

# Effective action for the Regge processes and double logarithmic asymptotics in gravity

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# 1 Pomeron in hadron interactions

High energy kinematics

$$s = 4E^2 \gg (-t) = \vec{q}^2 \sim m^2, \theta \ll 1$$

Froissart and Pomeranchuk theorems

$$\sigma_t = \frac{1}{s} \Im A(s, 0) < c \ln^2 s, \sigma_t^{pp} = \sigma_t^{p\bar{p}}$$

Pomeron contribution at high energies

$$A_P(s, t) \approx is \gamma^2(t) s^{\omega(t)}, \omega(t) \approx \Delta + \alpha' t, \sigma_t \sim s^\Delta$$

Mandelstam cut contribution

$$A_M(s, t) = -is \int \frac{d^2k}{(2\pi)^2} \Phi^2(k, q-k) s^{\omega(-k^2)} s^{\omega(-(q-k)^2)}$$

## 2 Gribov Pomeron calculus

Ordering of particle clusters in rapidities

$$y_r = \frac{1}{2} \ln \frac{\sqrt{|k|^2 + m_r^2} + |k|}{\sqrt{|k|^2 + m_r^2} - |k|}, \quad 1 \ll y_r - y_{r-1} \ll \ln s$$

Non-relativistic Pomeron propagator

$$G_0 = \frac{1}{E + \Delta - \frac{|k|^2}{2m}}, \quad E = -\omega, \quad \alpha' = \frac{1}{2m}$$

Gribov effective action for Pomeron interactions

$$S = \int dy d^2\rho \left( \phi^* \left( \partial_y - \frac{\vec{\partial}^2}{2m} - \Delta \right) \phi + i\lambda(\phi^* \phi^2 + \phi \phi^{*2}) + \dots \right)$$

Weak coupling solution and universality

$$\Delta = 0, \quad \gamma_{ii}(0) = \text{const}, \quad \gamma_{ir}(0) = 0$$

### 3 Gluon reggeization in QCD

Born amplitude at high energies  $s \gg t$

$$M_{AB}^{A'B'}|_{Born} = 2s g T_{A'A}^c \delta_{\lambda_{A'}\lambda_A} \frac{1}{t} g T_{B'B}^c \delta_{\lambda_{B'}\lambda_B}$$

Leading Logarithmic Approximation (LLA)

$$M(s, t) = M|_{Born} s^{\omega(t)}, \quad \alpha_s \ln s \sim 1, \quad \alpha_s = \frac{g^2}{4\pi} \ll 1$$

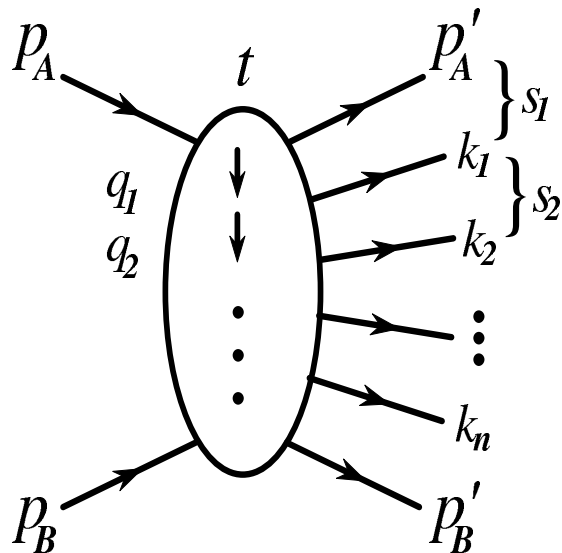
Gluon trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2} \int d^2k \frac{|q|^2}{|k|^2 |q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi} \ln \frac{|q|^2}{\lambda^2}$$

Bootstrap equation

$$\omega(-|q|^2) f = 1 + \left( \omega(-|k|^2) + \omega(-|q-k|^2) + \hat{K}_8 \right) f$$

## 4 Gluon production amplitude in LLA



$$M_{2 \rightarrow 2+n}^{BFKL} \sim \frac{s_1^{\omega_1}}{|q_1|^2} g T_{c_2 c_1}^{d_1} C(q_2, q_1) \frac{s_2^{\omega_2}}{|q_2|^2} \dots g T_{c_{n+1} c_n}^{d_n} C(q_{n+1}, q_n) \frac{s_{n+1}^{\omega_{n+1}}}{|q_{n+1}|^2},$$

$$C(q_2, q_1) = -q_1^\perp - q_2^\perp + p_A \left( \frac{q_1^2}{k p_A} + \frac{k p_B}{p_A p_B} \right) - p_B \left( \frac{q_2^2}{k p_B} + \frac{k p_A}{p_A p_B} \right)$$

