\hat 2\hat 3}}^{[-a]} - \mathsf{Q}_{12\hat 2\hat 4}^{[a]}\mathsf{Q}_{\overline{12\hat 2\hat 4}}^{[-a]} + \mathsf{Q}_{12\hat 3\hat 4}^{[a]}\mathsf{Q}_{\overline{12\hat 3\hat 4}}^{[-a]}$$

$$\mathsf{T}_{a,-1} = (-1)^{a+1}\left(\mathsf{Q}_{43\hat 4}^{[a]}\mathsf{Q}_{\overline{43\hat 4}}^{[-a]} - \mathsf{Q}_{43\hat 3}^{[a]}\mathsf{Q}_{\overline{43\hat 3}}^{[-a]} + \mathsf{Q}_{43\hat 2}^{[a]}\mathsf{Q}_{\overline{43\hat 2}}^{[-a]} - \mathsf{Q}_{43\hat 1}^{[a]}\mathsf{Q}_{\overline{43\hat 1}}^{[-a]}\right) \;,\quad a \geq 1$$

$$\mathsf{T}_{a,-2} = \mathsf{Q}_{43}^{[a]}\mathsf{Q}_{\overline{43}}^{[-a]} \;,\quad a \geq 2$$

$$\mathsf{T}_{2,s} = +\mathsf{Q}_{43}^{[-s]}\mathsf{Q}_{\overline{43}}^{[s]} \;,\quad s \leq -2$$

$$\mathsf{T}_{1,s} = +\mathsf{Q}_4^{[s]}\mathsf{Q}_{\bar 4}^{[-s]} - \mathsf{Q}_3^{[s]}\mathsf{Q}_{\bar 3}^{[-s]} \;,\quad s \leq -1$$

**Plücker relations express all 256 Q-functions through 8 independent ones**

definitions:

$$\mathsf{Q}_{\bar I} \equiv \mathsf{Q}_{\{1,2,3,4,\hat 1,\hat 2,\hat 3,\hat 4\}\setminus I}$$

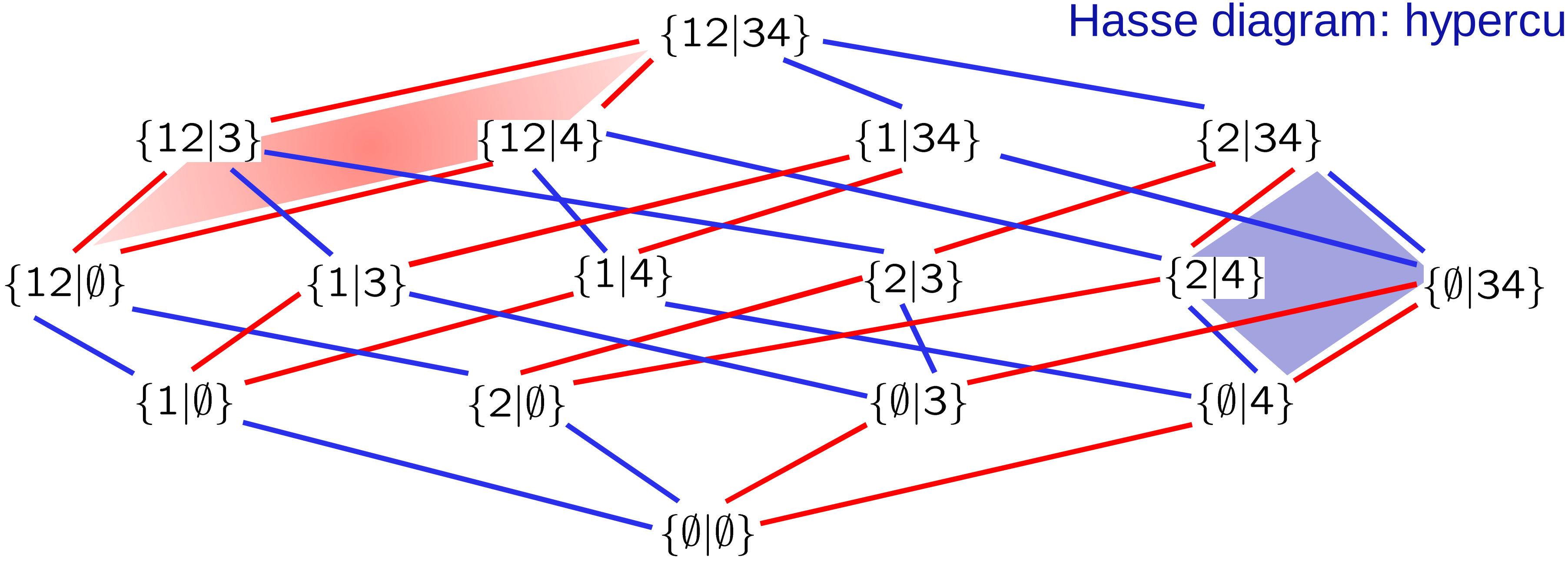
$$\mathsf{Q}^{[k]} \equiv \mathsf{Q}(u + i\,k/2)$$

