Symmetry protected topological phases and "orbifolds"

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Topological phases and anomalies

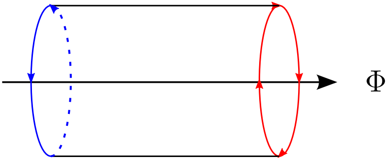
Topological phases (in broad sense):

no analogous phase in classical systems
(very quantum state of matter)

Anomalies:

breakdown of a classical symmetry by quantum effects
(nothing more quantum than this)

Laughlin's argument -- "large gauge transformation"



- Adiabatic process $\Phi \to \Phi + \Delta\Phi$

- When $\Delta\Phi = \mathrm{integer} \times \Phi_0$ system goes back to itself ("large gauge equivalent")

$$H(\Phi) = H(\Phi + n\Phi_0) \qquad Z(\Phi) = Z(\Phi + n\Phi_0)$$

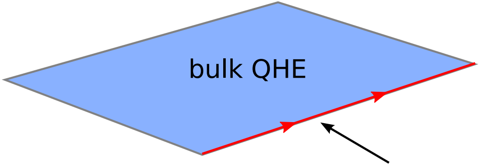
- However, by this adiabatic process, an integer multiple of charge is transported from the left (right) to right (left) edge.

- Charge is not conserved for a given edge.

Edge of QHE

- Chiral edge theory

$$\mathcal{L} = \frac{1}{2\pi}\psi_R^\dagger i(\partial_t + v\partial_x)\psi_R$$



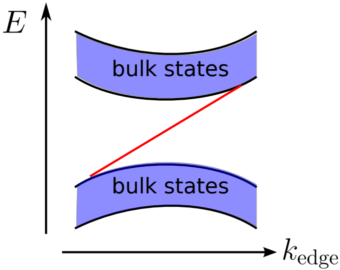
1D boundary (edge)
chiral edge state

- Partition function with fluxes

$$\psi_R(t, x+2\pi) = e^{2\pi i a}\psi_R(t,x),$$
$$\psi_R(t+2\pi\tau_2, x+2\pi\tau_1) = e^{2\pi i b}\psi_R(t,x).$$

gauge anomaly
$\langle \nabla^\mu J_\mu \rangle \neq 0$

$$Z_{[a,b]}(\tau) = \frac{1}{\eta(\tau)}\vartheta\begin{bmatrix} a-1/2 \\ -b+1/2 \end{bmatrix}(0,\tau)$$



- Large gauge anomaly

$$Z_{[a,b]} = Z_{[a+1,b]} = e^{2\pi(a-1/2)}Z_{[a,b+1]}$$

Diagnose (non)topological phases by anomalies

- Advantage:

  - Robust against interactions, e.g., Adler-Bardeen's theorem
  - (often) Observable: anomaly = "response"

- This talk: Symmetry protected topological phases (SPT)

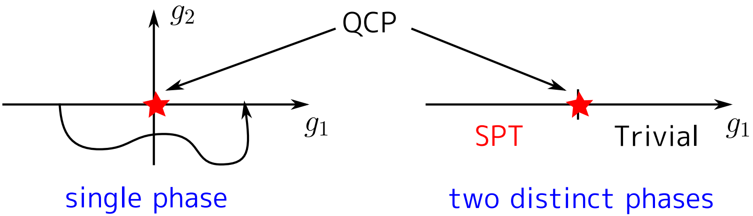
  Topological phases characterized by chiral edge states are by now familiar $\sigma_{xy} \neq 0$ $\kappa_{xy}/T \neq 0$

  How about topological phases without conserved charge and with non-chiral edge mode?

Symmetry protected topological phases

- Not a topological phase
  "deformable" to a trivial phase (state w/o entanglement)

- But sharply distinct from trivial state w/ symmetry



- E.g. quantum spin Hall effect, topological insulator

- (often) accompanied by non-chiral edge state
  "unstable" w/o symmetry

Questions

- Under which conditions does a SPT phase occur?

  What kind of symmetries are necessary?

In 2+1 dimensions,

- Under which conditions is a non-chiral edge "non-gappable"?

