A Spectral Parameter for $\mathcal{N} = 4$ Scattering Amplitudes

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My Collaborators

Ongoing project based on work with

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arXiv: 1212.0850, and to appear.
Plus further work with Rouven Frassek, Nils Kanning, Yumi Ko.

A Case for 3+1 Dimensions

Nature prefers Yang-Mills theory in exactly 1+3 dimensions:
Coordinates x^μ, momenta p^μ.

Split index $\mu = 0, 1, 2, 3$ into spinorial indices $\alpha = 1, 2$ and $\dot\alpha = \dot1, \dot2$.

Interesting bijection $\mathbb{R}^{1,3} \to H(2 \times 2), p^\mu \mapsto p_{\alpha\dot\alpha} = p_\mu \, (\sigma^\mu)_{\alpha\dot\alpha}$.
Reverse map $H(2 \times 2) \to \mathbb{R}^{1,3}, p^\mu \mapsto \frac{1}{2} \operatorname{Tr} p_{\alpha\dot\alpha} \, (\bar\sigma^\mu)^{\dot\alpha\alpha}$.
Here $\sigma^\mu = (\mathbb{1}, \vec\sigma)$ and $\bar\sigma^\mu = (\mathbb{1}, -\vec\sigma)$ with Pauli matrices $\vec\sigma$. Explicitly:

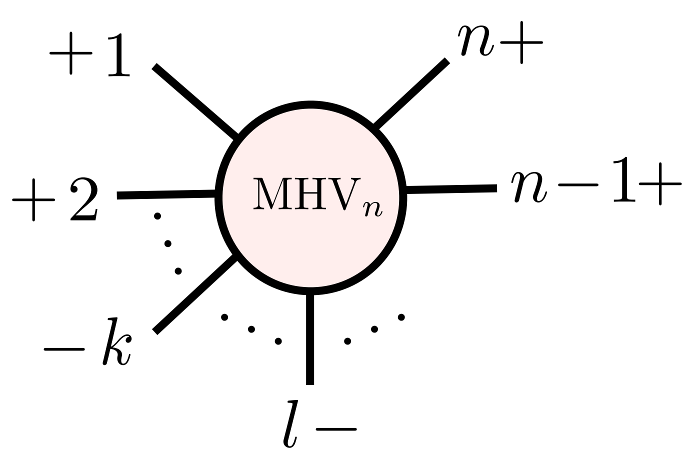
$$p_{\alpha\dot\alpha} = \begin{pmatrix} p_0 + p_3 & p_1 - i\,p_2 \\ p_1 + i\,p_2 & p_0 - p_3 \end{pmatrix}$$

Gluons are labeled by momenta p^μ with $p^2 = p^\mu p_\mu = \det p_{\alpha\dot\alpha} = 0$ and helicity ± 1. Momentum factors: $p_{\alpha\dot\alpha} = \lambda_\alpha \tilde\lambda_{\dot\alpha}$, shorthand for $= |\lambda_\alpha\rangle[\tilde\lambda_{\dot\alpha}|$.

Spinor-Helicity and Amplitudes, I

Good variables for YM$_{1+3}$.

For example, the (color stripped) MHV tree amplitudes for n particles are



$$= \frac{\langle k\ell\rangle^4}{\langle 12\rangle\langle 23\rangle\ldots\langle n-1,n\rangle\langle n1\rangle}\,\delta^4(P_{\alpha\dot\alpha}).$$

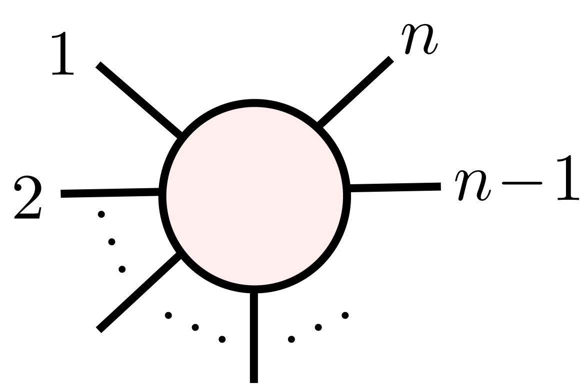
[Parke, Taylor '86]

Here $\langle k\ell\rangle = \epsilon^{\alpha\beta}\lambda_{k,\alpha}\lambda_{\ell,\beta}$ and $[k\ell] = \epsilon^{\dot\alpha\dot\beta}\tilde\lambda_{k,\dot\alpha}\tilde\lambda_{\ell,\dot\beta}$ and $P_{\alpha\dot\alpha} = \sum_j \lambda_{j,\alpha}\tilde\lambda_{j,\dot\alpha}$.

Otherwise, unbelievable mess in p^μ variables!

Spinor-Helicity and Amplitudes, II

There is a beautiful extension to maximally supersymmetric $\mathcal{N} = 4$ theory:
One introduces for each leg j a Graßmann spinor η_j^A where $A = 1, 2, 3, 4$.
With total "fermionic momentum" $Q_\alpha^A = \sum_j \lambda_{j,\alpha}\eta_j^A$ one gets



$$= \frac{\delta^4(P_{\alpha\dot{\alpha}})\delta^8(Q_\alpha^A)}{\langle 12\rangle\langle 23\rangle \ldots \langle n-1, n\rangle\langle n1\rangle}.$$

All external helicity configurations are generated by expansion in the η_j^A.

Graßmannian Integrals and Amplitudes

Build super-twistors $\mathcal{Z}_j^{\mathcal{A}} = (\tilde{\mu}_j^{\alpha}, \tilde{\lambda}_j^{\dot{\alpha}}, \eta_j^A)$ with Fourier conjugates $\lambda_j^{\alpha} \to \tilde{\mu}_j^{\alpha}$.
Graßmannian formulation of tree-level $\mathrm{N}^{k-2}\mathrm{MHV}_n$ amplitudes

$$\int \frac{\prod_{a=1}^{k} \prod_{i=k+1}^{n} dc_{ai}}{(1 \ldots k)(2 \ldots k+1) \ldots (n \ldots n+k-1)} \prod_{a=1}^{k} \delta^{4|4}\Big(\mathcal{Z}_a^{\mathcal{A}} + \sum_{i=k+1}^{n} c_{ai} \mathcal{Z}_i^{\mathcal{A}}\Big)$$

Integration is along "suitable contours". [Arkani-Hamed et.al. '09; Mason, Skinner '09]
The parameters c_{ai} are the non-trivial entries of a $k \times n$ matrix

$$C = \left(\begin{array}{c|cccc} & c_{1,k+1} & c_{1,k+2} & \cdots & c_{1,n} \\ \mathbb{I}_{k\times k} & \vdots & \vdots & \ddots & \vdots \\ & c_{k,k+1} & c_{k,k+2} & \cdots & c_{k,n} \end{array} \right)$$

A $\mathrm{GL}(k)$ symmetry fixes $\mathbb{I}_{k\times k}$. The $(i\, i+1 \ldots i+k-1)$ are $k \times k$ minors.

