Formal semiclassical field theories

Note Title 5/28/2013

(with A. Cattaneo, P. Mnev)

The formal semiclassical path integral:

$$Z_{M,a} = \int_{\alpha \in T_a F_M} e^{\frac{i}{\hbar} S_M(a+\alpha)} D\alpha$$

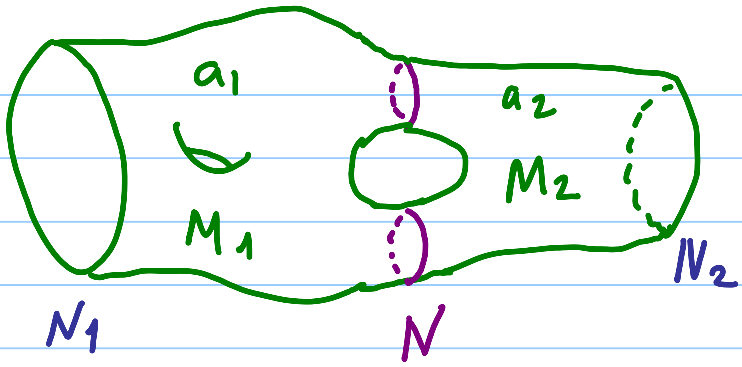
- path "integral" over fluctuations near the classical background (scale of α $\propto \sqrt{\hbar}$ renormaliz.?)

M is compact with $\partial M \neq \emptyset$

boundary conditions for fluctuations

The idea:

- define $Z_{M,a}$ as a formal power series with fixed b.c.
- prove locality (gluing prop. of $Z_{M,a}$)



$$\int_{\substack{\text{b.c. at} \\ N}} Z_{a_1, M_1} \; Z_{a_2, M_2} \, db =$$

$$= Z_{M_1 \cup_N M_2 ,\, a_{12}}$$

↑ dep. on b.c. at N_1, N_2

a_{12} – background on $M_1 \cup_N M_2$

a_1 – the restriction of a_{12} to M_1

a_2 – ——— || ——— to M_2

Boundary conditions and the Hamiltonian structure in 1st order classical field theories

(a) • M - space time , F_M - fields on M , $\partial M \hookrightarrow M \rightsquigarrow$ the projection $F_M \to F_{\partial M}$ (restr. to ∂M)

• $S_M : F_M \to \mathbb{R}$ the action linear in 1st order derivatives of fields

$$\delta S_M = \underbrace{\int_M (\text{EL-equation})\, \delta\varphi}_{\text{bulk}} + \int_{\partial M} \alpha(\varphi)\, \delta\varphi$$

bulk ↙ Euler-Lagrange equations and the set of all their solutions EL_M

↓ 1-form $\alpha_{\partial M}$ on $F_{\partial M}$

This produces:

(i) 2-form $\omega_{\partial M} = \delta \alpha_{\partial M}$, closed, in our examples nondegenerate

$(F_{\partial M}, \omega_{\partial M})$ is symplectic

(ii) $\pi(EL_M) = L_M \subset F_{\partial M}$

isotropic, in our examples

Lagrangian

(iii) $C_{\partial M} \subset F_{\partial M}$ boundary values

of solutions to EL equations

in an ε-nbd of the boundary

$C_{\partial M}$ - coisotropic

$$\boxed{\begin{array}{ccccccc} L_M = \pi(EL_M) & \subset & C_{\partial M} & \subset & F_{\partial M} \\ \text{Lagrangian} & & \text{coisotropic} & & \text{symplectic} \end{array}}$$

(iv) $\delta S \big|_{EL_M} = \pi^*(\alpha_{\partial M})\big|_{L_M}$

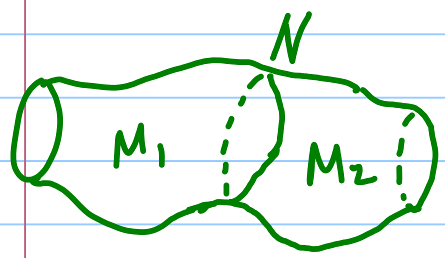
Hamilton-Jacoby equations

(B) • Boundary condition (variational)