  -- stable SPT phase

  When is it "gappable"?

  -- trivial case

Example: boson with Z_N symmetry

- Bosonic topological insulator [Lu-Vishwanath (12)]

$$S = \frac{1}{4\pi}\int d^2x d\tau\, \epsilon^{\mu\nu\lambda} a^I_\mu K_{IJ} \partial_\nu a^J_\lambda \qquad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- Edge theory: Compactified free boson;

$$S \sim \int dx d\tau\, \partial_x\phi \partial_\tau \theta \qquad [\phi, \theta] \sim i$$

$$\phi \equiv \phi + 2\pi$$
$$\theta \equiv \theta + 2\pi$$

$$S = \frac{1}{4\pi}\int dx d\tau \left[\frac{1}{v}(\partial_t\phi)^2 - v(\partial_x\phi)^2\right]$$

- Z_N symmetry (k=1,2,...,N-1)

$$G_k: \quad \phi \to \phi + \frac{2\pi k}{N}$$
$$\theta \to \theta + \frac{2\pi k q}{N}$$

q: controls asymmetry

Is the edge "gappable" or "ungappable" ?

- N=2, q=0 $\phi \to \phi + \pi \quad \theta \to \theta$

  $\cos\theta$ can gap the edge

- N=2, q=1 $\phi \to \phi + \pi \quad \theta \to \theta + \pi$

  $\cos\theta$:not allowed by the symmetry

  $\cos 2\theta$:allowed, but spontaneous symmetry breaking

  $\cos(\theta + \phi)$:allowed, but cannot gap the edge

  The bulk is non-deformable to a trivial state.

- N=2, q=1 with two copies N_f =2

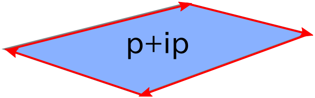
  Can condense $\phi_1 - \phi_2$ and $\theta_1 + \theta_2$

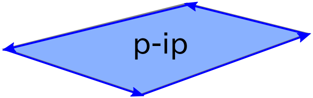
  $\phi_1 + \phi_2 \qquad \theta_1 - \theta_2$

Topological phase with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

- System of "real" fermions

- Can be build from two chiral p-wave SCs with opposite chiralities





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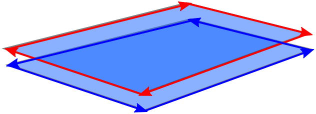
non-chiral edge



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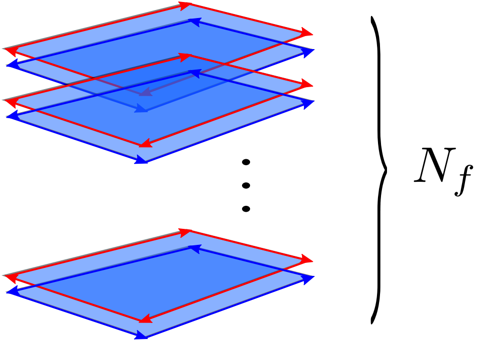


- Fermion number parities are conserved seperately for each chirality

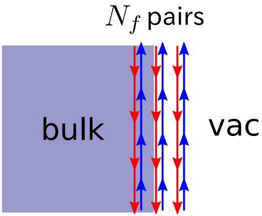
[Similar (same) models: Qi (2012), Gu-Wen (2012), Yao-SR (2012)]

Topological phase with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

- Edge theory: non-chiral

$$\mathcal{L} = \frac{1}{4\pi} \sum_{a=1}^{N_f} [\psi_L^a(\partial_\tau + iv\partial_x)\psi_L^a + \psi_R^a(\partial_\tau - iv\partial_x)\psi_R^a]$$

$$G_R = (-1)^{N_R} \qquad G_L = (-1)^{N_L}$$



- Topological classification of the phases at quadratic level:

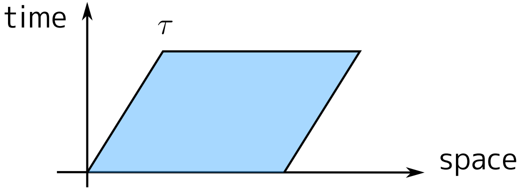
number of non-chiral edge modes
= bulk Chern number parity = $N_f \in \mathbb{Z}$

Edge is stable at quadratic level:

mass term $\psi_L^a \psi_R^b$ is odd under $\mathbb{Z}_2 \times \mathbb{Z}_2$

- No "perturbative anomalies" $\kappa_{xy}/T = 0 \qquad \sigma_{xy} = 0$
  stability against interactions ?