Spinor-Helicity, Graßmannian, and Amplitudes

Huge progress in last 10 years for MSYM$_{1+3}$, that is $\mathcal{N} = 4$ gauge theory.

Bonus:

- All tree level, n-particle, any helicity $n - 2k$ amplitudes known.
- Integrand of all L-loop n-particle amplitudes known (BCFW).
- Tree level Yangian invariance discovered, some evidence for 1-loop.

Malus:

- How to perform L-loop integrations? Infrared divergences!
- How to practically use Yangian invariance $=$ integrability?

Shortcomings of Current Approaches

1) Current approach to infrared divergences:

- Rewrite integrand in p^μ variables.
- Regulate $D = 4 \to D = 4 + 2\epsilon$.
- Extract appropriate finite pieces.

Drawback: $p = \lambda\tilde{\lambda}$ decomposition breaks down in $D = 4 + 2\epsilon$!

2) Current approach to Yangian invariance:

- Observe it. Admire it.

Drawback: Not clear how to make use of it!

Killing Two Birds with One Stone

Conclusions:

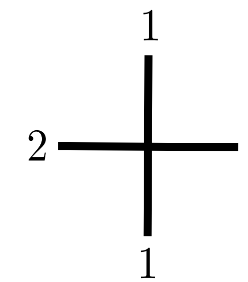
1) $D = 4 + 2\epsilon$, that is dimensional regularization, is too crude!
Stay in $D = 4$! But then, how to regulate?

2) Take inspiration from QISM (Quantum Inverse Scattering Method):
Introduce a spectral parameter z in order to practically use integrability.
But how?

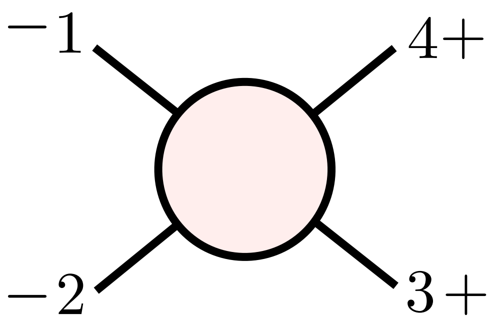
Idea:

Combine 1) and 2), find z, and use it instead of ϵ!
$\Rightarrow$ Replace dimensional regularization by spectral regularization.
Should be symmetry preserving!

A Puzzle: Three or Four?

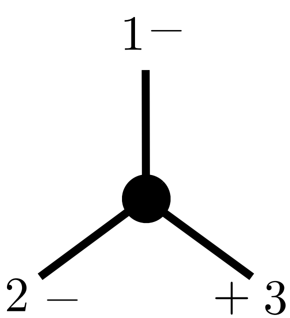
In the QISM, basic object is a 4-vertex, the R-matrix: $R(z) =$ 

Could try to relate this to 4-particle $\mathrm{MHV}_4 = \overline{\mathrm{MHV}}_4$:

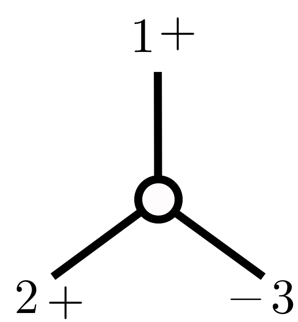


$$= \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \delta^4(\lambda_1 \tilde{\lambda}_1 + \ldots + \lambda_4 \tilde{\lambda}_4).$$

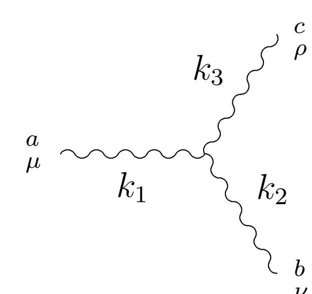
Puzzle: On-shell methods are rather based on 3-vertices: [Arkani-Hamed et.al. '12]



$$= \underbrace{\frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}}_{w_\bullet(1,2,3)} \delta^4(P),$$



$$= \underbrace{\frac{[12]^3}{[23][31]}}_{w_\circ(1,2,3)} \delta^4(P).$$

One can derive these directly from the YM Feynman rules: 

A Technical Problem with On-Shell 3-Vertices

In $\mathbb{R}^{1,3}$ cannot have $p_1^2 = p_2^2 = p_3^2 = 0$ along with $p_1 + p_2 + p_3 = 0$.
In spinor language, related to condition $(\lambda_\alpha)^* = \tilde{\lambda}_{\dot{\alpha}}$ of $\mathfrak{so}(1,3) = \mathfrak{sl}(2,\mathbb{C})$.

Way Out 1: Drop reality conditions, "complexify".
Somewhat confusing if we want to integrate over $\lambda, \tilde{\lambda}$.

Way Out 2: Wick-rotate $\mathfrak{so}(1,3) \longrightarrow \mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$.
Nice, since "everything stays real". So, with $p_{-1} = i\,p_2$, we have

$$p_{\alpha\dot{\alpha}} = \begin{pmatrix} p_0 + p_3 & p_1 - p_{-1} \\ p_1 + p_{-1} & p_0 - p_3 \end{pmatrix}$$

Now, $p^2 = 0 \quad \Leftrightarrow \quad p_0^2 + p_{-1}^2 - p_3^2 - p_1^2 = 0$.
Then $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ with independent $\lambda_\alpha \in \mathbb{R}^2$ and $\tilde{\lambda}_{\dot{\alpha}} \in \mathbb{R}^2$.