- $\varphi|_{\partial M} \in L$, Lagrangian
- $\alpha_{\partial M}|_L = 0$, L is $\alpha_{\partial M}$-exact

then $\delta S|_{EL_M + \text{b.c.}(L)} = 0$

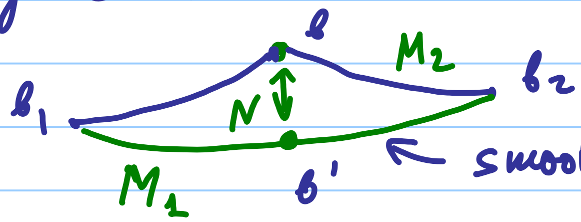
(c) Boundary conditions and locality



Locality:

- $F_{M_1 \cup_N M_2} = F_{M_1} \times_{F_N} F_{M_2}$
- $S_{M_1} + S_{M_2} = S_{M_1 \cup_N M_2}$

We need a family of b.c. to glue solutions to EL equations



← smooth gluing

- <u>Family of boundary conditions</u> (type of boundary value problem)

$$\begin{array}{c} F_{\partial M} \\ \downarrow \\ B_{\partial M} \end{array} \quad \leftarrow \text{ fibers are } \alpha_{\partial M}\text{-exact Lagrangian subman transversal to } L_M$$

Hamiltonian mechanics:

- $M = [t_1, t_2]$,
- $F_M = \{\gamma : [t_1, t_2] \to T^*\mathcal{M}_n\}$,
- $$S_M[\gamma] = \int_\gamma p\,dq - \int_{t_1}^{t_2} H(\gamma(t))\,dt$$

  $H(p,q)$ = the Hamiltonian of the system
- $$EL_M = \left\{ \dot{p} = -\frac{\partial H}{\partial q},\ \dot{q} = \frac{\partial H}{\partial p} \right\}$$
- $$F_{\partial M} = T^*\mathcal{M}^{(1)} \times T^*\mathcal{M}^{(2)}$$

  $$\alpha_{\partial M} = p_1\,dq_1 - p_2\,dq_2$$

  $$\omega_{\partial M} = dp_1 \wedge dq_1 - dp_2 \wedge dq_2$$
- $$C_{\partial M} = T^*\mathcal{M} \times T^*\mathcal{M}$$

  Cauchy data near t_1 and near t_2

- $L_M = \{ ((p_1, q_1), (p_2, q_2)) \in T^*\mathcal{M} \times T^*\mathcal{M} \mid$
  q_1 and q_2 are connected by $\{\gamma(t)\}_{t_1}^{t_2}$
  $\gamma(t)$ solves EL equations $\}$

- Dirichet boundary conditions:

$T^*\mathcal{M}$ fiber over q: $T_q^*\mathcal{M} = \{p\}$

$\downarrow$ $\quad \{\gamma(t)\}_{t_1}^{t_2} \in EL$, $\quad \gamma(t_1) = q_1$, $\gamma(t_2) = q_2$

$\mathcal{M}$

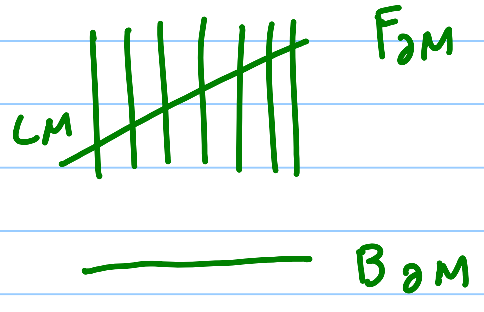
$\mathcal{M} = \mathbb{R}^n$, Neumann boundary problem

$T^*\mathbb{R}^n \quad (p, q) \quad \gamma(t_1) = p_1$

$\downarrow \qquad \downarrow$

$\mathbb{R}^n \quad p \quad \gamma(t_2) = p_2$

In both cases fibers are transversal to L_M



Classical Chern-Simons field theory:

- Space time M is compact, smooth, oriented, possibly $\partial M \neq \emptyset$
- Fields are F_M connections on $\begin{matrix} G \times M \\ \downarrow \\ M \end{matrix}$, can be identified with $\Omega^1(M, \mathfrak{g})$

