Superstrings in AdS backgrounds

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based on work with Ben Hoare:
arXiv:1303.1037, 1304.4099

- Introduction / Review

- S-matrix for string in AdS3 x S3 x T4 with RR and NSNS flux:
  I. Perturbation theory
  II. Exact dispersion relation and S-matrix from symmetry considerations

Superstring theory in AdS   backgrounds:

important for gauge-string duality

AdS5 x S5 = 4d N=4 SYM

AdS4 x CP3 = 3d CS + matter

related to quantum black hole models

AdS3 x S3 -- extremal 5d BH

AdS2 x S2 -- extremal 4d BH

solving AdS x S string sigma models

an important problem

may be as fundamental as WZW models

Some key examples:

1. AdS5 x S5 + RR 5-form flux:
limit of D3-brane supergravity background

2a. AdS3 x S3 xT4 + RR 3-form flux:
limit of D5-D1 supergravity background
2b. AdS3 x S3 xT4 + NS-NS 3-form flux:
limit of NS5-NS1 supergravity background
2c. AdS3 x S3 xT4 +RR and NS-NS flux:
limit of "mixed" background

3. AdS2 x S2 xT6 + RR 5-form flux:
limit of D3-D3-D3-D3 supergravity background

Classical string motion in curved background

$$I = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\mathrm{d}\tau\left[G_{MN}(Y)\partial_a Y^M \partial^a Y^N + \epsilon^{ab} B_{MN}(Y)\partial_a Y^M \partial_b Y^N\right]$$

$$G_{MN}\dot{Y}^M \acute{Y}^N = 0$$

$$G_{MN}(\dot{Y}^M \dot{Y}^N + \acute{Y}^M \acute{Y}^N) = 0$$

p-brane background:

$$\mathrm{d}s^2 = f^{-2}(r)\,\mathrm{d}x^\mu \mathrm{d}x^\nu \eta_{\mu\nu} + f^2(r)(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_k^2)\ ,$$

$$f(r) = \left(1 + \frac{Q}{r^n}\right)^m, \qquad n, m \neq 0$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, ..., +1), \qquad \mu, \nu = 0, ..., p$$

Example: String motion in NS5-F1 background

include non-zero B- coupling: string motion in background produced by fundamental strings delocalised inside NS5 branes

$$\mathrm{d}s^2 = H_1^{-1}(r)(-\mathrm{d}t^2 + \mathrm{d}z^2) + \mathrm{d}x^i\mathrm{d}x_i + H_5(r)(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_3^2) \ ,$$

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2 + \cos^2\theta\,\mathrm{d}\phi^2 \ ,$$

$$B = -H_1^{-1}(r)\,\mathrm{d}t\wedge\mathrm{d}z + Q_5\sin^2\theta\ \mathrm{d}\varphi\wedge\mathrm{d}\phi \ ,$$

$$H_5 = 1 + \frac{Q_5}{r^2}, \qquad H_1 = 1 + \frac{Q_1}{r^2} \ ,$$

Large $Q_1 = Q_5 = Q$: $\qquad H_1 = H_5 \to h = \frac{Q}{r^2}$

$$ds^2 = \frac{r^2}{Q}(-dt^2 + dz^2) + \frac{Q}{r^2}(dr^2 + r^2 d\Omega_3) + dx_1^2 + \ldots + dx_4^2$$

$AdS_3 \times S^3 \times T^4$ with NS-NS 3-form flux:
$SU(1,1) \times SU(2) \times T^4$ WZW model

Example: D3-D3-D3-D3 intersection

$$ds^2_{10} = (T_1T_2T_3T_4)^{-1/2}\big[- T_1T_2T_3T_4 \; dt^2$$

$$+ T_1T_2dy_1^2 + T_1T_3dy_2^2 + T_1T_4dy_3^2 + T_2T_3dy_4^2 + T_2T_4dy_5^2 + T_3T_4dy_6^2 + dx_sdx_s\big]$$

$$\mathcal{F}_5 = dt \wedge (dT_1 \wedge dy_1 \wedge dy_2 \wedge dy_3 + dT_2 \wedge dy_1 \wedge dy_4 \wedge dy_5$$

$$+ dT_3 \wedge dy_2 \wedge dy_4 \wedge dy_6 + dT_4 \wedge dy_3 \wedge dy_5 \wedge dy_6)$$