On-Shell Integration in $\mathbb{R}^{2,2}$

Let us see how this works. In “projective” coordinates t, x, y we have

$$p = (p_0, p_{-1}, p_3, p_1) = t\,(\tfrac{1+xy}{2}, \tfrac{x-y}{2}, \tfrac{1-xy}{2}, \tfrac{x+y}{2}).$$

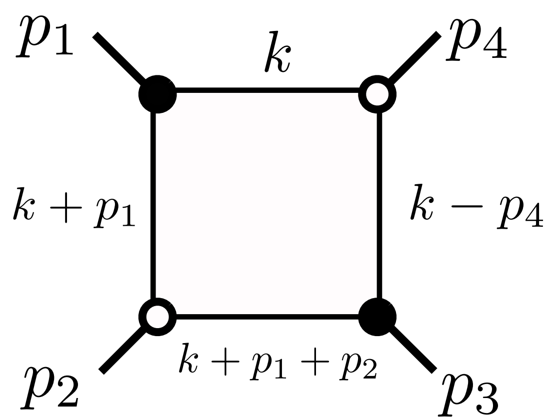
This yields

$$p_{\alpha\dot{\alpha}} = t\,\begin{pmatrix} 1 \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 & y \end{pmatrix} = \begin{pmatrix} t & ty \\ tx & txy \end{pmatrix}.$$

On-shell measure: $\int_{\mathbb{R}^{2,2}} d^4p\,\delta(p^2) = \int_{-\infty}^{\infty} \frac{|t|}{4} dt \int_{\mathbb{R}^2} dxdy.$

Four-Vertex Decomposition into Three-Vertices

Now the MHV_4 amplitude decomposes as [Arkani-Hamed et.al. '12]



$$p_1 + p_2 + p_3 + p_4 = 0.$$

We get

$$\int_{\mathbb{R}^{2,2}} d^4k\, \delta\left(k^2\right) \delta\left((k+p_1)^2\right) \delta\left((k+p_1+p_2)^2\right) \delta\left((k-p_4)^2\right)$$

$$\times w_\bullet(k, p_1, k+p_1)\tilde{w}_\circ(k+p_1, p_2, k+p_1+p_2)$$

$$\times w_\bullet(k+p_1+p_2, p_3, k-p_4)\tilde{w}_\circ(k-p_4, p_4, k)$$

Localization of Four-Point Amplitude, I

Four integrations, four delta functions, the integral completely localizes:

$$\int_{\mathbb{R}^{2,2}} d^4k \, \delta\left(k^2\right) \delta\left((k+p_1)^2\right) \delta\left((k+p_1+p_2)^2\right) \delta\left((k-p_4)^2\right) = \frac{-1}{st} \, ,$$

where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_4)^2$ are Mandelstam variables.

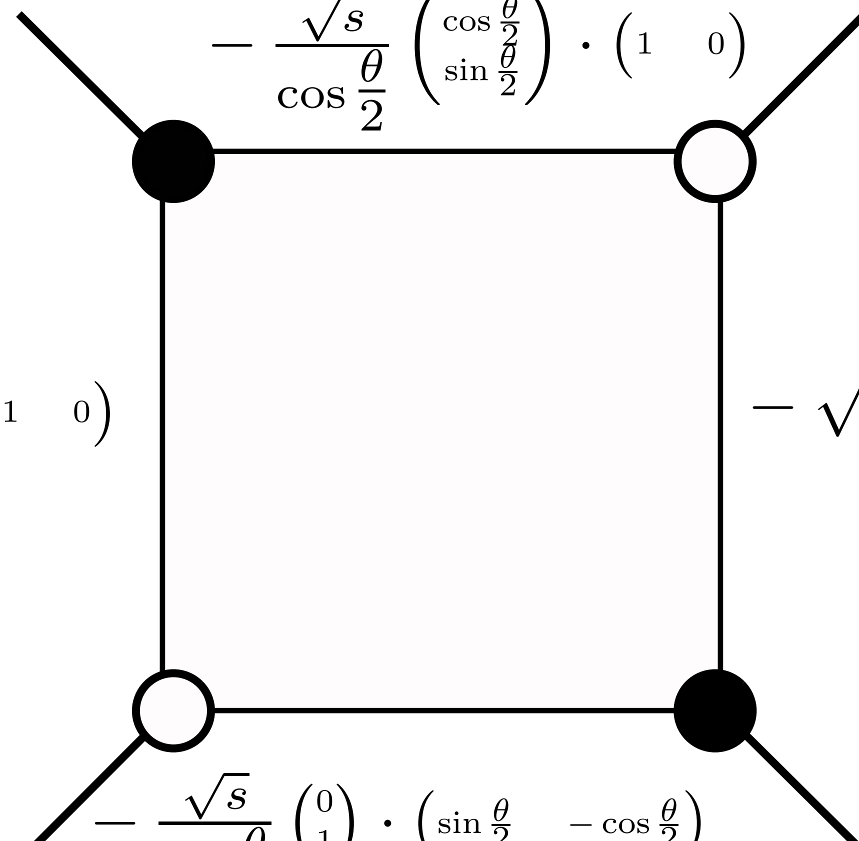
Multiplying this kinematical factor with the four 3-vertex weights, we get

$$\frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \, .$$

Parke-Taylor! Notice that all four components of k get fixed.

Localization of Four-Point Amplitude, II

Take in $\mathfrak{so}(1,3)$ momenta $p_1 = \frac{\sqrt{s}}{2}(1,0,0,1)$, $p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1)$, and $p_3 = \frac{\sqrt{s}}{2}(-1,\sin\theta,0,\cos\theta)$, $p_4 = \frac{\sqrt{s}}{2}(-1,-\sin\theta,0,-\cos\theta)$. Then



$$\sqrt{s}\begin{pmatrix}1\\0\end{pmatrix}\cdot\begin{pmatrix}1 & 0\end{pmatrix}$$

$$-\frac{\sqrt{s}}{\cos\frac{\theta}{2}}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}\cdot\begin{pmatrix}1 & 0\end{pmatrix}$$

$$-\sqrt{s}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}\cdot\begin{pmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\end{pmatrix}$$

$$-\sqrt{s}\tan\frac{\theta}{2}\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}1 & 0\end{pmatrix}$$

$$-\sqrt{s}\tan\frac{\theta}{2}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}\cdot\begin{pmatrix}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2}\end{pmatrix}$$

$$\sqrt{s}\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}0 & 1\end{pmatrix}$$