Large coordinate transformations (modular transformations)



- Spacetime manifold of the edge theory = 2d torus

- "Shape" of torus is parameterized by a single complex number

$$z \equiv z + 2\pi(m + \tau n)$$

- $\tau' = \tau + 1$ and $\tau' = -1/\tau$ represent the same torus since

$$(m,n) \to (m-n,n) \qquad (m,n) \to (n,-m)$$

Modular invariance/non-invariance in CFTs

- Any CFT derived as a continuum limit of a lattice system is expected to be modular invariant (i.e., anomaly free).

[Cardy, Cappelli-Itzykson-Zuber, Kato]

- Chiral CFTs are often (but not always) modular non-invariant (i.e., anomalous).

- "Gluing" left-moving and right-moving parts properly, non-chiral CFTs can usually be made modular invariant.

- However, demanding a CFT to be invariant under some symmetry (e.g. $Z_2 \times Z_2$ symmetry) may conflict with modular invariance

Generalized Laughlin argument

- Gapable <--> modular invariant
  Ingapable <--> modular non-invariant

- Symmetry Enforcement:

  Projection by symmetry group G --> orbifold part. function

$$Z^{\rm orb} = \frac{1}{|G|} \sum_{g_1, g_0 \in G} \epsilon(g_1, g_0) Z^{g_0}{}_{g_1}$$

time bc

spatial bc

Example: boson with Z_N symmetry

- Orbifold partition function ("shift orbifold"):

$$Z^{k_1}{}_{k_2} = \mathrm{Tr}_{k_2}\left[G_{k_1} q^{(L_0-1/24)} \bar{q}^{(\bar{L}_0-1/24)}\right]$$

$$Z^{\mathrm{orb}} = \frac{1}{|G|}\sum_{g_1,g_0\in G} \epsilon(g_1,g_0) Z^{g_0}{}_{g_1}$$

- Modular transformations:

$$Z^{k_1}{}_{k_2}(\tau+1) = e^{-\frac{2\pi i q k_1^2}{N^2}} Z^{k_1}{}_{k_1+k_2}(\tau)$$

$$Z^{k_1}{}_{k_2}(-1/\tau) = e^{\frac{4\pi i k_1 k_2}{N^2}} Z^{k_2}{}_{-k_1}(\tau)$$

- Modular invariance: e.g. when N=2, q=1 --> need N_f= 2

- Gapable when N_f =2

  Can condense $\phi_1 - \phi_2$ and $\theta_1 + \theta_2$

  $\phi_1 + \phi_2$ $\theta_1 - \theta_2$

simple example

- Free complex left-moving fermion $\mathcal{L}_L = \frac{1}{2\pi}\Psi_L^\dagger\left(\partial_\tau + v\partial_x\right)\Psi_L$

- Partition function with a given boundary condition

spatial bc

$$Z^\alpha{}_\beta(\tau) = \mathrm{Tr}_\alpha\left[e^{\pi i\beta N_L} q^{H_L}\right], \quad q = e^{2\pi i\tau} \qquad N_L := \int dx\,\Psi_L^\dagger \Psi_L$$

time bc

alpha, beta = 0 or 1 (anti PBC or PBC)