$$+ \; * dT_1^{-1} \wedge dy_4 \wedge dy_5 \wedge dy_6 + *dT_2^{-1} \wedge dy_2 \wedge dy_3 \wedge dy_6$$

$$+ \; * dT_3^{-1} \wedge dy_1 \wedge dy_3 \wedge dy_5 + *dT_4^{-1} \wedge dy_1 \wedge dy_2 \wedge dy_4 \; .$$

$$T_i^{-1} = 1 + \frac{Q_i}{r}$$

$$Q_i = Q, \quad \frac{Q}{r} \gg 1, \qquad T_i \to \frac{Q}{r} = h$$

$$ds^2_{10} = -h^{-2}dt^2 + h^2(dr^2 + r^2 d\Omega_3) + dy_1^2 + ... + dy_6^2$$

Bertotti-Robinson $\quad AdS_2 \times S_2 \times T^6$ with RR 5-form flux

What does it mean to solve string theory:

compute spectrum of energies of string states

find corresponding vertex operators and their correlations functions (scattering amplitudes)

$$\text{String in } \; AdS_5 \times S^5 :$$
$$\text{tension } T = \frac{R^2}{4\pi\alpha'} = \frac{\sqrt{\lambda}}{4\pi} \quad = \text{h}$$

Spectrum:
energy as function E of tension or λ
and conserved charges (mode numbers)

Solving string theory in curved backgrounds
is hard: very few solvable examples known

CFT's admitting effective free-field realization:
(gauged) WZW, etc. -- solved in conformal gauge
simple plane-wave type models -- solved in l.c. gauge

Examples in AdS/CFT: AdS x S spaces
described by integrable sigma models

Novel way of solving for string spectra
in integrable cases is gradually emerging:
find l.c. gauge S-matrix and then construct
corresponding (Thermodynamic) Bethe Ansatz

Solution of integrable 2d sigma models:
[Zamolodchikovs; Polyakov, Wiegmann; Faddeev-Reshetikhin,...]
find S-matrix and then use it in BA

String theory in curved space:
sigma models appear in conformal gauge but:
(i) S-matrix in conformal gauge is not unitary -
how to implement Virasoro condition?
(ii) complication of superstrings in RR backgrounds:
GS action requires to start with bosonic background
for perturbative expansion

Way out: use "light-cone" gauge
(or other "physical" gauge like static gauge)

String sigma-models as found in conformal gauge are conformally invariant: no new scale, no mass generation

But spectrum of elementary excitations in l.c. gauge may be massive -- residual conformal invariance is "spontaneously broken" by choice of $x^+ = p^+ \tau$

formal scale is introduced by choice of l.c. gauge: fixed value of p^+

Non-trivial AdS x S sectors of S-matrix are massive but there may be also massless sectors (they decouple at tree level due to integrability but in general are important)

Subtleties:

1. for strings in curved target space l.c. gauge S-matrix is not 2d Lorentz-invariant

2. l.c. gauge S-matrix has reduced global symmetry

3. choice of l.c. gauge is effectively choice of particular vacuum (usually BPS)

4. no 2d relativistic invariance: elementary string excitations (“magnons”) get quantum corrections to dispersion relation -- get “dressed”

$$e^2 = 1 + p^2 + c_2\, p^4 + \ldots$$

Example of S-matrix approach to string spectrum: flat space Nambu action in static gauge - expansion near long string vacuum

$$x^0 = \tau, \quad x^1 = R\sigma$$

standard bosonic string spectrum in D dimensions:

$$E^2 = T^2R^2 + 4\pi T(N + \bar{N} - \frac{D-2}{12})$$

$N = \bar{N}$ if no momentum along string

Nambu action in static gauge:

$$I = T\int d\tau d\sigma \sqrt{-\det(\eta_{ab} + \partial_a X^i \partial_b X^i)}$$

$$L = \frac{1}{2}T[\partial_+ X^i \partial_- X^i + (\partial_+ X^i)^2(\partial_- X^j)^2 + ...]$$