## 5 BFKL Pomeron

Balitsky-Fadin-Kuraev-Lipatov equation (1975)

$$E \Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12} \Psi(\vec{\rho}_1, \vec{\rho}_2), \quad \sigma_t \sim s^\Delta, \quad \Delta = -\frac{\alpha_s N_c}{2\pi} E_0$$

Hamiltonian for the Pomeron wave function

$$H_{12} = h_{12} + h_{12}^*, \quad \rho_{12} = \rho_1 - \rho_2, \quad \rho_r = x_r + iy_r,$$

$$h_{12} = \frac{1}{p_1} \ln(\rho_{12}) p_1 + \frac{1}{p_2} \ln(\rho_{12}) p_2 + \ln(p_1 p_2) - 2\psi(1)$$

Möbius invariance and Pomeron intercept

$$\rho_k \rightarrow \frac{a\rho_k + b}{c\rho_k + d}, \quad \Psi = \left( \frac{\rho_{12}}{\rho_{10}\rho_{20}} \right)^m \left( \frac{\rho_{12}^*}{\rho_{10}^*\rho_{20}^*} \right)^{\tilde{m}}, \quad E = \epsilon_m + \epsilon_{\tilde{m}},$$

$$m = \frac{1+n}{2} + i\nu, \quad \epsilon_m = \psi(m) + \psi(1-m) - 2\psi(1), \quad \Delta = \frac{g^2 N_c}{\pi^2} \ln 2 > 0$$

## 6 BKP equation in multi-color QCD

Bartels-Kwiecinski-Praszalowicz equation (1979)

$$E \Psi = H \Psi, \quad H = \sum_{k < l} \frac{\vec{T}_k \vec{T}_l}{-N_c} H_{kl}, \quad H = h + h^*, \quad \Delta = -\frac{\alpha N_c}{2\pi} E_0$$

Holomorphic factorization at large  $N_c$  (L. (1988))

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, \dots, \vec{\rho}_n) = \sum_{r,s} a_{r,s} \Psi_r(\rho_1, \dots, \rho_n) \Psi_s(\rho_1^*, \dots, \rho_n^*)$$

Integrability and duality symmetry (L. (1993, 1999))

$$t(u) = L_1 L_2 \dots L_n = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \quad L_k = \begin{pmatrix} u + \rho_k p_k & p_k \\ -\rho_k^2 p_k & u - \rho_k p_k \end{pmatrix},$$

$$T(u) = A(u) + D(u), \quad [T(u), h] = 0, \quad p_k \rightarrow \rho_{k,k+1} \rightarrow p_{k+1}$$

# 7 Effective action for reggeized gluons

Locality in the rapidity space

$$y = \frac{1}{2} \ln \frac{\epsilon_k + |k|}{\epsilon_k - |k|}, \quad |y - y_0| < \eta, \quad \eta \ll \ln s$$

Gluon and reggeized gluon fields

$$v_\mu(x) = -iT^a v_\mu^a(x), \quad A_\pm(x) = -iT^a A_\pm^a(x), \quad \partial_\mp A_\pm(x) = 0$$

Effective action for the reggeon interactions (L., 1995)

$$S = \int d^4x (L_{QCD} + Tr(V_+ \partial_\mu^2 A_- + V_- \partial_\mu^2 A_+)) ,$$

$$V_+ = -\frac{\partial_+}{g} P \exp \left( -g \int_{-\infty}^{x^+} v_+(x') d(x')^+ \right) = v_+ - gv_+ \frac{1}{\partial_+} v_+ + \dots$$



## 8 BFKL equation in $N = 4$ SUSY

BFKL kernel in two loops (F., L. and C.,C. (1998))

$$\omega = 4\hat{a} \chi(n, \gamma) + 4\hat{a}^2 \Delta(n, \gamma), \quad \hat{a} = g^2 N_c / (16\pi^2),$$

Hermitian separability in  $N = 4$  SUSY (K.,L. (2000))

$$\Delta(n, \gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega}, \quad M = \gamma + \frac{|n|}{2},$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2), \quad \beta'(z) = \frac{1}{4} \left[ \Psi'\left(\frac{z+1}{2}\right) - \Psi'\left(\frac{z}{2}\right) \right]$$

Maximal transcendentality (K.,L. 2002)

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M) \left( \Psi(1) - \Psi(M) \right),$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M} \left( \Psi'(k+1) - \frac{\Psi(k+1) - \Psi(1)}{k+M} \right)$$

## 9 Twist-2 anomalous dimension $\gamma$

Maximal transcendentality of  $\gamma$ -expansion (K.,L.)