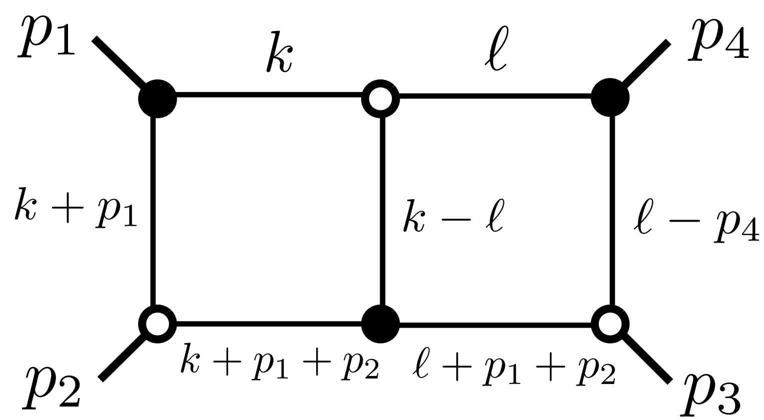
$$-\frac{\sqrt{s}}{\cos\frac{\theta}{2}}\begin{pmatrix}0\\1\end{pmatrix}\cdot\begin{pmatrix}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2}\end{pmatrix}$$

$$-\sqrt{s}\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}\cdot\begin{pmatrix}\sin\frac{\theta}{2} & -\cos\frac{\theta}{2}\end{pmatrix}$$

On-Shell Loop Diagrams

So we have seen that 1-loop on-shell = 0-loop off-shell! [Arkani-Hamed et.al. '12]

Now proceed:

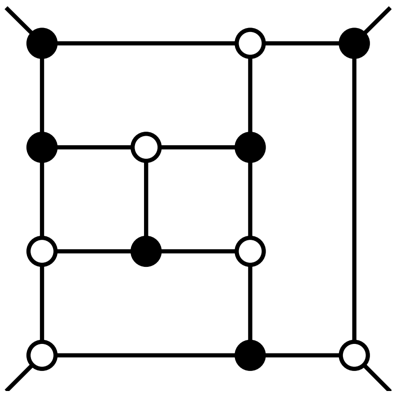


$$\underbrace{\int_{\mathbb{R}^{2,2}} d^4k \int_{\mathbb{R}^{2,2}} d^4\ell}_{8-\mathrm{dim}} \underbrace{\delta\left(k^2\right)\ldots\delta\left(\ell^2\right)}_{7-\mathrm{dim}} w_\bullet w_\circ \ldots w_\bullet w_\circ$$

$\Rightarrow$ One integration remains: 2-loop on-shell = $\frac{1}{4}$-loop off-shell.

Comment: Even $\frac{1}{4}$-loop is already infrared divergent!

One-Loop Off-Shell from On-Shell Diagrams



$$\underbrace{\int d^4k_1 \ldots \int d^4k_5}_{20-\mathrm{dim}} \; \underbrace{\delta(...)\ldots\delta(...)}_{16-\mathrm{dim}} \, w_\bullet \, w_\circ \ldots w_\bullet \, w_\circ$$

$\Rightarrow$ Four integrations remain: 5-loop on-shell = 1-loop off-shell. Result:

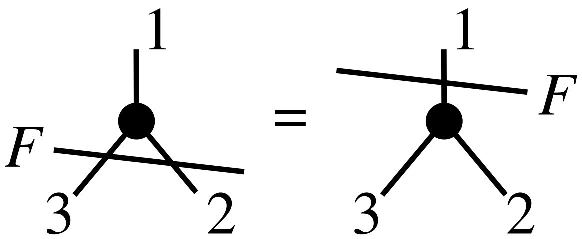
$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \times st \int d^4k \, \frac{1}{k^2(k+p_1)^2(k+p_1+p_2)^2(k-p_4)^2}$$

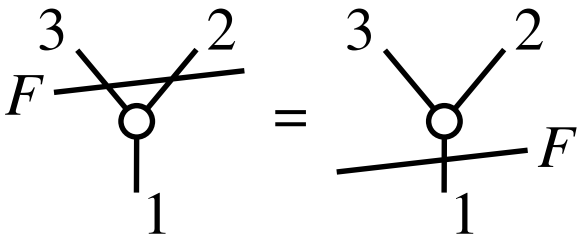
Infrared divergent!

Bootstrap Equations

Back to integrability and our question: Do 3-vertices appear in the QISM?

Answer: Yes, in bootstrap equations!



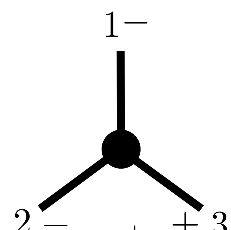


Similar but different from Yang-Baxter equations!

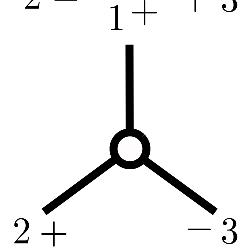
The equations involve spectral parameters.

Deformed Three-Point Vertices

The bootstrap equations may be solved, resulting in 3-point "R-matrices". With z_3 such that $z_1 + z_2 + z_3 = 0$, we found



$$= \frac{\langle 12 \rangle^4}{\langle 12 \rangle^{1+z_3} \langle 23 \rangle^{1+z_1} \langle 31 \rangle^{1+z_2}} \, \delta^4(P),$$



$$= \frac{[12]^4}{[12]^{1-z_3} [23]^{1-z_1} [31]^{1-z_2}} \, \delta^4(P).$$

Physical interpretation of the spectral parameters z_i :

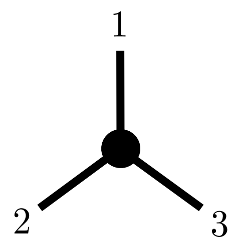
Helicity h_i is assigned to the spinors as $h(\lambda_i) = -\frac{1}{2}$ and $h(\tilde{\lambda}_i) = \frac{1}{2}$. Then

$$\frac{\langle 12 \rangle^4}{\langle 12 \rangle^{1+z_3} \langle 23 \rangle^{1+z_1} \langle 31 \rangle^{1+z_2}} = \frac{1}{\langle 12 \rangle^{h_1+h_2-h_3} \langle 23 \rangle^{h_2+h_3-h_1} \langle 31 \rangle^{h_3+h_1-h_2}}.$$

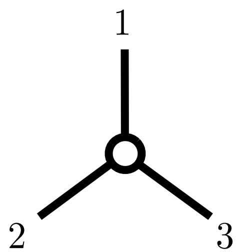
Analogy to 3-pt functions in CFT_2! In projective coord.: $\langle ij \rangle \simeq (x_i - x_j)$.