$T_{++}T_{--}$ integrable perturbation of free theory
corresponding massless $2 \to 2$ S-matrix?
Perturbation theory: tree + 1-loop:
$S = 1 + iM, \quad M = \frac{1}{4}\ell^2 s - \frac{i}{32}\ell^4 s^2 \;, \qquad T \equiv \ell^{-2}$
Exact guess [Dubovsky, Flauger, Gorbenko]

$$S = e^{\frac{i}{4}\ell^2 s} \;, \qquad s = (p + p')^2$$

Consistent with general expectations for massless LR S-matrix
(massless RG flow between two CFT's) [Al.Zamolodchikov]
$S = \frac{1+ia^2 s}{1-ia^2 s} e^{iP(s)} \;, \qquad P \sim s^3 + ...$
Use this S-matrix in TBA equations to find energy spectrum
$e(p) = Rp - \int_0^\infty \frac{dp'}{2\pi}\phi(p,p')\bar{K}(p')$
$\bar{e}(p) = Rp - \int_0^\infty \frac{dp'}{2\pi}\phi(p,p')K(p')$
$\phi = -i\partial_{p'} \ln S(p,p') \;,\; K(p) = \ln(1 + e^{-e(p)}) \;,\; \bar{K}(p) = \ln(1 + e^{-\bar{e}(p)})$
$E(R) = -\int_0^\infty \frac{dp'}{2\pi}\; [K(p) + \bar{K}(p)]$
similar equations for excited states [also Caselle,Fioravanti,Gliozzi,Tateo]
reproduce string energy spectrum

$$E = \sqrt{T^2 R^2 + 4\pi T(N + \bar{N} - \frac{D-2}{12})}$$

Comments:

S-matrix is fixed by unitarity, crossing, analyticity
-- here simplest case of purely diagonal scattering

l.c. S-matrix here is trivial; in static-gauge it is non-trivial; gauge-equivalence only if gauge transformations are trivial at infinity

Nambu action is not well-defined at quantum level: to preserve integrability will require specific counter-terms; they will ensure equivalence with conformal-gauge (free-field) quantization/spectrum

Apply similar strategy
in integrable curved space string models:
physical-gauge S-matrix -- TBA -- spectrum

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2c. AdS3 x S3 xT4 +RR and NS-NS flux:
limit of "mixed" background (duality-rotated)

3. AdS2 x S2 xT6 + RR 5-form flux:
limit of D3-D3-D3-D3 supergravity background

Superstring actions (or their truncations) are related to supercoset sigma models

$\hat{F}/G$: $\hat{F}$= supergroup of superisometries of background bosonic part F: F/G is $AdS_n \times S^n$

$AdS_5 \times S^5$: $\hat{F} = PSU(2,2|4)$

$$\frac{SU(2,2)\times SU(4)}{Sp(2,2)\times Sp(4)} = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$$

$AdS_3 \times S^3$: $\hat{F} = PSU(1,1|2) \times PSU(1,1|2)$

$$\frac{[SU(1,1)\times SU(2)]^2}{SU(1,1)\times SU(2)} = \frac{SO(2,2)}{SO(1,2)} \times \frac{SO(4)}{SO(3)}$$

$AdS_2 \times S^2$: $\hat{F} = PSU(1,1|2)$

$$\frac{SU(1,1)\times SU(2)}{SO(1,1)\times U(1)} = \frac{SO(2,1)}{SO(1,1)} \times \frac{SO(3)}{SO(2)}$$

superalgebras admit Z_4 grading:
algebra of G + fermions + coset F/G part + fermions
$J = \hat{f}^{-1} d\hat{f} = J^{(0)} + J^{(1)} + J^{(2)} + J^{(3)}$
superstring action:

$$I = T \int d^2\sigma \; \mathrm{Str}\Big[\sqrt{g} g^{ab} J_a^{(2)} J_b^{(2)} + \epsilon^{ab} J_a^{(1)} J_b^{(3)}\Big]$$

Classical integrability (Lax pair), κ-symmetric, UV finiteness

Note: central charge does not receive curvature corrections
in low-dim models cancellation of conformal anomaly
still requires full $D = 10$ string superstring theory
[alternatively like in flat space extra modes will be required
in l.c. gauge to check analog of target space Lorentz symmetry]

inclusion of massless modes may be important
for realisation of symmetry at quantum level

Key example: AdS5 x S5 superstring

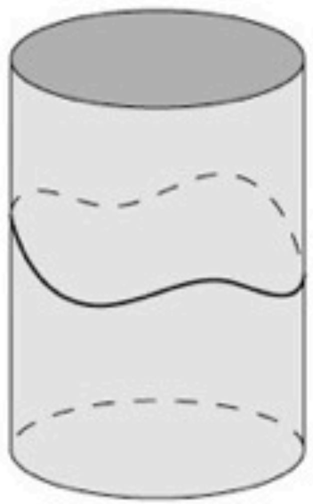
PSU(2,2|4) invariant supercoset GS sigma model:
string spectrum in light-cone gauge is given by TBA
corresponding to massive S-matrix that can be
constructed using symmetry / bootstrap methods
(unitarity, YBE, crossing)