  $\mathfrak{g}$ - Lie algebra of a simple compact Lie group G (assume matrix, i.e. $G \subset SU(N)$ and Killing form $= \mathrm{tr}$)
- We have a natural projection:

  $\pi: F_M \to F_{\partial M}$, π: the restriction to ∂M
- The action (topological)

$$S(A) = \int_M \mathrm{tr}\left(\tfrac{1}{2} A \wedge dA + \tfrac{1}{3} \wedge^3 A\right)$$

$$\delta S(A) = \int_M \mathrm{tr}(\delta A \wedge F(A)) - \frac{1}{2} \int_{\partial M} \mathrm{tr}(A \wedge \delta A)$$

i) Euler-Lagrange equations

$$F(A) = 0 \qquad F(A) = dA + A \wedge A$$

EL_M = all solutions, i.e. all flat connections on $M \times G$

ii) $$\alpha_{\partial M} = -\frac{1}{2} \int_{\partial M} \operatorname{tr}(A \wedge \delta A)$$

1-form on boundary fields

$F_{\partial M}$ = connections on $\begin{matrix} G \times \partial M \\ \downarrow \\ \partial M \end{matrix} \simeq \Omega^1(M, \mathfrak{g})$

$$\omega_{\partial M} = \delta \alpha_{\partial M} = -\frac{1}{2} \int_{\partial M} \operatorname{tr}(\delta A \wedge \delta A)$$

is a symplectic structure on $F_{\partial M}$ (Atiyah-Bott)

Thm $L_M = \pi(EL_M) \hookrightarrow F_{\partial M}$ is a Lagrangian submanifold.

Proof: isotropic (easy), coisotropic (more difficult, reduction)

- $\partial M_\varepsilon = [-\varepsilon, \varepsilon] \times \partial M$ small ε

$C_{\partial M} = \pi(EL_{\partial M_\varepsilon}) =$ flat connections on ∂M

$(C_{\partial M} \simeq EL_{\partial M_\varepsilon})$

Thm $C_{\partial M}$ is a coisotropic subspace

Thus, on the boundary we have:

$$\boxed{\underset{\text{Lagrangian}}{L_M = \pi(EL_M)} \hookrightarrow \underset{\text{coisotropic}}{C_{\partial M}} \hookrightarrow \underset{\text{sympl.}}{F_{\partial M}}}$$

Gauge symmetry

For trivial principal G-bundle

$G_M = Maps(M, G)$

Restriction to the boundary gives:

$$1 \to \ker(\tilde{\pi}) \to G_M \to G_{\partial M} \to 1$$

$\ker(\tilde{\pi})$: gauge transforms on M which act trivially on ∂M

$G_{\partial M}$: gauge transforms on the boundary

The action is almost gauge invariant:

$$S_M(A^g) = S_M(A) +$$

$$+ \frac{1}{2}\int_{\partial M} tr(g^{-1}Ag \wedge g^{-1}dg) - \underbrace{\frac{1}{6}\int_M tr(\wedge^3 g^{-1}dg)}_{W_M(g)}$$

Assume integrality of the Maurer-Cartan form on G:

$$\theta = -\tfrac{1}{2}\,\mathrm{tr}(\Lambda^3 g^{-1}dg)$$

i.e. $[\theta] \in H_3(G, \mathbb{Z})$

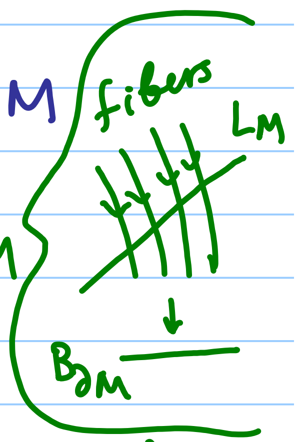
<u>Thm</u> $W_M(g) \bmod \mathbb{Z}$ depends only on $g|_{\partial M}$.