Deformed Supersymmetric Three-Point R-Matrices

In supersymmetric formulation, the deformed three-point amplitudes are



$$= R_\bullet = \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle^{1+z_3}\langle 23\rangle^{1+z_1}\langle 31\rangle^{1+z_2}}\,,$$



$$= R_\circ = \frac{\delta^4(P)\delta^4([12]\eta_3^A + [23]\eta_1^A + [31]\eta_2^A)}{[12]^{1-z_3}[23]^{1-z_1}[31]^{1-z_2}}\,.$$

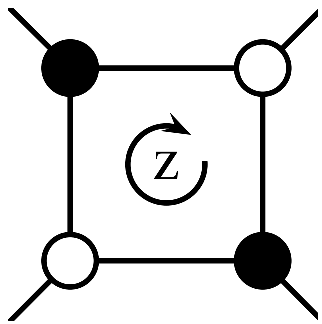
Mathematical interpretation of the spectral parameters z_i in terms of central charges: $C_j = \frac{1}{2}\left(\lambda_j\partial_{\lambda_j} - \tilde{\lambda}_j\partial_{\tilde{\lambda}_j} - \eta_j\partial_{\eta_j}\right) + 1$, since

$$C_j\, R_\bullet = -\frac{z_j}{2}\, R_\bullet\,, \qquad\qquad C_j\, R_\circ = -\frac{z_j}{2}\, R_\circ\,.$$

Four-Point R-Matrix

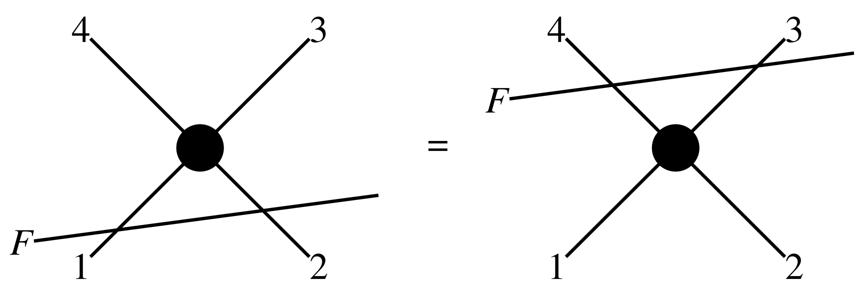
Back to 4-point functions. Glue together 4 deformed 3-point functions:

$R(z) :=$



$$= \frac{\delta^4(P)\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \times \underbrace{\left(\frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 14\rangle}\right)^z}_{\left(\frac{s}{t}\right)^z}$$

Satisfies the Yang-Baxter equation:



In fact, $R(z)$ turns out to be (the kernel of) the R-matrix of the one-loop spin chain of $\mathcal{N} = 4$! Generalization of an insight due to [Zwiebel '11].

Yang-Baxter Equation

Let $\mathbb{R}_{ij}(z)$ act non-trivially on $\mathbb{V}_i \otimes \mathbb{V}_j$ only. Then on $\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3$

$$\mathbb{R}_{23}(z_1)\mathbb{R}_{13}(z_2)\mathbb{R}_{12}(z_2 - z_1) = \mathbb{R}_{12}(z_2 - z_1)\mathbb{R}_{13}(z_2)\mathbb{R}_{23}(z_1).$$

Let $\mathbb{V}_3 = \mathbb{C}^{4|4}$. Let $J_{i\,\mathcal{B}}^{\mathcal{A}}$ be the generators of $\mathfrak{gl}(4|4)$ written in rep i and $e_{\mathcal{A}}^{\mathcal{B}}$ the ones of the fundamental rep. Put

$$\mathbb{R}_{i3}(z) \mapsto \mathbf{L}_i(z) := z\,\mathbb{I} + \sum_{\mathcal{A},\mathcal{B}} (-1)^{\mathcal{B}}\, J_{i\,\mathcal{B}}^{\mathcal{A}}\, e_{\mathcal{A}}^{\mathcal{B}}\,, \qquad \mathbb{R}_{12}(z) \mapsto \mathbf{R}_{12}(z)\,,$$

Linear equation for $\mathbf{R}_{12}$! Implies

$$[\mathbf{R}_{12}, J_1 + J_2] = 0\,,$$

$$(-1)^{\mathcal{B}}\mathbf{R}_{12}\, J_{1\mathcal{B}}^{\mathcal{A}}\, J_{2\mathcal{C}}^{\mathcal{B}} - (-1)^{\mathcal{B}}\, J_{2\mathcal{B}}^{\mathcal{A}}\, J_{1\mathcal{C}}^{\mathcal{B}}\mathbf{R}_{12} = z\big((J_2)_{\mathcal{C}}^{\mathcal{A}}\mathbf{R}_{12} - \mathbf{R}_{12}(J_2)_{\mathcal{C}}^{\mathcal{A}}\big).$$

Oscillator Representations

Schwinger oscillator realization of $\mathfrak{gl}(4|4)$

$$J^{\mathcal{A}}_{\mathcal{B}} = \bar{\mathbf{a}}^{\mathcal{A}}\,\mathbf{a}_{\mathcal{B}}\,, \qquad \text{where} \qquad [\mathbf{a}_{\mathcal{A}}, \bar{\mathbf{a}}^{\mathcal{B}}\} = \delta^{\mathcal{B}}_{\mathcal{A}}\,.$$