Original construction was heavily guided
by spin chain picture which applies at weak coupling as
implied by duality to N=4 SYM theory

Idea is to reformulate solution so that it refers only to
string sigma model and then use similar strategy
to other cases where no spin chain picture is known

Strings on $AdS_5 \times S^5$

IIB superstrings on the curved $AdS_5 \times S^5$ superspace



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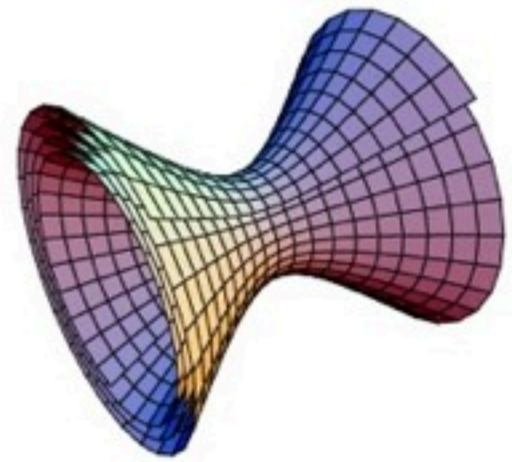

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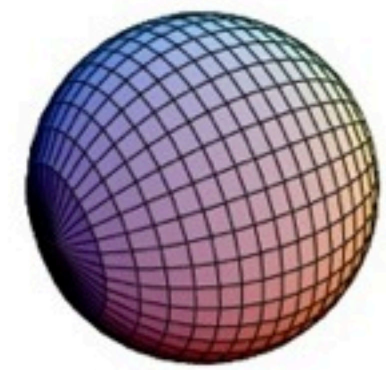


$\times$ fermions

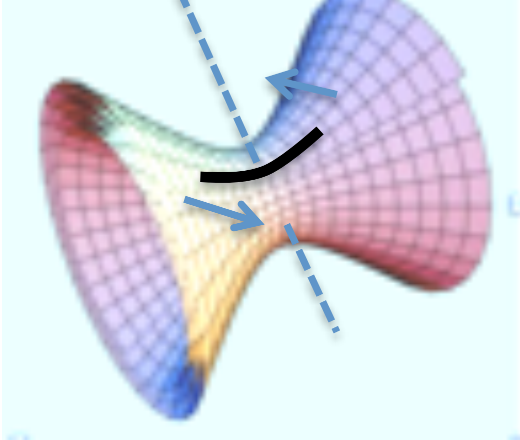
Coset space

$$AdS_5 \times S^5 \times \text{fermi} = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{Sp}(1,1) \times \mathrm{Sp}(2)}\,.$$

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \Big[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \Big]$$

$$-Y_0^2 - Y_5^2 + Y_1^2 + \ldots + Y_4^2 = -1\,, \qquad X_1^2 + \ldots + X_6^2 = 1$$

$$I = -\frac{\sqrt{\lambda}}{2\pi} \int d^2\xi \, [L_B(x,y) + L_F(x,y,\theta)] \, , \qquad \sqrt{\lambda} \equiv \frac{R^2}{\alpha'}$$



$$L_B = \frac{1}{2}\sqrt{-g}g^{ab}[G^{(AdS_5)}_{mn}(x)\partial_a x^m \partial_b x^n + G^{(S^5)}_{m'n'}(y)\partial_a y^{m'}\partial_b y^{n'}]$$

- Very **non-trivial** L_F (already in flat space). **Quadratic part**

$$\boxed{L_F = i(\sqrt{-g}g^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ})\bar{\theta}^I \varrho_a D_b \theta^J + O(\theta^4)}$$

$$\varrho_a \equiv \Gamma_A E^A_M \partial_a X^M = (\Gamma_p E^p_M + \Gamma_{p'} E^{p'}_M)\partial_a X^M$$

RR five form

$$D_a = \partial_a X^M \, D_M$$

$$D^{IJ}_M = (\partial_M + \tfrac{1}{4}\omega^{AB}_M \Gamma_{AB})\delta^{IJ} - \tfrac{1}{8\cdot 5!} F_{A_1\ldots A_5}\Gamma^{A_1\ldots A_5}\Gamma_M \epsilon^{IJ}$$

