Some remarks on Chern-Simons and Bott-Chern forms

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Euler Symposium on Theoretical and Mathematical Physics
St. Petersburg, July 12–17, 2013

Dedicated to the memory of Dmitri Diakonov

Plan

1. Chern-Simons forms

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2. Bott-Chern forms

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2. Bott-Chern forms
3. Results

Data

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- Chern-Weil theory: characteristic forms are closed, and their cohomology classes do not depend on the choice of A.

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- For Φ that corresponds to ch, $\tilde{\Phi}$ is denoted by cs.

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- $\operatorname{tr} F(A)^k = \mathrm{d}\, \mathrm{cs}_k(A)$, k-th CS action:

$$
\mathrm{CS}_k(A) = \int_M \mathrm{cs}_k(A), \quad \dim M = 2k - 1.
$$

Equations of motion: $F^{k-1}(A) = 0$.

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- **Proposition**

The image of the Chern character map contains all exact forms. Specifically, for every exact even form ω there is a trivial vector bundle $V=X\times\mathbb{C}^r$ with connection $\nabla=\mathrm{d}+A$ such that

$$\mathrm{ch}(V,A)-r=\omega.$$

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- Put $\widetilde{\mathcal{A}}^{\mathrm{odd}}(X) = \mathcal{A}^{\mathrm{odd}}(X)/d\mathcal{A}^{\mathrm{even}}(X)$ and $\mathrm{CS}(A_1, A_0) = \mathrm{cs}(A_1, A_0) \bmod \mathrm{d}\,\mathcal{A}^{\mathrm{even}}(X)$.

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  - For details see J. Simons–D. Sullivan, arXiv:0810.4935, and V. Pingali–L.T., arXiv:1102.1105.

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- Main question: describe the image of the Chern character map of virtual Hermitian bundles.

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satisfies the equation

$$\Phi(E, h_1) - \Phi(E, h_0) = \frac{\sqrt{-1}}{2\pi}\,\bar{\partial}\partial\,\widetilde{\Phi}(E; h_0, h_1)$$

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- For short exact sequence $\mathscr{E}$ of Hermitian holomorphic vector bundles

$$0 \longrightarrow F \xrightarrow{\ i\ } E \xrightarrow{\ p\ } H \longrightarrow 0$$

the Bott-Chern form satisfies

$$\Phi(E, h_E) - \Phi(F \oplus H, h_F \oplus h_H) = \frac{\sqrt{-1}}{2\pi}\,\bar{\partial}\partial\,\widetilde{\Phi}(\mathscr{E}; h_E, h_F, h_H),$$

and the functorial property. It vanishes when the exact sequence $\mathscr{E}$ holomorphically splits and $h_E = h_F \oplus h_H$.

Homotopy formula

- In the smooth manifold case, for the linear homotopy of connections $A_t = (1-t)A_0 + tA_1$, it is possible to integrate over t in the homotopy formula and obtain explicit formulas for the Chern-Simons forms, like $\mathrm{tr}(A\,\mathrm{d}\,A + \frac{2}{3}A^3)$, etc.

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- In the complex manifold case the situation is more complicated. There is a Bott-Chern homotopy formula

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but even for a linear homotopy $h_t = (1-t)h_0 + th_1$ of Hermitian metrics, it contains the inverse metrics through $F_t = \bar{\partial}(h_t^{-1}\partial h_t)$, which does not allow to integrate over t in a closed form. As the result, it is difficult to get explicit formulas for the Bott-Chern forms in terms of the Hermitian metrics h_0 and h_1 only.

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- Here we explicitly compute Chern and Bott-Chern forms for some special non-diagonal Hermitian metrics. Such formulas were not known before; this is a joint work with V. Pingali, see V. Pingali–L.T. arXiv:1102.1105.

Linear algebra over a ring with nilpotents

Lemma

Let A be a matrix over $\mathbb{C}$ or over a commutative algebra $\mathcal{A}$ over $\mathbb{C}$, where in the latter case all its matrix elements are nilpotent. Suppose that $A^2 = aA$ for some $a \in \mathcal{A}$, and that $1 - \lambda a$ is invertible for λ in some domain $D \subset \mathbb{C}$ containing 0. Then for such λ we have

$$(I - \lambda A)^{-1} = I + \frac{\lambda}{1 - \lambda a} A,$$

and

$$\det(I - \lambda A) = \exp\left\{\frac{\operatorname{tr} A}{a} \log(1 - \lambda a)\right\}.$$

In particular, if α_i, β_i, $i = 1, \ldots, k$, are odd elements in some graded-commutative algebra over $\mathbb{C}$ (e.g., the algebra of complex differential forms on X), and $A_{ij} = \alpha_i \beta_j$, then $A^2 = aA$ where $a = -\operatorname{tr} A = -\sum_{i=1}^{k} \alpha_i \beta_i$, and

$$\det(I - \lambda A) = \frac{1}{1 - \lambda a}.$$

Special Hermitian metrics

Proposition

Let $E_r = X \times \mathbb{C}^r$ be a trivial rank r vector bundle over X with a Hermitian metric $h = h(\sigma, f_1, \dots, f_{r-1}) = g^* g$, where

$$g = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \bar{f}_1 \\ 0 & 1 & 0 & \dots & 0 & \bar{f}_2 \\ 0 & 0 & 1 & \dots & 0 & \bar{f}_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \bar{f}_{r-1} \\ 0 & 0 & 0 & \dots & 0 & e^{\sigma/2} \end{pmatrix},$$

and $f_1, \dots, f_{r-1} \in C^\infty(X, \mathbb{C})$, $\sigma \in C^\infty(X, \mathbb{R})$. Then

$$\mathrm{c}(E_r, h) = \mathrm{c}(E_1, e^\sigma) + \frac{\sqrt{-1}}{2\pi} \bar{\partial}\partial \log\left(1 - \frac{\sqrt{-1}}{2\pi} U\right),$$

where

$$U = e^{-\sigma} \sum_{i=1}^{r-1} \partial f_i \wedge \bar{\partial}\bar{f}_i.$$

