From quantum current algebras to q-Virasoro and q-string theory

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Introduction

- *Main discovery of the late 1970 – early 1980's :* Quantum groups (a byproduct of the study of solvable models in QFT and Quantum Statistical Physics)
- *Question* : can we apply these new objects to the study of real world ($\dim > 2$) ?
- *Obvious obstruction:* Braiding $\Rightarrow$ Nonconventional statistics.
- *One case when this is not quite bad* : string theory.
- Basically, a string is a 2-dimensional QFT with conformal invariance and with "external" Poincaré symmetry.

Quantum current algebra

- Schwinger commutation relations for the current algebra

$$[j^a(x),\ j^b(y)] = f_c^{ab} j^c(x)\delta(x-y) + c\delta^{ab}\delta'(x-y)$$

- Set $j(x) = j^a x \otimes e_a$, $j_V(x) = j^a x \otimes \pi_V(e_a)$. Then

$$[j^1_U(x),\ j^2_V(y)] = [t^{12}_{UV},\ j^1_U(x) - j^2_V(y)] + t^{12}_{UV}\delta'(x-y),$$

where

$$t^{12}_{UV} = \pi_U(e_a) \otimes \pi_V(e_a).$$

- Quantum (or *q-deformed*) generalization of the Schwinger formula uses the fundamental object in Quantum Groups Theory, the *quantum R-matrix*.

Quantum current algebra

- Quantum R-matrices are defined for pairs of finite-dimensional representations of quantum algebra and satisfy QYBE:

$$R^{12}_{UV}(s)R^{13}_{UW}(st)R^{23}_{VW}(t) = R^{23}_{VW}(t)R^{13}_{UW}(st)R^{12}_{UV}(s)$$

- Quantum current $J_V(s)$ satisfies commutation relations:

$$J^1_V(s)R^{12}_{UV}(st^{-1}q^c)J^2_V(t)R^{12}_{UV}(st^{-1})^{-1} =$$
$$R^{12}_{UV}(st^{-1})J^2(t)R^{12}_{UV}(st^{-1}q^{-c})J^1_V(s).$$

- Exponential parametrization $s = e^{2\pi ix/L}, q^c = e^{2\pi il/L}$. Hence central charge together with q introduce an ultraviolet scale.

Sugawara construction

- We have $j(x) = j_+(x) - j_-(x)$.
- q-deformed counterpart: $J(s) = J_+(sq^{c/2})J_-(s)^{-1}$ (already normal ordered!)
- Sugawara current:

$$S(x) =: \langle j(x),\ j(x)\rangle :,$$

$$[S(x),\ S(y)] = (c+n)\left((S(x) + S(y))\delta'(x-y) + k\delta'''(x-y)\right).$$

- For critical value of c, $c = -n$, Sugawara operators commute with each other.
- The would-be q-deformed counterpart: $S_V(s) = \mathrm{tr}_V J_V(s)$.

Sugawara construction

- **Theorem.** For the critical value of the central charge $S_V(s)$ lies in the center of the quantum current algebra; this is valid for any V.
- Unfortunately, for other values of c the commutation relations for $S_V(s)$ do not close up.
- *Partial remedy:* quasiclasscal computation for $c \simeq c_{crit}$. This yields the definition of q-deformed Poisson Virasoro algebra (Reshetikhin–E.Frenkel)

$$\{T(z), T(w)\} = \phi(\frac{z}{w})T(z)T(w) + \delta\left(\frac{wq}{z}\right) - \delta\left(\frac{w}{zq}\right),$$

where $\phi(z) = \sum_{m \in \mathbb{Z}} \frac{1 - q^m}{1 + q^m} z^m$

Drinfeld–Sokolov construction

- The formula above is not very transparent (but suggests the important role of elliptic functions in the matter!)
- *Alternative approach:*
  - Poisson Virasoro algebra arises in the study of 2nd order differential operators.
  - Natural way to get 2nd order differential operators: by Hamiltonian reduction of the space of first order 2×2 matrix differential operators.
    - *Gauge group:* lower triangular matrices.
    - Natural constraint:

$$L = \begin{pmatrix} * & 1 \\ * & * \end{pmatrix}$$

Drinfeld–Sokolov construction

- Two cross-sections:

$$L \simeq \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}, \quad \text{or} \quad L \simeq \begin{pmatrix} v & 1 \\ 0 & -v \end{pmatrix}$$