<u>Corollary</u> When $\partial M = \emptyset$, $S_M \bmod \mathbb{Z}$ is gauge invariant, $S_M = \deg(g: M \to G)$

Boundary conditions

Lagrangian fibration
$$\begin{array}{c} F_{\partial M} \\ \downarrow \\ B_{\partial M} \end{array}$$

- agrees with $G_{\partial M}$ action (possible to reduce)
- fibers are transversal to L_M
- $\alpha_{\partial M}|_{\text{fibers}} = 0$



Not easy to find (does not exist)

The formal semiclassical path integral:

Classical field theory

F_M, S_M, $\pi : F_M \to F_{\partial M}$,

$$\begin{array}{c} F_{\partial M} \\ \downarrow p_\partial \\ B_{\partial M} \\ (\text{b.c.}) \end{array}$$

Assume there is no gauge invariance, S_M has isolated (nondeg.) crit pnts for each $b \in B_{\partial M}$

$$Z_{M,a} \overset{\text{def}}{=} \int^{\text{formal}}_{\substack{\alpha \in T_a F \\ (dp_\partial)_{\pi(a)} (d\pi(\alpha)) = 0}} e^{\frac{i}{\hbar} S_M(a + \alpha\sqrt{\hbar})} D\alpha =$$

$$= C \frac{\exp(\frac{i}{\hbar} S_M(a))}{\sqrt{\det' S^{(2)}_a}} \Big(1 + \sum_{n \geq 1} \hbar^n \, \mathrm{Feyn}_{(n)}(a)\Big)$$

formal power series, Feynman diagr.

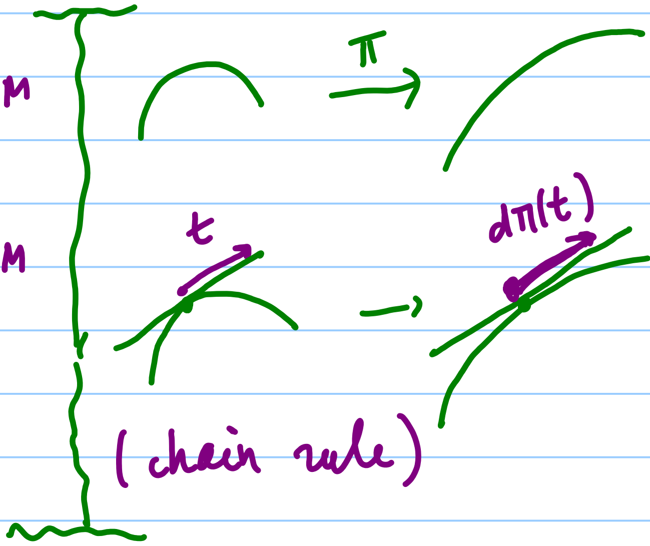
Here $a \in EL_M$, $\pi : F_M \to F_{\partial M}$

and $p_\partial : F_{\partial M} \to B_{\partial M}$ is a

boundary fibration

$d\pi : TF_M \to TF_{\partial M}$

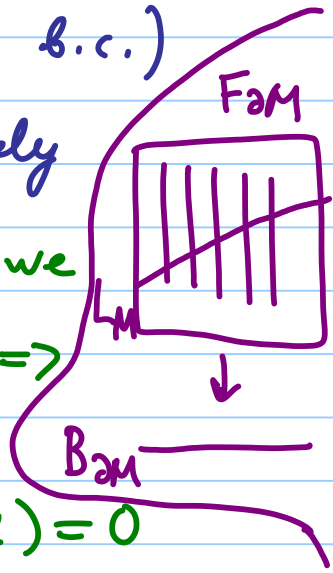
$dp_\partial : TF_{\partial M} \to TB_{\partial M}$



<u>Assumption:</u> <u>fibers of p_∂ are</u>
<u>transversal to L_M</u> ("good" b.c.)

i.e. L_M projects surjectively
to $B_{\partial M}$. $\Rightarrow$ varying a we
can span $T_{p_\partial \pi(a)} B_{\partial M}$ $\Rightarrow$