Remarks

- The following identities hold

$$\sum_{l=0}^{r}(-1)^l \operatorname{ch}_k(\Lambda^l E_r^*) = -\frac{\delta_{kr}}{r-1}\left(\frac{\sqrt{-1}}{2\pi}\right)^r \bar{\partial}\partial\left(U^{r-1}\right),$$

$k = 0, \cdots, r$, as it follows from Proposition and the following general formula

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$$\sum_{l=0}^{r}(-1)^l \operatorname{ch}(\Lambda^l E^*, \wedge^l h_*) = \operatorname{td}(E,h)^{-1}\operatorname{c}_r(E,h).$$

- For $r = 2$,

$$h = h(\sigma, f) = \begin{pmatrix} 1 & \bar{f} \\ f & |f|^2 + e^{\sigma}\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ f & e^{\sigma/2}\end{pmatrix}\begin{pmatrix} 1 & \bar{f} \\ 0 & e^{\sigma/2}\end{pmatrix},$$

  the identity takes the form

$$\operatorname{ch}_2(E_2, h(\sigma, f)) - \operatorname{ch}_2(E_1, e^{\sigma}) = -\frac{1}{(2\pi)^2}\bar{\partial}\partial(e^{-\sigma}\partial f\wedge\bar{\partial}\bar{f}),$$

  and can be verified by a straightforward computation.

Remarks cont.

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- For $r = 2$ put $\theta = h^{-1}\partial h$ and $\bar{\theta} = h^{-1}\bar{\partial} h$. Then

$$\mathrm{Tr}(\theta \wedge \bar{\theta}) = e^{-\sigma}\partial f \wedge \bar{\partial}\bar{f} + e^{-\sigma}\partial\bar{f} \wedge \bar{\partial} f.$$

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- The simplest nontrivial Bott-Chern form

$$\mathrm{bc}_2(h, I) = e^{-\sigma}\partial f \wedge \bar{\partial}\bar{f},$$

  where I is a trivial Hermitian metric on E_2, is obtained by adding the $(1,1)$-component of the "Wess-Zumino term" to the "kinetic term" $\mathrm{Tr}(\theta \wedge \bar{\theta})$.

Remarks cont.

- The exponents of Bott-Chern forms already appeared in the work of Losev-Moore-Nekrasov-Shatashvili on supersymmetric quantum field theories, as ratios of non-chiral partition functions for the higher dimensional versions of the *bc*-systems.
- For $r = 2$ put $\theta = h^{-1}\partial h$ and $\bar{\theta} = h^{-1}\bar{\partial} h$. Then

$$\mathrm{Tr}(\theta \wedge \bar{\theta}) = e^{-\sigma}\partial f \wedge \bar{\partial}\bar{f} + e^{-\sigma}\partial\bar{f} \wedge \bar{\partial} f.$$

- The simplest nontrivial Bott-Chern form

$$\mathrm{bc}_2(h, I) = e^{-\sigma}\partial f \wedge \bar{\partial}\bar{f},$$

  where I is a trivial Hermitian metric on E_2, is obtained by adding the $(1,1)$-component of the “Wess-Zumino term” to the “kinetic term” $\mathrm{Tr}(\theta \wedge \bar{\theta})$.
- This procedure was first used by Alekseev-Shatashvili, where for the case of the Minkowski signature the decomposition $h = g^* g$ is replaced by the Gauss decomposition for $\mathrm{SL}(2, \mathbb{C})$.

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Two versions of Bott-Chern action for $V = X \times \mathbb{C}^r$ — trivial bundle.

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- Equations of motion:

$$F(h)^k = 0.$$

Chern character map

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  For every $\bar{\partial}\partial$-exact form $\omega \in \mathcal{A}(X, \mathbb{C}) \cap \mathcal{A}^{\mathrm{even}}(X, \mathbb{R})$ on a complex manifold X there is a trivial vector bundle E over X with two Hermitian metrics h_0 and h_1 such that

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  - See V. Pingali-L.T. for the proofs.

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  - See V. Pingali-L.T. for the proofs.
  - Image of ch is an outstanding problem (related to the Hodge conjecture).

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- **Proposition**

Let $\mathscr{E}$ be a short exact sequence of holomorphic vector bundles over X, equipped with Hermitian metrics h_F, h_E and h_H, where the metric h_F on F is the restriction of the metric h_E on $i(F) \subset E$, and the metric h_H on H is defined by the C^∞ isomorphism between H and the orthogonal complement $i(F)^\perp$ of $i(F)$ in E. Let A be the second fundamental form of $i(F) \subset E$. In the case when F is a line bundle, the generating function for the Bott-Chern forms $\tilde{c}_t(\mathscr{E}; h)$ is given by the following explicit formula ($\kappa = \frac{\sqrt{-1}}{2\pi}$)

$$\tilde{c}_t(\mathscr{E}; h) = -c_t(H, h_H) \log\left(1 + \kappa t \operatorname{tr}(I + \kappa t \Theta_H)^{-1} A \wedge A^*\right).$$