# Hasse diagram and QQ-relations (Plücker id.)

Tsuboi
V.K.,Sorin,Zabrodin
Tsuboi,Bazhanov

- gl(2|2) example: classification of all Q-functions $I \in \{1,2|3,4\}$

Hasse diagram: hypercub




- E.g. $Q_{12|\emptyset}\, Q_{12|34} = Q^{+}_{12|3}\, Q^{-}_{12|4} - Q^{-}_{12|3}\, Q^{+}_{12|4}$ - bosonic QQ-rel.

$Q_{2|4} Q_{\emptyset|34} = Q^{-}_{\emptyset|4} Q^{+}_{2|34} - Q^{+}_{\emptyset|4} Q^{-}_{2|34}$ - fermionic QQ rel.

- Nested Bethe ansatz equations follow from polynomiality of $Q_I$ along a nesting path

- All Q's expressed through a few basic ones by determinant formulas

Maldacena
Gubser, Polyakov, Klebasnov
Witten

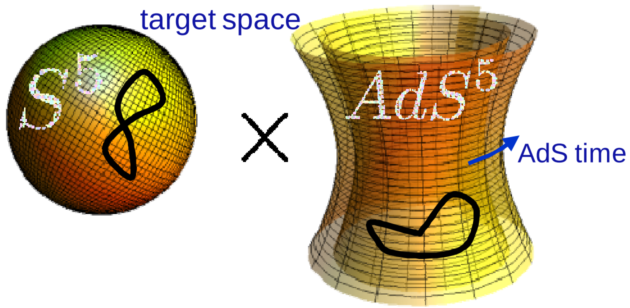
# SYM is dual to supersting σ-model on $AdS_5 \times S^5$

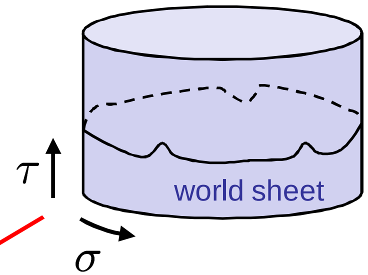
Super-conformal N=4 SYM symmetry PSU(2,2|4) → isometry of string target space

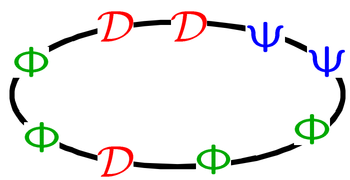
- 2D σ-model on a coset $\frac{PSU(2,2|4)}{SO(1,4)\times SO(5)}$ $\qquad g = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in SU(2,2|4)$

fermions

$$J = -g^{-1}dg = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)} \in su(2,2|4)$$




- Metsaev-Tseytlin action

$$S_{MT} = \sqrt{\lambda}\ \mathrm{str}\int_{\mathcal{M}_2}\left[\left(J^{(2)}\right)^2 - J^{(1)}\wedge J^{(3)}\right]$$

Dimension of YM operator $\Delta_A(\lambda)$ **=** Energy of a string state

# Classical integrability of superstring on $AdS_5 \times S^5$

- String equations of motion and constraints can be recasted into zero curvature condition

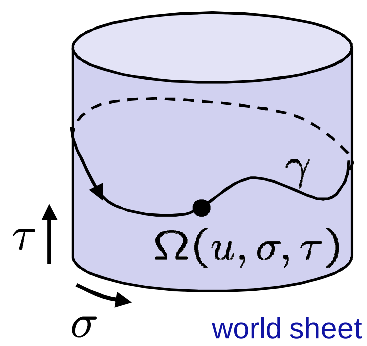
Mikhailov,Zakharov
Bena,Roiban,Polchinski

$$(d+\mathcal{A}(u))\wedge(d+\mathcal{A}(u))=0,$$

for Lax connection - ***double valued*** w.r.t. spectral parameter $u$

$$\mathcal{A}(u)=J^{(0)}+\frac{u}{\sqrt{u^2-4}}J^{(2)}+\frac{1}{\sqrt{u^2-4}}*J^{(2)}+\left(\frac{u+2}{u-2}\right)^{1/4}J^{(1)}+\left(\frac{u-2}{u+2}\right)^{1/4}J^{(3)}$$