$\Rightarrow$ a - fixed, we can
choose $(dp_\partial \circ d\pi)(\alpha) = 0$



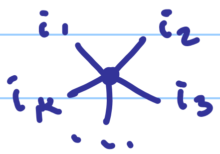
Feynman diagrams:

Here: $$\int_{\mathbb{R}^n} e^{i\left(\frac{1}{2}(x,Bx)+\sum_{k=3}^{\infty}\hbar^{\left(\frac{k}{2}-1\right)}V_k(x)\right)}\, d^n x =$$

$$=(2\pi)^{\frac{n}{2}}\, \frac{e^{\frac{i}{4}\operatorname{sign}(B)}}{\sqrt{|\det(B)|}}\left(1+\sum_{n\geq 1}\hbar^n\, \mathrm{Feyn}_{(n)}\right)$$

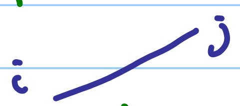
$$\mathrm{Feyn}_{(n)} = \sum_{\Gamma,\ \mathrm{ord}(\Gamma)=n} \frac{w(\Gamma)}{|\mathrm{Aut}(\Gamma)|},$$

$w(\Gamma)$:



$\sim\ V_{(k)}^{i_1 \dots i_k}$

star of a vertex



$\sim\ (B^{-1})_{ij}$

edge

$$w(\Gamma) = \sum_{\text{"states" } \{i\}} \prod_e (B^{-1})_{i_e j_e} \prod_v V_{(k_v)}^{i_1 \dots i_{k_v}}$$

$$\mathrm{ord}(\Gamma) = \sum_v \left(\frac{k_v}{2}-1\right)$$

Remark:

When B is degenerate (group action, ...) we should account for ghost fermions in determinants and Feynman diagrams (Faddeev-Popov, BRST, BV)

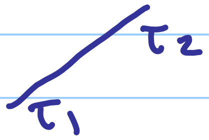
Example: Quantum mechanics

$$Z_{M,\gamma_c} = (2\pi\hbar)^{\frac{n}{2}} \sqrt{\det \frac{\partial^2 S[\gamma_c]}{\partial q_1 \partial q_2}} \; \cdot \exp\left(\frac{i}{\hbar} S[\gamma_c]\right)\left(1 + \sum_{n \geq 1} \hbar^n F_n(\gamma_c)\right)$$

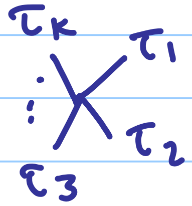
sum of Feynman diagrams of order n

- $B = -\frac{d^2}{dt^2} + V''(\gamma_c(t))$, $\quad t \in [t_1, t_2]$

Dirichlet



$\longrightarrow \quad B^{-1}(\tau_1 \tau_2)$, $\tau_1, \tau_2 \in [t_1, t_2]$

- 

$\longrightarrow \quad \prod_{a=1}^{K-1} \delta(\tau_a - \tau_{a+1}) \; V(\gamma_c(\tau_1))$

- Integrals convergent

- $$\sqrt{\left|\det\left(\frac{\partial^2 S(\gamma_c)}{\partial q_1 \partial q_2}\right)\right|} = \frac{1}{\sqrt{\det'(B)}}$$

Thm formal

$$\int Z_{[t_1,t_2],\gamma_{12}}(q_1 \, q_2) \, Z_{[t_2,t_3],\gamma_{23}}(q_2 \, q_3) \, dq_2 =$$

$$= Z_{[t_1,t_3],\gamma_{13}}(q_1, q_3)$$

(not 50 years ago but ~ 3 ?
T. Johnson - Freyd 2010)

Usual "proof":

$$
\begin{tikzcd}
\begin{pmatrix} t_1 \quad t_2 \\ \delta_1\, \delta_2 \dots \\ \text{discrete} \\ \text{version} \end{pmatrix} \arrow[r, "\hbar \to 0"] \arrow[d, "\delta \to 0"'] \arrow[dr, phantom, "?"] & \begin{pmatrix} \text{Feynman} \\ \text{diagrams} \end{pmatrix} \arrow[d, "\delta \to 0"] \\
\begin{pmatrix} \text{Schrödinger} \\ \text{propagator} \end{pmatrix} \arrow[r, "\hbar \to 0"] & \begin{pmatrix} \text{QM Feynman} \\ \text{diagrams} \end{pmatrix}
\end{tikzcd}
$$

- Ultraviolet divergencies for $d > 1$ non-topological models

- No divergencies for TFT (theory of nothing in the bulk) Interesting because of gauge symmetry

Rmrks ① Semiclassical and scaling limit ?