- Miura transform: $u = v^2 + v'$.
- One has $\{v(x), v(y)\} = \delta'(x-y)$ (inherited from the current algebra), and

$$\{u(x), u(y)\} \, (u(x) + u(y))\delta'(x-y) + \delta'''(x-y).$$

q-difference case

- To get q-deformed Poisson Virasoro algebra we start with 1st order difference equations of the form

  $\psi(qz) = L(z)\psi(z),$ where both L and ψ are 2×2 matrices.

  - *Important:* Now q is the modulus of the difference operator (Planck constant appears at the next stage!)
  - Gauge group again consists of lower triangular matrices.
  - *New features:*
    - 1st order difference operators carry non-trivial Poisson bracket; its choice depends on the choice of a classical r-matrix.
    - Gauge action is not Hamiltonian.
    - Reduction becomes non-trivial.

q-difference case

- Constraint

$$L = \begin{pmatrix} * & 1 \\ * & * \end{pmatrix}$$

  remains the same.
- Two natural cross-section of the gauge action are

$$L \simeq \begin{pmatrix} 0 & 1 \\ -1 & u \end{pmatrix}, \quad \text{or} \quad L \simeq \begin{pmatrix} v & 1 \\ 0 & v^{-1} \end{pmatrix}$$

- Miura transform: $u(x) = v(x) + v^{-1}(qx)$.
- The constraint is of 2nd class according to Dirac (which is very bad!)
- *But there is the way to save the construction* : appropriate choice of r-matrix!

q-difference case

- Result of computation:
  - $\phi(z) = \sum_{m \in \mathbb{Z}} z^m \frac{1-q^m}{1+q^m}$ appears very naturally as a component of new r-matrix.
  - One gets Poisson bracket for v:

$$\{v(x), v(y)\} = \phi(x/y)v(x)v(y).$$

  - The Poisson brackets for reproduce the q-deformed Poisson Virasoro algebra.
- *"What I say three times is true."*
- *Quantization:* Using the Poisson brackets for v (which give basically Heisenberg algebra) and the q-Miura transform.

q-deformed Virasoro $Vir_{q,t}$

- The result (known for 17 years):

$$f(w/z)T(z)T(w) - T(w)T(z)f(z/w) =$$
$$\frac{(1-q)(1-t^{-1})}{1-p}\left[\delta(pw/z) - \delta(p^{-1}w/z)\right], \quad (1)$$

where $p = qt^{-1}$ and

$$f(z) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \frac{(1-q^n)(1-t^{-n})}{1+p^n} z^n.$$

Why this does not fit?

- Technically, all this is rather difficult. In particular, the algebra is not finitely generated (like ordinary Virasoro).
- More importantly, there are restrictions on q: $|q| < 1$, while one would like to have $q = e^{ih}$.
- **So, what to do ?**
- In standard string theory much can be guessed from the residual projective invariance (in particular, Veneziano amplitude).
  - Standard Veneziano amplitude $A(s,t) = \frac{\Gamma(1-\alpha s)\Gamma(1-\alpha t)}{\Gamma(2-\alpha s-\alpha t)}$.
  - q-deformed Veneziano amplitude derived from $SU_q(1,1)$-covariance (A.Leclerc, 1989): $\Gamma \to \Gamma_q$
- This is also not the correct version!

On the role of Γ-function

- Early results: Harish-Chandra theory of spherical functions.
- Modern theory: Quantum Separation of variables (Sklyanin, Lebedev & Gerasimov, . . .
- Baxter equation, Mellin-Barnes integrals.
- Principal series representations for quantum group $U_q(sl(2,\mathbb{R})$
- Quantum torus and quantum dilogarithm.