- Monodromy matrix $\Omega(u)=P\exp\oint_\Gamma \mathcal{A}(u) \quad \in PSU(2,2|4)$

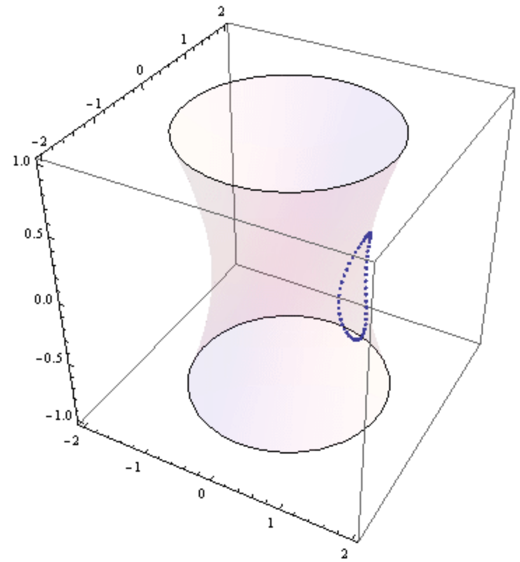



encodes infinitely many conservation lows

- Algebraic curve for quasi-momenta

$$\mathcal{P}(p,u)=\text{sdet}\left(\mathbf{I}-e^{-ip}\cdot\Omega(u)\right)\sim\frac{0}{0}$$

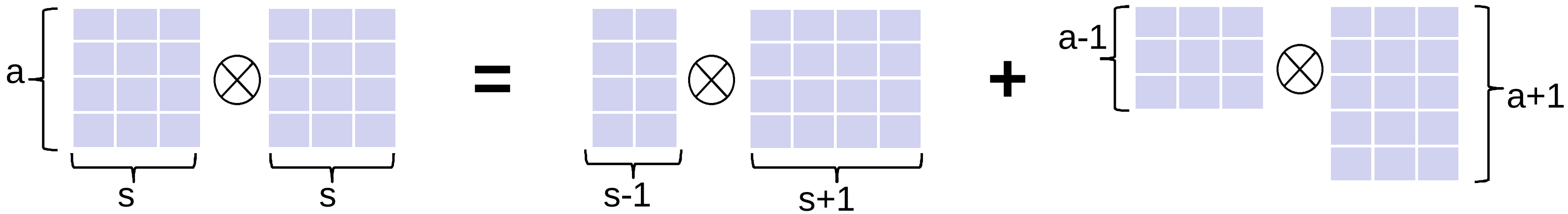
generates finite gap solutions:



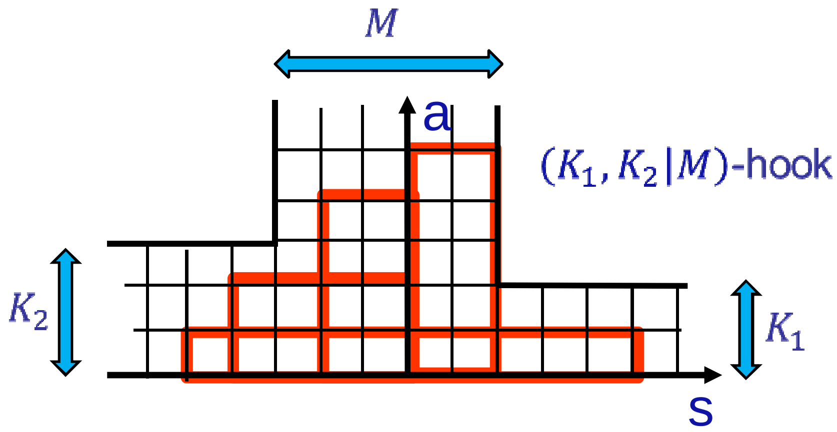
Its,Matweev,Dubrovin,Novikov,Krichever
V.K.,Marshakov,Minahan,Zarembo
Beisert,V.K.,Sakai,Zarembo

# (Super-)group theoretical origins of Y- and T-systems

- A curious property of gl(N|M) representations with rectangular Young tableaux:




- For characters – simplified Hirota eq.: $\chi_{a,s}^2 = \chi_{a+1,s}\chi_{a-1,s} + \chi_{a,s+1}\chi_{a,s-1}$
- Boundary conditions for Hirota eq. for $U(K_1,K_2|M)$T-system (from $\chi$-system): ∞ - dim. unitary highest weight representations of the "T-hook" !