Strategy:

(i) symmetry algebra and its reduction in l.c. gauge

(ii) represent l.c. symmetry on particle states and find exact dispersion relation from algebra

(iii) construct 2-particle S-matrix consistent with symmetry, Yang-Baxter equation, unitarity, crossing; symmetry of l.c. S-matrix as Hopf algebra

(iv) check against perturbative S-matrix found directly from superstring action

special features: no 2d relativistic invariance; basic objects are "dressed" elementary excitations; non-locality of fermionic Noether charges: non-trivial co-product depending on 2d momentum

Exact S-matrices in relativistic 2d integrable models:

- no particle production; same initial/final sets of momenta;
- factorization of scattering into $2 \to 2$ processes
- $S_{12} = S(\theta), \;\; \theta = \theta_1 - \theta_2, \quad E = m\cosh\theta, \; p = m\sinh\theta$
- crossing: $S_{12}(\theta) = S_{\bar{2}1}(i\pi - \theta)$
- YBE: $\quad S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}$
- bound-state bootstrap [Zamolodchikovs]

Quantum-deformed symmetry: Hopf algebra

symmetry may act on 2-partcle states / S-matrix

with non-trivial co-product

$S = PR: \quad R: \; V_1 \times V_2 \to V_1 \times V_2,$

V: representation of algebra A

$\Delta: \;\; A \to A \times A\,, \quad [\Delta(a), \Delta(b)] = \Delta([a, b])$

$\Delta(J_i) = J \times 1 + U^{k_i} \times J_i$

$AdS_5 \times S^5$:

l.c. gauge symmetry: centrally extended $psu(2|2) \oplus psu(2|2)$
from spin chain perspective [Beisert 05]
dynamical spin chain (due to presence of fermions) –
interpreted as non-trivial Hopf algebra with $U = e^{ip}$
[Gomez,Hernadez; Plefka,Spill,Torrielli 06]

from string world-sheet perspective:
supersymmetry generators are non-local in l.c. gauge
[Arutyunov,Frolov,Plefka,Zamaklar; Klose,McLoughlin,Roiban,Zarembo]

$$Q = \int_{-\infty}^{\infty} d\sigma \mathcal{J}(\sigma) \exp i \int_{-\infty}^{\sigma} d\sigma' \partial x^-(\sigma')$$

semiclassical scattering state as two separated soliton excitations:
[cf: Luscher,Pohlmeyer]
action of Q: $Q(1+2) \sim Q(1) + e^{ip} Q(2)$
may be interpreted as $\Delta(Q) = Q \times 1 + e^{ip} \times Q$

Quantum corrected dispersion relation

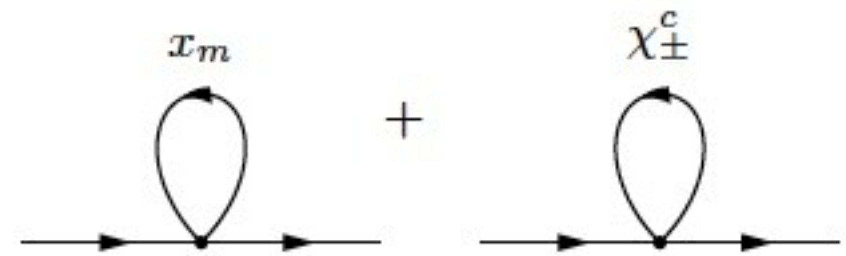
no relativistic invariance: BMN disperaion relation may receive quantum corrections

$$E = \sqrt{1 + 4h(g)^2 \sin^2 \frac{p_1}{2}}, \qquad h(g) = \frac{g}{2\pi} + \dots$$