② Gluing with corners ?

Quantum Chern - Simons theory

Witten 1989

outline of the path integral and conject. formulae via TQFT $\leftrightarrow$ CFT correspondence

R, Turaev 1989

combinatorial construction based on $U_q(\mathfrak{g})$ at roots of 1

"Solvable" Gauge theories

- QCD, dim=2
- CS, dim=3

Axelrod-Singer

acyclic connections

$\partial M = \emptyset$

Kontsevich

Bott - Cattaneo

Cattaneo - Mnev

non- acyclic connections

J.E. Andersen

M = mapping class cylinder

$I \times_g \Sigma$ (Berezin-Teplitz quantization)

Quantization

The goal:

$$Z_a = \int_{\Lambda_a}^{\text{formal}} e^{\frac{i}{\hbar} S_M(a+\sqrt{\hbar}\,\alpha)} \mathcal{D}\alpha$$

$$T_a \Omega^1(M, \text{b.c.}) = \Lambda_a \oplus T_a EL_M$$

- The background field is a flat connection $a \in EL_M$: covariant derivatives
$$d_a\alpha = d\alpha + [a \wedge \alpha], \quad d_a^2 = 0$$

- Fluctuations $\hbar^{1/2}\alpha$, $\alpha \in \Omega^1(M, \mathfrak{g})$
(in formal nbd of a)

$$S_M(a + \hbar^{1/2}\alpha) = S_M(a) + \sqrt{\hbar}\int_{\partial M} \operatorname{tr}(a \wedge \alpha)$$

$$+ \hbar \int_M \operatorname{tr}\left(\tfrac{1}{2}\alpha \wedge d_a\alpha + \hbar^{1/2}\tfrac{1}{3}\wedge^3\alpha\right)$$

Boundary conditions:

Choose $L \subset F_{\partial M}$

- $i^*(a) \in L$,
- gauge invariant

Choose
$$T_{i^*(a)} F_{\partial M} = \overset{T_{i^*(a)}L \;=\!\!/}{\mathcal{L}_+} \oplus \mathcal{L}_-$$

Lagrangian transversal subspaces, transversal to $\mathcal{L}_M = T_{i^*(a)} L_M$

boundary values of flat connections on M $\Big\} \rightarrow L_M$

We impose:
$$i^*(\alpha)_+ = 0$$

Lagrangian subspaces
$$\omega_+ + \mathcal{L}_- \,, \quad \omega_+ \in \mathcal{L}_+$$

are not exact for
$$\alpha_{\partial M} = \int_{\partial M} tr(a \wedge \delta a)$$

Modified action:

$$\tilde{S}(a+\sqrt{\hbar}\,\alpha) = S(a+\sqrt{\hbar}\,\alpha) +$$

$$+\frac{1}{2}\int_{\partial M} \mathrm{tr}\,\underbrace{(a+\sqrt{\hbar}\,\alpha_+)\wedge(a+\sqrt{\hbar}\,\alpha_-)}_{A_+\wedge A_- \text{ in vicinity of } a} =$$

$$= \tilde{S}_M(a) + \sqrt{\hbar}\int_{\partial M}\mathrm{tr}(a\wedge\alpha_+)$$

$$+\hbar\int_M \mathrm{tr}\left(\frac{1}{2}\alpha\wedge d_a\alpha + \hbar^{1/2}\frac{1}{3}\wedge^3\alpha\right) +$$

$$+\frac{\hbar}{2}\int_{\partial M}\mathrm{tr}(\alpha_+\wedge\alpha_-)$$

With boundary conditions $\alpha_+=0$ we have:

$$\tilde{S}(a+\sqrt{\hbar}\,\alpha) = \tilde{S}_M(a) +$$

$$+\hbar\int_M \mathrm{tr}\left(\frac{1}{2}\alpha\wedge d_a\alpha + \hbar^{1/2}\frac{1}{3}\wedge^3\alpha\right)$$