Kwon
Cheng,Lam,Zhang
Gromov, V.K., Tsuboi
Gunaydin, Volin

- Full quantum Hirota equation: extra variable – spectral parameter
- "Classical limit": eq. for characters as functions of classical monodromy

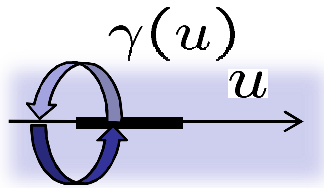
$$T_{a,s}\left(u+\frac{i}{2}\right) T_{a,s}\left(u-\frac{i}{2}\right) = T_{a,s-1}(u)\, T_{a,s+1}(u) + T_{a+1,s}(u) T_{a-1,s}(u)$$

Gromov
Gromov,V.K.,Tsuboi

# Classical $\mathbb{Z}_4$ symmetry

$$\Omega(u^\gamma) = C(\Omega^{ST})^{-1}(u)C^{-1} \quad \text{where} \quad C = \left(\begin{array}{c|c} E & 0 \\ \hline 0 & -iE \end{array}\right), \qquad E = \left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)_{4\times 4}$$

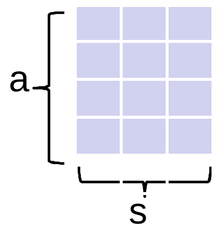
- $\mathbb{Z}_4$ symmetry, together with unimodularity of $\Omega(u)$ induces a monodromy




$$\tilde{p}_{1,2,3,4}(u^\gamma) = -\tilde{p}_{2,1,4,3}(u), \qquad \hat{p}_{1,2,3,4}(u^\gamma) = -\hat{p}_{2,1,4,3}(u)$$

- Trace of classical monodromy matrix is a psu(2,2|4) character. We take it in irreps for $a \times s$ rectangular Young tableaux: Gromov,V.K.,Tsuboi

$$T_{a,s}(u) = \mathrm{Tr}_{a,s}\,\Omega(u)$$




- $\mathbb{Z}_4$ symmetry:

$$T_{a,s}(u) = (-1)^s T^c_{a,-s}(u^\gamma), \qquad \text{if } |s| \geq a$$
$$T_{a,s}(u) = (-1)^a T^c_{-a,s}(u^\gamma), \qquad \text{if } a \geq |s|$$

# Global Charges

- Conserved charges: angular momenta, spins $J_1, J_2, J_3, S_1, S_2$

  and energy *E*: defined from the asymptotics at x=0,∞.

$$\mathbf{S^5}: \quad \tilde{p}_1(x) = -\frac{2\pi}{\sqrt{\lambda}}(J_1 + J_2 - J_3)\frac{1}{x} + \ldots, \quad etc.$$

$$\mathbf{AdS^5}: \quad \hat{p}_1(x) = \frac{2\pi}{\sqrt{\lambda}}(E + S_1 - S_2)\frac{1}{x} + \ldots, \quad \text{etc.}$$

- Energy E of state → dimension Δ of operator in SYM.