$$\gamma(\omega) = \sum_{r=1}^{\infty} \gamma_r(\omega) (4\hat{a})^r$$

Pole contributions at small  $\omega$  from BFKL (KLOV)

$$\gamma_1 = \frac{1}{\omega}, \quad \gamma_2 = 0, \quad \gamma_3 = \frac{\zeta_3}{2\omega^2}, \quad \gamma_4 = \frac{16\zeta_3 + 5\zeta_4\omega}{8\omega^4}, \quad \gamma_5 = \frac{2\zeta_2\zeta_3 + 16\zeta_5}{2\omega^4}$$

Convergency radius for  $\gamma$  in  $N = 4$  SUSY at  $N_c \rightarrow \infty$

$$\gamma(\omega) = \sum_{k=1}^{\infty} \lambda^k c_k(\omega), \quad \lambda = g^2 N_c, \quad \lim_{k \rightarrow \infty} c_k(\omega) = \frac{\lambda_{cr}^{-k}}{k^{3/2}} \frac{1}{2\sqrt{\pi}} a,$$

$$\omega = \omega_0(\lambda_{cr}) = 1 - \Delta(\lambda_{cr}), \quad a = \sqrt{\frac{\lambda_{cr} \omega'_0(\lambda_{cr})}{D(\lambda_{cr})}}$$

# 10 Pomeron and graviton in N=4 SUSY

Eigenvalue of the BFKL kernel in a diffusion approximation

$$j = 2 - \Delta - D\nu^2, \quad \gamma = 1 + \frac{j-2}{2} + i\nu$$

AdS/CFT relation for the graviton Regge trajectory

$$j = 2 + \frac{\alpha'}{2} t, \quad t = E^2/R^2, \quad \alpha' = \frac{R^2}{2} \Delta$$

Pomeron intercept at large  $\lambda = g^2 N_c$  (KLOV, BPST, CGP, KL)

$$j_0 = 2 - \Delta, \quad \Delta = 2\lambda^{-1/2} + \lambda^{-1} - 1/4 \lambda^{-3/2} - 2(1+3\zeta_3)\lambda^{-2} + O(\lambda^{-5/2})$$

Slope of  $\gamma$  at  $j = 2$  (KLOV, Basso)

$$\gamma'(2) = -\frac{\lambda}{24} + \frac{1}{2} \frac{\lambda^2}{24^2} - \frac{2}{5} \frac{\lambda^3}{24^2} + \frac{7}{20} \frac{\lambda^4}{24^4} - \frac{11}{35} \frac{\lambda^5}{24^5} + \dots = -\frac{\sqrt{\lambda}}{4} \frac{I_3(\sqrt{\lambda})}{I_2(\sqrt{\lambda})}$$

# 11 Reggeized graviton effective theory

Graviton and reggeized graviton fields

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \partial_+ A^{++}(x) = \partial_- A^{--}(x) = 0$$

Effective action for the reggeon interactions (L. 2011)

$$S = -\frac{1}{2\kappa} \int d^4x \left( \sqrt{-g} R + \frac{1}{2} (\partial_+ j^- \partial_\mu^2 A^{++} + \partial_- j^+ \partial_\mu^2 A^{--}) \right)$$

Hamilton-Jacobi equation for  $j^\pm = 2x^\pm - \omega^\pm$

$$g^{\mu\nu} \partial_\mu \omega^\pm \partial_\nu \omega^\pm = 0, \quad \partial_\pm j^\mp = h_{\pm\pm} - \left( h_{\rho\pm} - \frac{1}{2} \frac{\partial_\rho}{\partial_\pm} h_{\pm\pm} \right)^2 + \dots$$

Hamilton-Jacobi functional

$$\omega = \min_{x^\nu, p_\mu} \int_{-\infty}^{\tau} d\tau L, \quad L = p_\mu x^\mu - \frac{e(\tau)}{2} g^{\mu\nu} p_\mu p_\nu$$

# 12 High energy amplitudes in gravity

Production amplitudes in LLA (L.L. (1982))

$$A_{2 \rightarrow n} = -s^2 \Gamma_{\mu\nu}^{\mu'\nu'} \frac{s_1^{\omega(q_1^2)}}{q_1^2} \Gamma_{\rho_1\sigma_1} \frac{s_2^{\omega(q_2^2)}}{q_2^2} \Gamma_{\rho_2\sigma_2} \dots \Gamma_{\rho\sigma}^{\rho'\sigma'}$$

Graviton-graviton-reggeon vertex

$$\Gamma_{\mu\nu}^{\mu'\nu'} = \frac{\kappa}{4} (\Gamma_{\mu\mu'} \Gamma_{\nu\nu'} + \Gamma_{\mu\nu'} \Gamma_{\nu\mu'})$$

Gluon-gluon-reggeized gluon vertex

$$\Gamma_{\mu\mu'} = -\delta_{\mu\mu'} + \frac{p_{\mu'}^A p_{\mu}^B + p_{\mu}^{A'} p_{\mu'}^B}{p^A p^B} + \frac{q^2}{2} \frac{p_{\mu}^B p_{\mu'}^B}{(p^A p^B)^2}$$