Example: quantum sl_2

- Commutation relations

$$KE = q^2 EKn, \; KF = q^{-2} FK, \; EF - FE = \frac{K - K^{-1}}{q - q^{-1}},$$

- Family of homomorphisms into quantum torus (associative algebra $\mathbb{A}_q$ generated by u, v with commutation relation $uv = q^2 vu$):

$$K \mapsto zu^{-1}, E \mapsto \frac{v^{-1}}{q - q^{-1}}(1 - u^{-1}), F \mapsto \frac{qv}{q - q^{-1}}(z - z^{-1}u),$$

It sends Casimir element $C = qK + q^{-1}K^{-1} + (q - q^{-1})^2 FE$ into $z + z^{-1}$.

Example: quantum sl_2

- Involution: $K^* = K$, $E^* = -E$, $F^* = -F$; it is compatible with the commutation relations, iff $q = e^{\pi i \tau}$, $\tau = \omega_1/\omega_2$.
- Second copy of quantum sl_2: it is associated with $\tilde{q} = e^{-\pi i/\tau}$ and is mapped into the torus $\mathbb{A}_{\tilde{q}}$.
- Hilbert space representation: put

$$
\begin{aligned}
T_{i\omega_1}\varphi(t) &= \varphi(t + i\omega_1), \qquad & T_{i\omega_2}\varphi(t) &= \varphi(t + i\omega_2), \\
S_{\omega_1}\varphi(t) &= e^{\frac{2\pi t}{\omega_1}}\varphi(t), \qquad & S_{\omega_2}\varphi(t) &= e^{\frac{2\pi t}{\omega_2}}\varphi(t).
\end{aligned}
$$

Define the dual representations of $\mathbb{A}_q$ and $\mathbb{A}_{\tilde{q}}$ in $\mathcal{H} = L_2(\mathbb{R})$ by

$$
\begin{aligned}
\rho: \quad & u \mapsto T_{\omega_1}, \qquad & v \mapsto S_{-i\omega_2}, \\
\widetilde{\rho}: \quad & \widetilde{u} \mapsto T_{\omega_2}, \qquad & \widetilde{v} \mapsto S_{-i\omega_1}.
\end{aligned}
$$

Example: quantum sl_2

- **Claim:** This provides a unitary representation of $U_q(\mathfrak{sl}(2,\mathbb{R}))$ and $U_{\tilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$. Moreover, both algebras centralize each other in $L_2(\mathbb{R})$.

Basic special function: q-dilogarithm

- $$\log \mathcal{G}(z|\tau) = \int\limits_{\mathbb{R}+i0} \frac{e^{zt}}{(e^{\omega_1 t}-1)(e^{\omega_2 t}-1)} \frac{dt}{t}.$$
- Difference equations:
$$\frac{\mathcal{G}(z+\omega_1|\tau)}{\mathcal{G}(z|\tau)} = \frac{1}{1-e^{\frac{2\pi i z}{\omega_2}}}, \quad \frac{\mathcal{G}(z+\omega_2|\tau)}{\mathcal{G}(z|\tau)} = \frac{1}{1-e^{\frac{2\pi i z}{\omega_1}}}$$
- Important feature: $\mathcal{G}$ is *double-quasi-periodic*.
- Poles at $z = n_1\omega_1 + n_2\omega_2,\ n_1,\ n_2 \geq 1$, zeros at $z = n_1\omega_1 + n_2\omega_2,\ n_1, n_2 \leq 0$.
- Case when ω_1 and ω_2 are aligned is not excluded: no accumulation of zeros and poles unless τ is real negative (the only bad case).

Main problem

- **Claim:** q-dilogarithm is ubiquitous and provides a basis for Representation theory of $U_q(\mathfrak{g})$ which accounts for modular duality.
- Main problem:
  - Extend this assertion to quantum current algebras
  - Define the correct version of q-Virasoro which makes sense for $|q| = 1$ and is compatible with modular duality.
  - Is there a chance this will yield a new string theory?

- **Thank you for your attention!**