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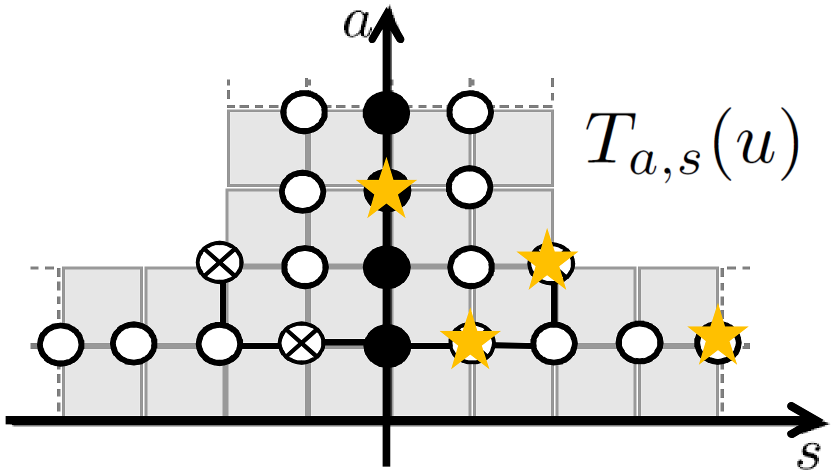
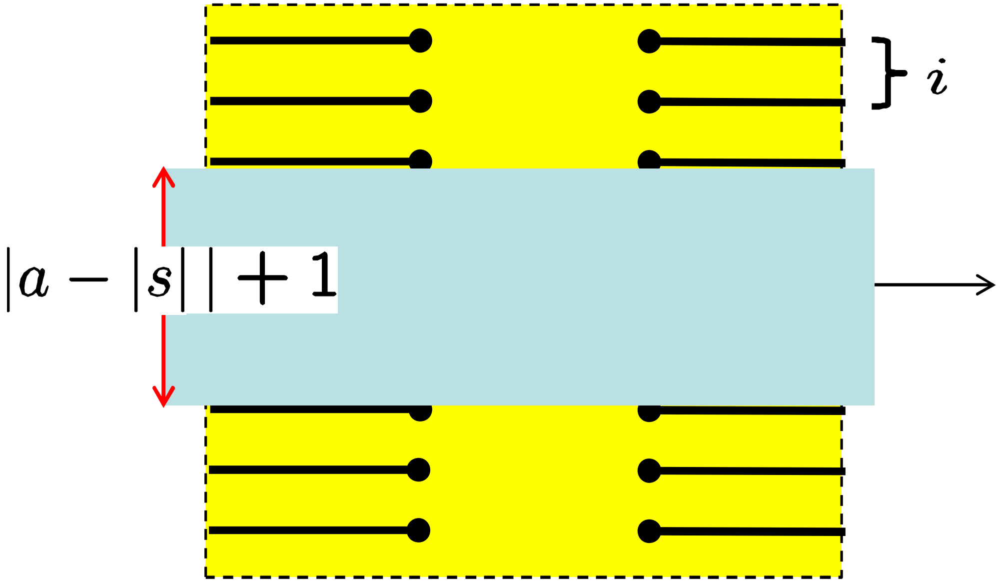
# Full quantum case: spectral AdS/CFT Y-system

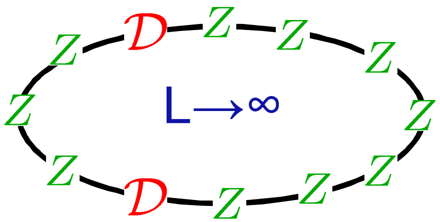
$PSU(2,2|4)$ T-hook




cuts in complex $u$-plane



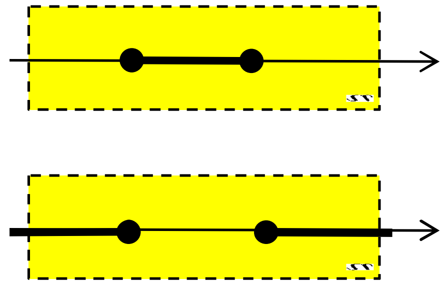

- Analyticity from large L or u asymptotics via one-particle dispersion relation:




$$\Delta_a^2 = a^2 + \lambda \sin^2 \frac{p_a}{2}$$

$$Y_{a,0}(u) \simeq e^{iLp_a(u)} = \left[\frac{z\left(u-\frac{ia}{2}\right)}{z\left(u+\frac{ia}{2}\right)}\right]^L$$

cuts in u -plane

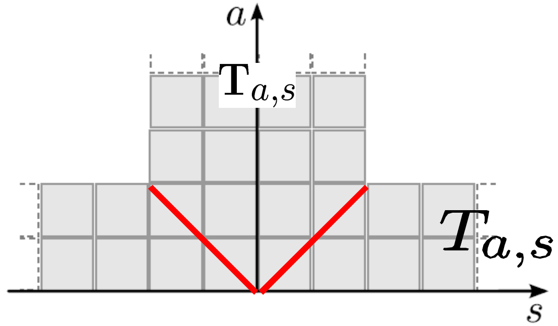


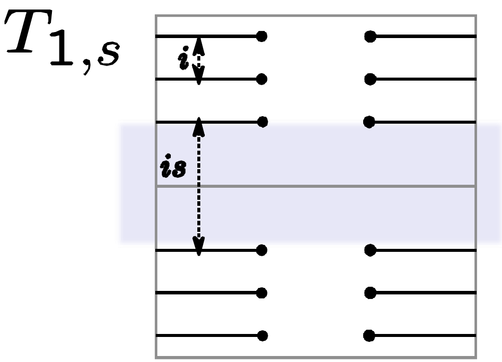
Zhukovsky map: $u = \sqrt{\lambda}\left(z+\frac{1}{z}\right) \Longleftrightarrow z = \frac{1}{2\sqrt{\lambda}}\left(u + i\sqrt{4\lambda - u^2}\right)$