Reggeon-reggeon-graviton vertex

$$\Gamma_{\rho\sigma} = \frac{\kappa}{4} (C_{\rho} C_{\sigma} - N_{\rho} N_{\sigma}), \quad N = \sqrt{q_1^2 q_2^2} \left( \frac{p^A}{k p^A} - \frac{p^B}{k p^B} \right)$$

# 13 Graviton trajectory in supergravity

Graviton Regge trajectory (L. (1982))

$$\omega(q^2) = \frac{\alpha}{\pi} \int \frac{q^2 d^2k}{k^2(q-k)^2} f(k, q), \quad \alpha = \frac{\kappa^2}{8\pi^2},$$

$$f(k, q) = (k, q-k)^2 \left( \frac{1}{k^2} + \frac{1}{(q-k)^2} \right) - q^2 + \frac{N}{2}(k, q-k)$$

Gravitino action

$$S_{3/2} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \int d^4x \sum_{r=1}^N \bar{\psi}_\mu^r \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma^r$$

Divergencies of the graviton Regge trajectory

$$\omega(q^2) = -\alpha |q|^2 \left( \ln \frac{|q|^2}{\lambda^2} + \frac{N-4}{2} \ln \frac{|\Lambda|^2}{|q|^2} \right)$$

# 14 Double-logarithms in (super) gravity

Mellin representation for the scattering amplitude

$$A(s, t) = A_{Born} s^{-\alpha|q|^2 \ln \frac{|q|^2}{\lambda^2}} \Phi(\xi), \quad \Phi(\xi) = \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i \omega} \left( \frac{s}{|q|^2} \right)^\omega f_\omega$$

Infrared evolution equation for supergravity (BLS (2012))

$$f_\omega = 1 + b \frac{d}{d\omega} \frac{f_\omega}{\omega} - b \frac{N-6}{2} \frac{f_\omega^2}{\omega^2}, \quad b = \alpha|q|^2, \quad \alpha = \frac{\kappa^2}{8\pi^2}, \quad \xi = \alpha|q|^2 \ln^2 \frac{s}{|q|^2}$$

Perturbative expansion

$$\Phi(\xi) = 1 - \frac{N-4}{2} \frac{\xi}{2} + \frac{(N-4)(N-3)}{2} \frac{\xi^2}{4!} - \frac{N-4}{8} (5N^2 - 26N + 36) \frac{\xi^3}{6!} + \dots$$

Solution in terms of the parabolic cylinder function

$$\frac{f_\omega^{(N)}}{\omega} = \frac{2}{6-N} \frac{1}{\sqrt{b}} \frac{d}{dx} \ln d^{(N)}(x), \quad d^{(N)}(x) = e^{\frac{x^2}{4}} D_{\frac{6-N}{2}}(x), \quad x = \frac{\omega}{\sqrt{b}}$$

# 15 Double-logarithmic eikonal phase

Scattering amplitude in the eikonal picture

$$A_{DL}(s, t) = -2is s^{-\alpha|q|^2 \ln \frac{|q|^2}{\mu^2}} \int d^2\rho e^{i\vec{q}\vec{\rho}} \left( e^{i\delta_{DL}(\vec{\rho}, \ln s)} - 1 \right),$$

Double logarithmic approximation for the phase

$$\delta_{DL}(\vec{\rho}, \ln s) = \frac{s}{2} \frac{\kappa^2}{(2\pi)^2} \int \frac{d^2q}{|q|^2} e^{-i\vec{q}\vec{\rho}} \Phi(\xi)$$

Eikonal phase in  $N = 8$  SUSY at small impact parameters

$$\delta_{DL}^{N=8}(\vec{\rho}, \ln s) = \frac{s}{2} \frac{\kappa^2}{(2\pi)^2} \pi \ln \frac{1}{\alpha\lambda^2 \ln^2(\rho^2 s)}, \quad \rho^2 \ll \alpha \ln^2 s$$

Two loop result in an agreement with exact calculations

$$A_4^{N=8} = \frac{\kappa^2 s^2}{|q|^2} (-i\pi s) \alpha^2 |q|^2 \frac{\ln^3 \frac{s}{|q|^2}}{3}$$



# 16 Discussion

1. Locality in rapidity and Gribov effective action.
2. BFKL Pomeron in QCD.
3. Integrability of the BKP equation.
4. Effective action for high energy QCD.
5. Pomeron and graviton in Maldacena model.
6. Effective action for high energy gravity.
7. Graviton trajectory and effective vertices.
8. Double logarithmic amplitudes in (super) gravity.
9. Eikonal amplitude with double logarithmic phase.