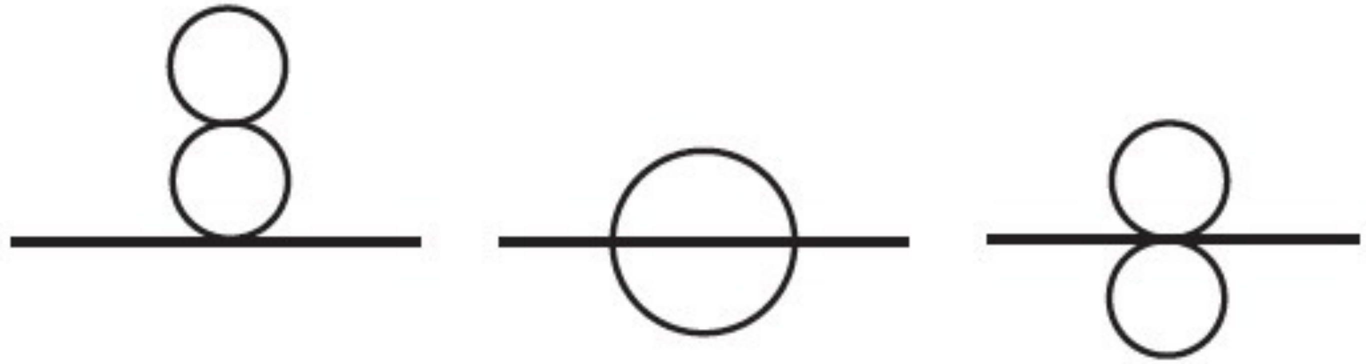
$$h(g) = \frac{g}{2\pi} + c \qquad\qquad g = \sqrt{\lambda}$$

$$E = \sqrt{1 + 4(\frac{g}{2\pi} + c)^2 \sin^2 \frac{p_1}{2}} = \sqrt{1+p^2} + \frac{2\pi c\, p^2}{g\sqrt{1+p^2}} + \mathcal{O}(g^{-2})$$

If c is non-zero, then the one-loop correction to the two-point functions is also non-zero.



$$c = 0\text{: no shift of } h$$



two-loop correction shifts the bare pole of the propagator

$$\frac{i}{\mathbf{p}^2 - 1} \rightarrow \frac{i}{p_0^2 - 1 - p_1^2 + \frac{\gamma^2}{192} p_-^4} + \ldots$$

$$E = \sqrt{1 + 4h^2 \sin^2 \frac{p_1}{2}} = \sqrt{1 + p_1^2 - \frac{1}{192h^2} p_-^4 + \ldots}$$

[Klose,McLoughlin,Minahan,Zarembo 07]

$AdS_5 \times S^5$:

centrally extended l.c. gauge symmetry, coproduct,
and general requirements on S-matrix:
enough to fix the exact dispersion relation and string S-matrix
for 8+8 physical massive excitations ("dressed" BMN fluctuations)
including dressing phase [Beisert, Eden, Staudacher, ...]

then "diagonalize" S-matrix to find asymptotic BA
which matches result of spin-chain construction
[Beisert,Staudacher; Plefka et al; de Leeuw]

same approach can be applied to superstring in $AdS_4 \times CP^3$
dual to 3d ABJM Chern-Simons + matter theory

AdS3/CFT2 and AdS2/CFT1 cases?
solve string theory from first principles using integrability:
shed light on dual CFTs which are not well understood

Strings in AdS3 x S3 x T4 with RR background:

limit of D5-D1 system, expected dual CFT is deformation of symmetric product orbifold SymN(T4) 2d model with (4,4) superconformal symmetry
[Maldacena,Strominger 98; Seiberg,Witten 99; Larsen,Martinec 99]

PSU(1,1|2) x PSU(1,1|2) invariant GS supercoset action is classically integrable -- suggests BA based solution
[Babichenko,Stefanski,Zarembo 09; David,Sahoo 10; Ohlsson Sax,Stefanski 10, Sundin,Wulff 12, ...]

integrability of dual CFT is a question [Pakman,Rastelli,Razamat 09; ...]

Plan: consider l.c. gauge and concentrate on massive modes only (ignore first T4 and massless fermions)

(i) use residual psu(1|1) x psu(1|1) BMN vacuum symmetry and bootstrap method to construct massive S-matrix for exact ("dressed") excitations;
check vs string S-matrix for elementary string excitations

(ii) find asymptotic BA "diagonalising" this S-matrix
[Borsato,Ohlsson Sax,Sfondrini,Stefanski,Torrielli 13]

[(iii) TBA for full spectrum may require massless modes]

Close analogy with AdS5 x S5 and AdS4 x CP3 cases:
suggests effective "discrete world sheet" or "spin chain" picture behind quantum string sigma model solution

Finding the right TBA generalization may require

inclusion of massless modes

[attempt to include massless modes from limit of AdS3 x S3 x S3 x S1 case [Ohlsson Sax,Stefanski,Torrielli 12]]