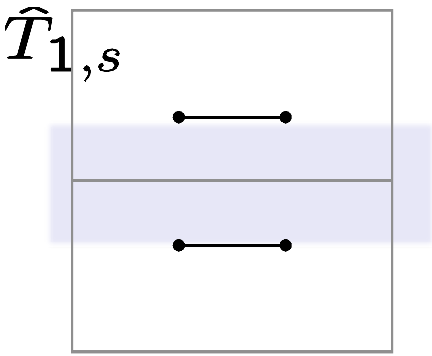
- Extra "corner" equations: $Y_{2,\pm 2}(u+i0) \cdot Y_{1,\pm 1}(u-i0) = 1$

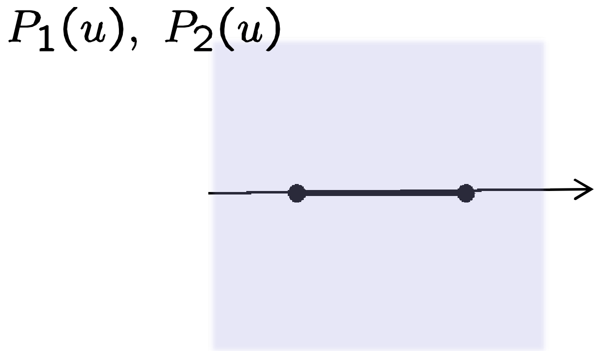
# Magic sheet and solution for the right band

- The property $Y_{2,\pm 2}(u+i0)\cdot Y_{1,\pm 1}(u-i0)=1$ suggests that certain T-functions are much simpler on the physical sheet, with only short cuts:




- Wronskian solution for the right band in terms of two Q-functions with one magic cut on $\mathbb{R}$

$$\widehat{T}_{1,s}(u) = P_1(u+\frac{is}{2})P_2(u-\frac{is}{2}) - P_1(u-\frac{is}{2})P_2(u+\frac{is}{2})$$

- $Z_4$ symmetry is satisfied authomatically!

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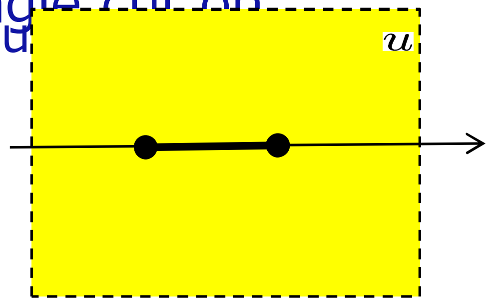
# Basic Q-functions and their asymptotics $u \to \infty$

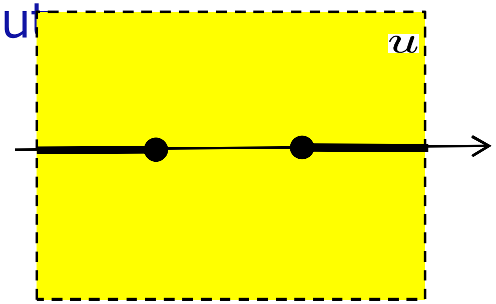
- Consider SL(2) operators $\mathrm{Tr}(Z\nabla^S Z^{L-1})$ (general case in work)
- We found two 4-vectors of Q-functions with a single cut on defining sheet:

$P_j = Q_{j|0}, \quad j = 1,2,3,4$ with one short, "physical" cut




and asymptotics $P = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} \sim \begin{pmatrix} u^{-\frac{L}{2}} \\ u^{-\frac{L+2}{2}} \\ u^{\frac{L}{2}} \\ u^{\frac{L-2}{2}} \end{pmatrix}$

$P_j = Q_{j|0}, \quad j = 1,2,3,4$ with one long, "mirror" cut




and asymptotics $Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} \sim \begin{pmatrix} u^{\frac{L+E-S}{2}} \\ u^{\frac{L+E+S-2}{2}} \\ u^{-\frac{L+E+S}{2}} \\ u^{-\frac{L+E-S+2}{2}} \end{pmatrix}$

Asymptotics are found by comparing to the classical and one loop approximat

- All other Q-functions can be found through these basic ones. But what about other sheets?
- We propose $P-\mu$ Riemann-Hilbert eqs to continue through the cuts

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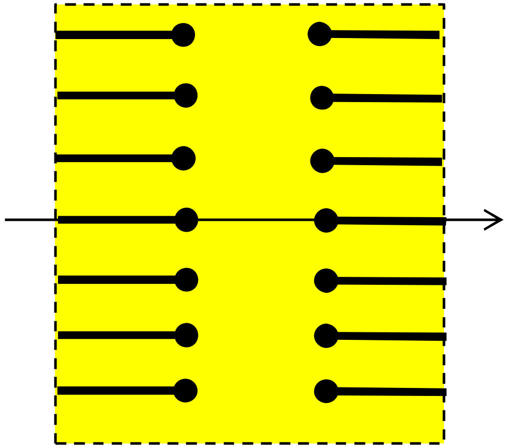
# μ-matrix