- Modular trsf:

$$S : \begin{cases} Z^0{}_0 & \to & Z^0{}_0, \\ Z^1{}_1 & \to & Z^1{}_1, \\ Z^1{}_0 & \to & Z^0{}_1, \\ Z^0{}_1 & \to & Z^1{}_0, \end{cases} \qquad T : \begin{cases} Z^0{}_0 & \to & e^{i\pi/12} Z^0{}_1, \\ Z^1{}_1 & \to & e^{-i\pi/6} Z^1{}_1, \\ Z^1{}_0 & \to & e^{-i\pi/6} Z^1{}_0, \\ Z^0{}_1 & \to & e^{i\pi/12} Z^0{}_0, \end{cases}$$

$$\tau \to -1/\tau \qquad\qquad \tau \to \tau + 1$$

- Combining left- and right-moving sectors, we acheive modular inv.:

$$Z = |Z^0{}_0|^2 + |Z^0{}_1|^2 + |Z^1{}_0|^2 + |Z^1{}_1|^2$$

Enforcing symmetries by projection

- Is the modular invariance "consistent" with the Z2xZ2 symmetry?

- Let's enforce Z2xZ2 symmetry by projection $(N = 2N_f)$

$$P = \frac{1+(-1)^{N_L}}{2}\frac{1+(-1)^{N_R}}{2}$$

$$N_L = \sum_{i=1}^{N} N_L^i$$
$$N_R = \sum_{i=1}^{N} N_R^i$$

- Projected partition function:

$$\mathrm{Tr}_0 \left[P q^{H_L}\right] = \frac{1}{2}\left[{Z^0}_0(\tau)^N \pm {Z^0}_1(\tau)^N\right] \quad \text{(apbc)}$$

$$\mathrm{Tr}_1 \left[P q^{H_L}\right] = \frac{1}{2}\left[{Z^1}_0(\tau)^N \pm {Z^1}_1(\tau)^N\right] \quad \text{(pbc)}$$

- Projected partition function is non-modular invariant in general.
  the edge theory is anomalous
  --> the bulk topolgoical phase

Case of N_f=8 real fermions (N=4 complex fermions)

- When N=4 (8 flavors of Majorana fermions), however, we acheive modular invariance !

$$Z(\tau) = \frac{1}{2}\left[{Z^0}_0(\tau)^4 - {Z^0}_1(\tau)^4 - {Z^1}_0(\tau)^4 \pm {Z^1}_1(\tau)^4 \right]$$

$$Z^{\mathrm{tot}}(\tau) = |Z_L(\tau)|^2$$

Suggesting there is no bulk topological phase when N =4.

- In fact, we can find interactions which destabilize the edge theory when N = 4.

Interactions in terms of "spinors":

$$\Psi^i_L \simeq e^{i\varphi^i_L}$$

$$e^{\frac{i}{2}\left[\pm\varphi^1_L \pm\varphi^2_L \pm\varphi^3_L \pm\varphi^4_L\right]}$$

Haldane phase,
two-leg ladder Hubbard model
[Tsvelik, Lin-Balents-Fisher, ...
Fidkowski-Kitaev]

Single particle classifiction Z --> Z_8.

- 10 = 8 + 2 is the critical dimenion of superstring theory.

Summary

- Proposed strategy;
  (i) Projection by symmetry group G --> "orbifold" part. function

$$Z^{\mathrm{orb}} = \frac{1}{|G|} \sum_{g_1, g_0 \in G} \epsilon(g_1, g_0) Z^{g_0}{}_{g_1}$$

time bc (g_0)

spatial bc (g_1)

  (ii) Look for anomaly (of any kind) by adiabatically changing moduli

- SPT phases = (asymmetric) "orbifolds"

- "Gappable"/"ingappable" <--> modular invariant/non-invariant

- Related works:
  Diagnosing edge theory by bulk fractional statistics
  "Duality" between SPT and topological order
  Case without any symmetry: "Lagrangian subgroup" criterion

[Levin-Gu (2012), Levin (2013), Kapustin and Saulina (2011)
Barkeshil-Jian-Qi (2013), Wang-Wen (2012), etc.]

-Related works:

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Summary

- Gapable <--> modular invariant
  Ingapable <--> modular non-invariant

- Proposed strategy; SPTs = asymmetric "orbifolds"

  (i) Projection by symmetry group G --> orbifold part. function
  (ii) Look for anomaly

- Tested for different systems.
  Consistent with (same as) the fractional statistics argument.