Particle-Hole transform to get to $\mathfrak{gl}(2,2|0+4)$

$$\left(\bar{\mathbf{a}}^{\mathsf{A}}\,,\,\mathbf{a}_{\mathsf{A}}\right) := \left(\bar{\mathbf{a}}^{\mathsf{A}}\,,\,\mathbf{a}_{\mathsf{A}}\right)\,, \qquad \left(\bar{\mathbf{b}}_{\dot{\mathsf{A}}}\,,\,\mathbf{b}^{\dot{\mathsf{A}}}\right) := \left(\mathbf{a}_{\dot{\mathsf{A}}}\,,\,-(-1)^{\dot{\mathsf{A}}}\,\bar{\mathbf{a}}^{\dot{\mathsf{A}}}\right)\,,$$

with Fock-vacuum

$$\mathbf{a}_{\mathsf{A}}\,|0\rangle = 0\,, \qquad \mathbf{b}^{\dot{\mathsf{A}}}\,|0\rangle = 0\,.$$

Interpretation in term of spinor-helicity variables:

$$\bar{\mathbf{a}}^{\mathsf{A}} \sim \lambda^{\alpha} \quad \bar{\mathbf{b}}_{\dot{\mathsf{A}}} \sim \{\,\tilde{\lambda}^{\dot{\alpha}},\; \eta^{A}\,\} \quad \mathbf{a}_{\mathsf{A}} \sim \frac{\partial}{\partial\,\lambda^{\alpha}} \quad \mathbf{b}^{\dot{\mathsf{A}}} \sim \{\,\frac{\partial}{\partial\,\tilde{\lambda}^{\dot{\alpha}}}\,,\,\frac{\partial}{\partial\,\eta^{A}}\,\}\,.$$

Yang-Baxter Equation and Oscillator Representations

Ansatz, with $\mathbf{N}$= total number operator,

$$\mathbf{R}_{12}(z) = \sum_{k,l,m,n} \alpha^{(\mathbf{N})}_{k,l,m,n}(z)\,\mathbf{Hop}_{k,l,m,n}$$

where $\mathbf{Hop}_{k,l,m,n}$ is

$$\frac{1}{k!\,l!\,m!\,n!}\,\bar{\mathbf{a}}_2^{\mathsf{A}_1}\ldots\bar{\mathbf{a}}_2^{\mathsf{A}_k}\,\bar{\mathbf{b}}^2_{\dot{\mathsf{A}}_1}\ldots\bar{\mathbf{b}}^2_{\dot{\mathsf{A}}_l}\,\bar{\mathbf{a}}_1^{\mathsf{B}_1}\ldots\bar{\mathbf{a}}_1^{\mathsf{B}_m}\,\bar{\mathbf{b}}^1_{\dot{\mathsf{B}}_1}\ldots\bar{\mathbf{b}}^1_{\dot{\mathsf{B}}_n}\times$$

$$\times\mathbf{b}_2^{\dot{\mathsf{B}}_n}\ldots\mathbf{b}_2^{\dot{\mathsf{B}}_1}\,\mathbf{a}^2_{\mathsf{B}_m}\ldots\mathbf{a}^2_{\mathsf{B}_1}\,\mathbf{b}_1^{\dot{\mathsf{A}}_l}\ldots\mathbf{b}_1^{\dot{\mathsf{A}}_1}\,\mathbf{a}^1_{\mathsf{A}_k}\ldots\mathbf{a}^1_{\mathsf{A}_1}$$

One finds, with $I = \frac{k+l+n+m}{2}$,

$$\alpha^{(\mathbf{N})}_I \simeq \frac{\Gamma(I+1)}{\Gamma(I+1-z-\frac{1}{2}\mathbf{N})\Gamma(z+\frac{1}{2}\mathbf{N}+1)}\,,$$

“Harmonic” R-matrix of the one-loop spin chain of $\mathcal{N}=4$.

Yang-Baxter Equation and Graßmannian Integral Rep

Super-twistors as $\mathfrak{gl}(4|4)$ oscillators, with $\mathcal{A} = 1, \ldots, 8$:

$$\bar{a}^{\mathcal{A}} \leftrightarrow \mathcal{Z}^{\mathcal{A}}, \qquad a_{\mathcal{A}} \leftrightarrow \frac{\partial}{\partial \mathcal{Z}^{\mathcal{A}}}.$$

R-matrix kernel:

$$(\mathbf{R}_{12}(z) \circ g)(\mathcal{Z}_3, \mathcal{Z}_4) = \int d^{4|4}\mathcal{Z}_1 d^{4|4}\mathcal{Z}_2 \, \mathcal{R}(z; \mathcal{Z}_3, \mathcal{Z}_4 | \mathcal{Z}_1, \mathcal{Z}_2) \, g(\mathcal{Z}_1, \mathcal{Z}_2)$$

C-matrix

$$\begin{pmatrix} 1 & 0 & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \end{pmatrix}$$

Solving Yang-Baxter yields for $\mathcal{R}(z)$ the deformed 4-pt amplitude:

$$\int \frac{dc_{13}\, dc_{14}\, dc_{23}\, dc_{24}}{c_{13}\, c_{24}\, \det C} \left(\frac{c_{13} c_{24}}{\det C}\right)^z \delta^{4|4}\Big(\mathcal{Z}_1 + \sum_{k=3}^{4} c_{1k} \mathcal{Z}_k\Big)\, \delta^{4|4}\Big(\mathcal{Z}_2 + \sum_{k=3}^{4} c_{2k} \mathcal{Z}_k\Big)$$

Recall: Graßmannian Integral Representation

Fourier conjugates $\lambda_j^\alpha \rightarrow \tilde{\mu}_j^\alpha$. Graßmannian formulation of tree-level $\mathrm{N}^{k-2}\mathrm{MHV}_n$ amplitudes

$$\int \frac{\prod_{a=1}^{k}\prod_{i=k+1}^{n} dc_{ai}}{(1\ldots k)(2\ldots k+1)\ldots(n\ldots n+k-1)} \prod_{a=1}^{k} \delta^{4|4}\Big(\mathcal{Z}_a^{\mathcal{A}} + \sum_{i=k+1}^{n} c_{ai}\mathcal{Z}_i^{\mathcal{A}}\Big)$$

Integration is along "suitable contours". The parameters c_{ai} are the non-trivial entries of a $k \times n$ matrix

$$C = \left(\begin{array}{c|cccc} & c_{1,k+1} & c_{1,k+2} & \cdots & c_{1,n} \\ \mathbb{I}_{k\times k} & \vdots & \vdots & \ddots & \vdots \\ & c_{k,k+1} & c_{k,k+2} & \cdots & c_{k,n} \end{array} \right)$$

A $\mathrm{GL}(k)$ symmetry fixes $\mathbb{I}_{k\times k}$. The $(i\, i+1 \ldots i+k-1)$ are $k \times k$ minors.