- We need another object representing certain Q-functions:
  - 4×4 matrix $\mu = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & d \\ -b & -c & 0 & e \\ -c & -d & -e & 0 \end{pmatrix}$

$\mu$ is $i$ -periodic in mirror, constrained by $\mu \cdot \mu = \mathbb{I}$

where we defined a scalar product: $(A \cdot B) \equiv A_1 B_4 - A_4 B_1 - A_2 B_3 + A_3 B_2$

- As we will see from the $P-\mu$ equations, it has the asymptotic $u \to \infty$

$$\mu = \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} \sim \begin{pmatrix} u^{\Lambda - L} \\ u^{\Lambda+1} \\ u^{\Lambda} \\ u^{\Lambda-1} \\ u^{\Lambda+L} \end{pmatrix}, \qquad \Lambda = 0,\ \pm\Delta,\ \pm(S-1)$$

- $i$ –periodic $\mu$ represents a set of Wronskians of underlying Baxter equations

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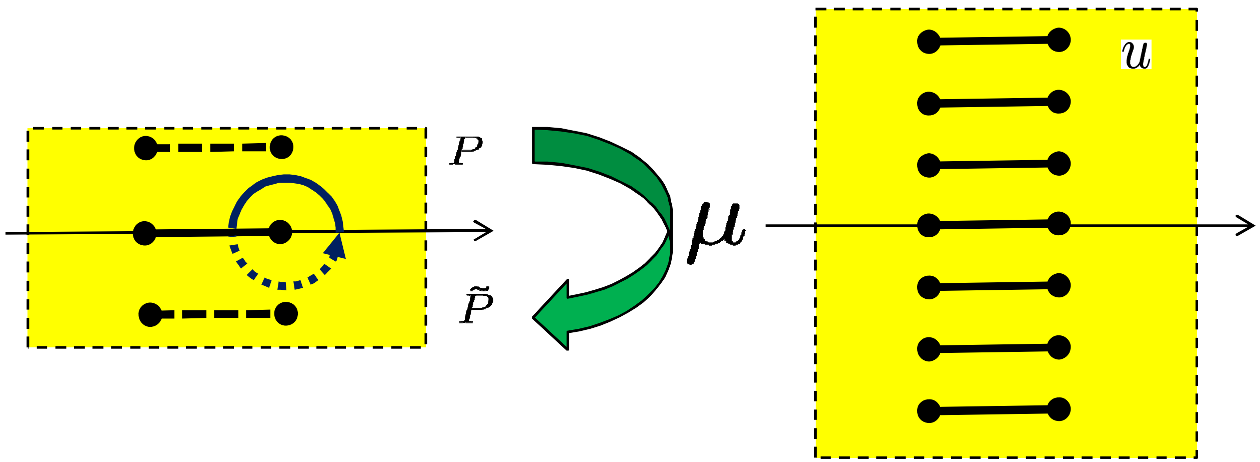
# AdS/CFT spectrum as a Riemann-Hilbert problem

- Using Hirota integrability, some symmetry and analyticity we br
  the spectral AdS/CFT problem to a finite number of non-liner int
  equations (FiNLIE), or, equivalently, to a Riemann-Hilbert proble

- AdS/CFT spectral problem reduces to Riemann-Hilbert-type relati
  on 4-vector $P(u)$ and 4×4 matrix $\mu(u)$

$$\tilde{P} + \mu \cdot P = 0$$




- Asymptotics of $\mu$ established can be established
  by knowing (from TBA) that $\mu_{1,2} \sim u^{\Delta-L}$
  and comparing various terms in $P-\mu$ equation.