Formal semiclassical partition function:

$$Z_a = e^{\frac{i}{\hbar} S_M(a)} .$$

$$\int_{\Lambda_a}^{\text{formal}} \exp\left(i \int_M tr\left(\tfrac{1}{2} \alpha \wedge d_a \alpha\right)\right) \exp\left(\hbar^{1/2} \tfrac{1}{3} \int_M tr(\wedge^3 \alpha)\right) \mathcal{D}\alpha$$

$$\Omega^1_D(M, \mathfrak{g}, \mathcal{L}_-) = \Lambda_a \oplus \Omega^1_D(M, \mathfrak{g}, \mathcal{L}_-)_{d_a^* = d}$$

$$\uparrow$$

$$T_a EL_M \text{ (b.c.)}$$

Formal integral should take into accound all ingradients of the gauge fixing

Remark Complex polarization:

$$\Omega^1(\partial M, \mathfrak{g})_{\mathbb{C}} = \Omega^{0,1}(\partial M, \mathfrak{g}) \oplus \Omega^{1,0}(\partial M, \mathfrak{g})$$

$$A = a(z, \bar{z})\, dz + \bar{a}(z, \bar{z})\, d\bar{z}$$

Good for geometric quantization.

Not good for b.c. in path integral

Convenient choice of polarization:

Choose a metric on M

$L \subset \Omega^1(\partial M, \mathfrak{g})$, Lagrangian "submanif"

$\mathcal{L} \subset \Omega^1(\partial M, \mathfrak{g}) = T_a L$

$\mathcal{L}_+ = \mathcal{L}$, $\mathcal{L}_- = \mathcal{L}_+^\perp$,

In the bulk:

$$\Lambda_a = T_a EL_M(b.c.)^\perp$$

Hodge-DR theorem

$$\Omega^1(M, \mathfrak{g}) = d^* \Omega^2_N(M, \mathcal{L}) \oplus \underbrace{\Omega^1_D(M, \mathcal{L}^\perp)_{cl}}_{T_a EL_M(b.c.)^\perp}$$

$$\Lambda_a = d^* \Omega^2_N(M, \mathcal{L}) \cap \Omega^1_D(M, \mathcal{L}^\perp)_{cl}$$

Thm $\Lambda_a \simeq \Omega^1_D(M, \mathcal{L}^\perp)/\ker(B)$

$$B(\alpha, \beta) = \int_M \mathrm{tr}(\alpha \wedge d_a \beta)$$

Work in progress:

$$Z_{M,\mathcal{L},a} = C \, T_{a,\mathcal{L}}^{1/2} \, e^{\frac{i S(a)}{\hbar} + i\eta(M)_{\mathcal{L}}} \cdot$$

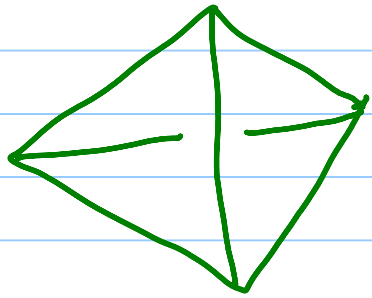
$$\cdot \left(1 + \sum_{n\geq 1} \hbar^n F_n(a,\mathcal{L})\right)$$

Thm (with A. Cattaneo, P. Mnev)

(1) $Z_{M,\mathcal{L},a}$ does not depend on metric on $M \setminus \partial M$
(topological $\Leftarrow$ gauge invariant)

(2) $Z_{M,\mathcal{L},a}$ satisfy gluing

$G = SU(2)$, $\quad M = S^2 \setminus \hat{T}$

$\hat{T} =$  (tub. nbd.)

$L \subset F_{\partial M}$ = flat connection which extend as flat inside of $\hat{T}$

$a \in \pi_1(M) \to SU(2)$

$Z_{M, \mathcal{L}, a} \sim$ the semiclassical asymptotic of q6j-symbols