Bootstrap Equations and Graßmannian Integral Reps

There are four equations for the two deformed 3-pt amplitudes. E.g.

$$\big(\delta^{\mathcal{A}}_{\mathcal{C}} + (-1)^{\mathcal{C}}\frac{(J_3)^{\mathcal{A}}_{\mathcal{C}}}{\tilde{z}_1}\big)\,\mathcal{R}_\bullet = \mathcal{R}_\bullet\,\big(\delta^{\mathcal{A}}_{\mathcal{B}} + (-1)^{\mathcal{B}}\frac{(J_1)^{\mathcal{A}}_{\mathcal{B}}}{\tilde{z}_1}\big)\left(\delta^{\mathcal{B}}_{\mathcal{C}} + (-1)^{\mathcal{C}}\frac{(J_2)^{\mathcal{B}}_{\mathcal{C}}}{\tilde{z}_2}\right)$$

$$\big(\delta^{\mathcal{A}}_{\mathcal{C}} + (-1)^{\mathcal{C}}\frac{(J_3)^{\mathcal{A}}_{\mathcal{C}}}{\tilde{z}_3}\big)\,\mathcal{R}_\bullet = \mathcal{R}_\bullet\,\big(\delta^{\mathcal{A}}_{\mathcal{B}} + (-1)^{\mathcal{B}}\frac{(J_2)^{\mathcal{A}}_{\mathcal{B}}}{\tilde{z}_4}\big)\big(\delta^{\mathcal{B}}_{\mathcal{C}} + (-1)^{\mathcal{C}}\frac{(J_1)^{\mathcal{B}}_{\mathcal{C}}}{\tilde{z}_3}\big)$$

Solving them yields the black and white "vertices" in Graßmannian form

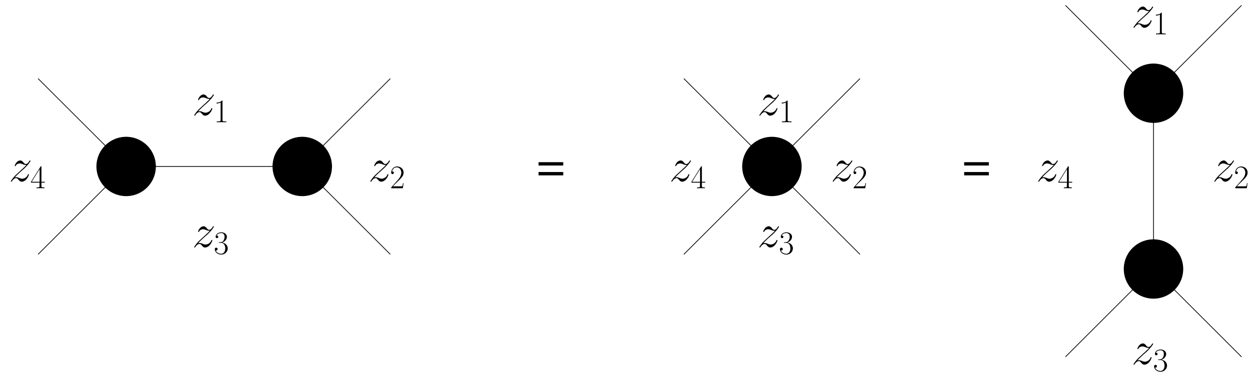
$$\mathcal{R}_\bullet = \int \frac{dc_{13}dc_{23}}{c_{13}c_{23}}\frac{1}{c_{13}^{z_1}c_{23}^{z_2}}\delta^{4|4}(\mathcal{Z}_1 + c_{13}\mathcal{Z}_3)\delta^{4|4}(\mathcal{Z}_2 + c_{23}\mathcal{Z}_3).$$

$$\mathcal{R}_\circ = \int \frac{dc_{12}dc_{13}}{c_{12}c_{13}}\frac{1}{c_{12}^{z_2}c_{13}^{z_3}}\delta^{4|4}(\mathcal{Z}_1 + c_{12}\mathcal{Z}_2 + c_{13}\mathcal{Z}_3),$$

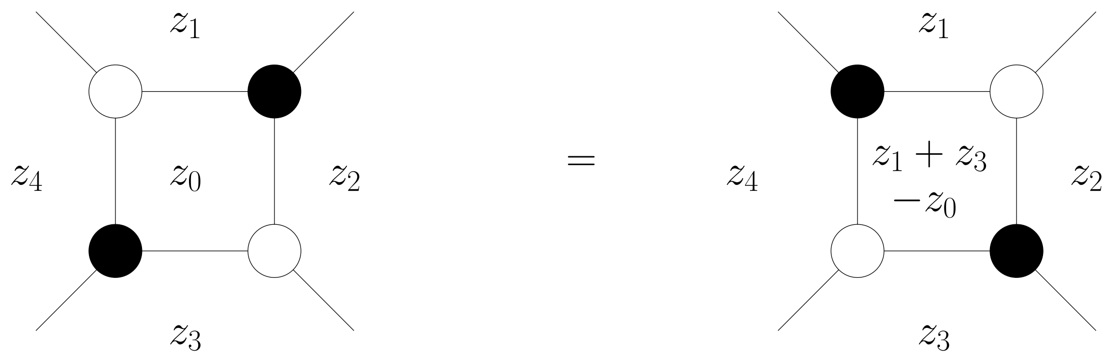
with the corresponding C-matrices $C_\bullet = \left(\begin{smallmatrix} 1 & 0 & c_{13} \\ 0 & 1 & c_{23}\end{smallmatrix}\right)$ and $C_\circ = \left(\begin{smallmatrix} 1 & c_{12} & c_{13}\end{smallmatrix}\right)$.