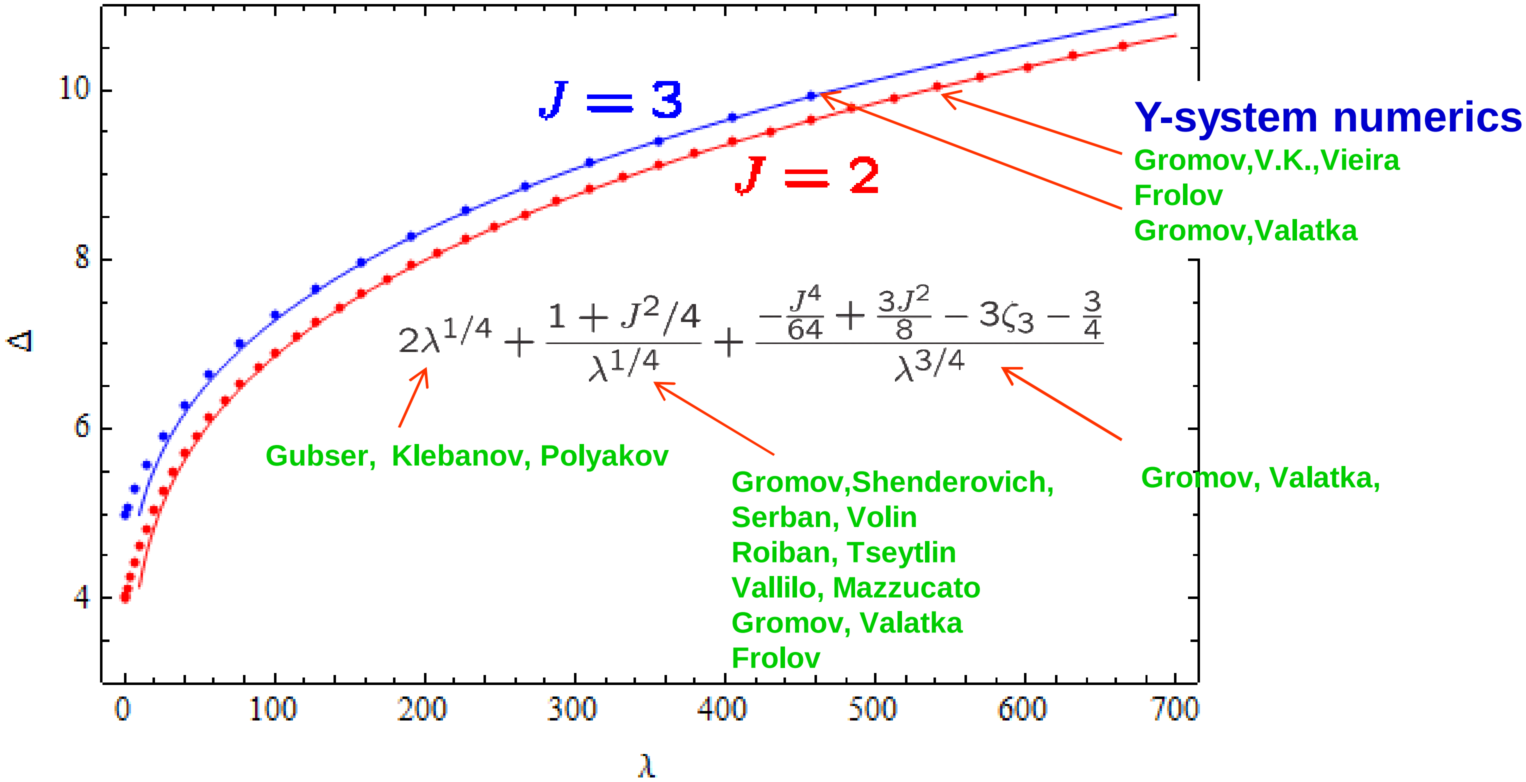
# AdS string quasiclassics and numerics in SL(2) sector: twist-J operators of spin S $\mathrm{Tr}\,\mathcal{D}^S Z^J$

- 3 leading strong coupling terms were calculated for any S and L
- Numerics at any coupling:

  - for Konishi operator $S=2, \quad J=2, \quad n=1$

  - and twist-3 operator $S=2, \quad J=3, \quad n=1$

  They perfectly reproduce the TBA/Y-system or FiNLIE numerics




- AdS/CFT Y-system passes all known tests!

# Conclusions

- We proposed a system of matrix Riemann-Hilbert equations for the exact spectrum of anomalous dimensions of planar N=4 SYM theory in 4D.
- Great possibilities for precise calculations of spectrum, numerically and analytically
- This P-μ system defines the full quantum spectral curve of $AdS_5 \times S^5$ duality
- Works for Wilson loops and quark-antiquark potential in N=4 SYM

Correa, Maldacena, Sever
Drucker
Gromov, Sever

## Future directions

- Why is N=4 SYM integrable?
- What lessons for less supersymmetric SYM and QCD?
- 1/N – expansion integrable?
- Gluon amlitudes, correlators, Wilson loops …integrable?
- BFKL (Regge limit) from P-μ -system?

END