Face Moves and Square Moves

Face Moves:

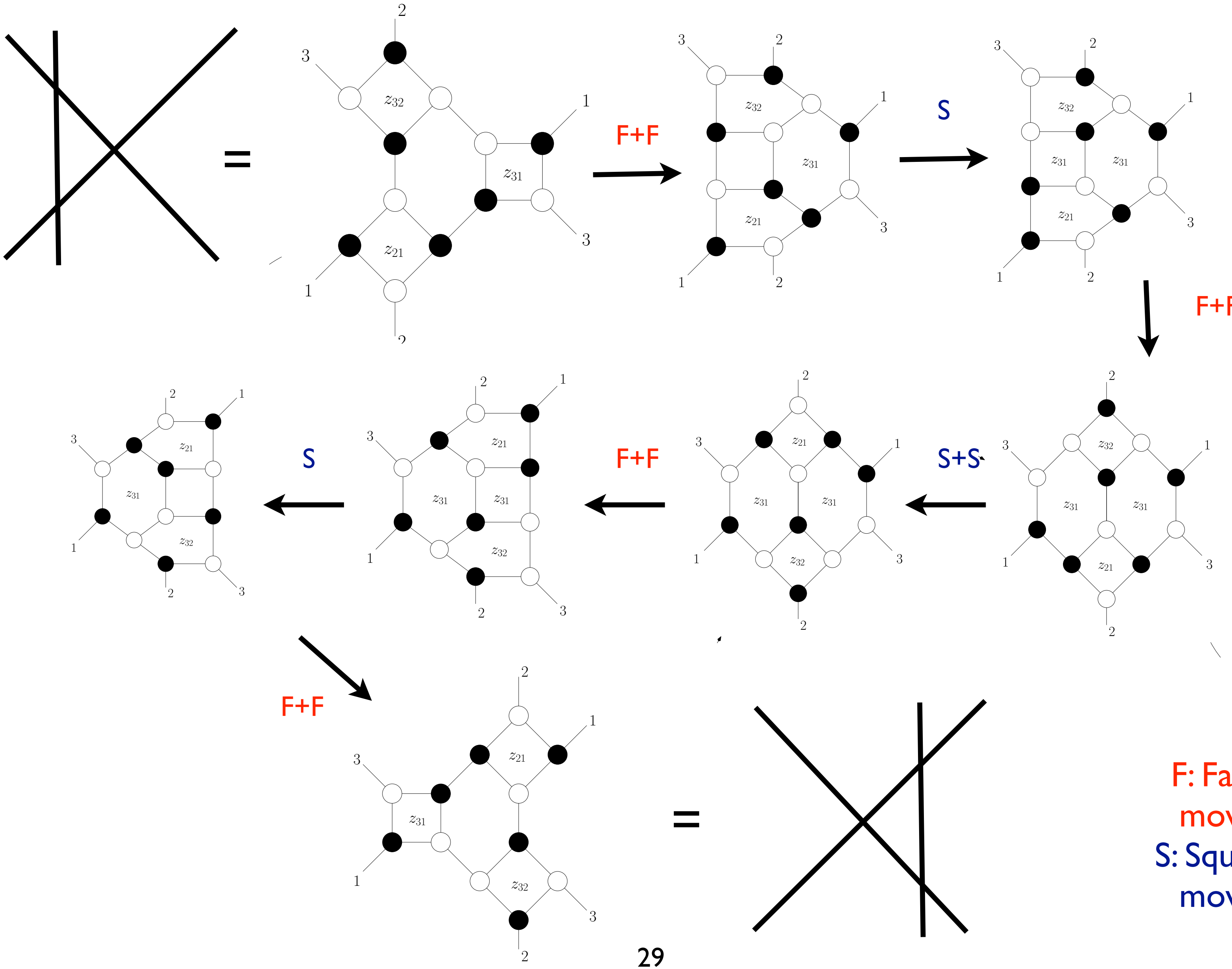


Square Moves



The face moves and the square moves are respected by the deformation!

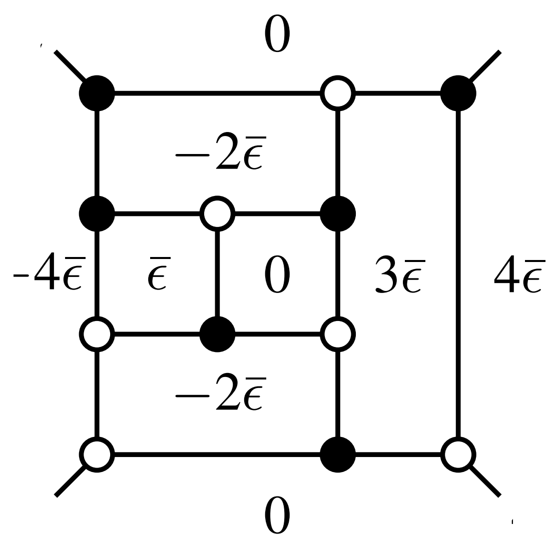
Yang-Baxter from Three-Vertices



F: Face move
S: Square move

Spectral Regularization of One-Loop Off-Shell Diagram

Finally, let us deform our earlier on-shell five-box diagram:



This deforms the off-shell one-box integral as

$$\int d^4k \, \frac{\left(\langle 34\rangle [21]\right)^{-4\bar{\epsilon}}}{k^{2(1-\bar{\epsilon})}(k+p_1)^{2(1-\bar{\epsilon})}(k+p_1+p_2)^{2(1-\bar{\epsilon})}(k-p_4)^{2(1-\bar{\epsilon})}}$$

Infrared convergent for $\bar{\epsilon} > 0$! We are integrating in exactly 4 dimensions!
Reminiscent, but different from so-called "analytic regularization".

Conclusions

- Spectral parameter for scattering amplitudes in $\mathcal{N} = 4$ SYM proposed.
- Physical interpretation as locally complexified helicity.
- Mathematical interpretation as locally complexified central charge.
- Evidence that it may serve as infrared regulator in 3+1 dimensions.
- Evidence that it respects all symmetries: superconformal, dual superconformal, Yangian.
- First concrete connection between the spectral problem and amplitudes.

Work in Progress

- Establish that spectral regularization works beyond one loop.

- Use it as an efficient calculational tool replacing dim reg by spec reg.

- Find a Bethe Ansatz for amplitudes = Yangian invariants?

  [Frassek, Kanning, Ko, MS to appear]

- Find all-loop cusp directly from amplitudes?

- Is spec reg also useful as ultraviolet regulator in exactly 3+1 dimensions?