Massive theory is formally consistent;

would be no problem in conformal gauge:

correct description of relevant part of spectrum

l.c. gauge string theory in D < 10 is formally consistent

on cylinder but analog of full

D=10 Lorentz symmetry is not realized

inconsistency will show up

at string 1-loop order (torus topology)

S-matrix is fixed by symmetries and YBE up to
two scalar phases which are determined
from crossing relations
[Borsato,Ohlsson Sax,sfondrini,Stefanski,Torrielli 12]

matches string perturbation theory results
[Beccaria,Levkovich-Maslyuk,Macorini,AT 12; Sundin,Wulff 13; Abbott 13]

Generalization: AdS3 x S3 x T4 string with RR+NSNS 3-form flux

type IIB background: near-horizon limit of non-threshold bound state of D5-D1 and NS5-NS1 system

$$\text{RR flux} \sim \hat{q} \qquad\qquad \text{NSNS flux} \sim q$$

$$q^2 + \hat{q}^2 = 1 \qquad\qquad 0 \leq q \leq 1$$

supergravity solutions with different q related by S-duality: supersymmetry and BPS states are same but perturbative string spectrum is not - q is non-trivial parameter [can not use S-duality to find dual CFT beyond BPS states]

PSU(1,1|2) x PSU(1,1|2) invariant GS superstring action is integrable for any q [Cagnazzo,Zarembo 12]

Aim: find string spectrum
for any value of string tension h and q

One particular motivation:
understand interpolation between spectra of
q=0 theory (pure RR flux)
solved in l.c. gauge using integrability/S-matrix/BA
and q=1 theory (pure NSNS flux)
solved in conformal gauge using chiral decomposition
of (super)WZW model [..., Maldacena, Ooguri 01, ...]

Plan: [Hoare, AT 13]

1. Lagrangian and tree-level S-matrix in l.c. gauge
2. Exact S-matrix from symmetry considerations

Superstring in AdS3 x S3 with mixed 3-flux

$$\frac{PSU(1,1|2)_L \times PSU(1,1|2)_R}{SU(1,1) \times SU(2)}$$

algebra has $\mathbb{Z}_4$ orthogonal decomposition

$$J = g^{-1}dg = J_0 + J_1 + J_2 + J_3 \ , \qquad dJ + J \wedge J = 0$$

$$\mathrm{S} = \frac{\sqrt{\lambda}}{2\pi}\Big[\int d^2x \ \tfrac{1}{2}\mathrm{STr}\Big[J_{2+}J_{2-} + \tfrac{1}{2}\sqrt{1-q^2}(J_{1+}J_{3-} - J_{1-}J_{3+})\Big] \\ - q\int d^3x \ \epsilon^{abc}\,\widetilde{\mathrm{STr}}\Big[\tfrac{2}{3}J_{2a}J_{2b}J_{2c} + J_{1a}J_{3b}J_{2c} + J_{3a}J_{1b}J_{2c}\ \Big]$$

[Pesando 99; Cagnazzo,Zarembo 12]

$q = 0$: superstring action for pure RR background

$q = 1$: superstring action for pure NSNS background

– GS version of super $SU(1,1) \times SU(2)$ WZW model

$$S = AdS_3 + S^3 \text{+fermions}$$

bosonic $q \neq 1$ model: principal chiral model with WZ term
RR fermionic couplings make it conformal for any q

Bosonic theory: $SU(1,1) \times SU(2)$

$$\mathrm{S} = \tfrac{1}{2}\mathrm{h}\Big[\int d^2\sigma\ \tfrac{1}{2}\mathrm{tr}(J^a J_a) + q\int d^3\sigma\ \tfrac{1}{3}\epsilon^{abc}\mathrm{tr}(J_a J_b J_c)\Big] ,$$

$$J_a = g^{-1}\partial_a g , \qquad \mathrm{h} = \frac{R^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi} .$$

equations of motion can be written as ($\partial_\pm = \partial_0 \pm \partial_1$)

$$(1-q)\partial_+ J_- + (1+q)\partial_- J_+ = 0 , \qquad \partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0 ,$$

the Lax pair is given by

$$\mathcal{L}_\pm = \tfrac{1}{2}(1 \pm q + z^{\pm 1}\sqrt{1-q^2})\, J_\pm$$

$$k = 2\pi h q = \text{ integer}$$

$q = 1$ is WZW theory case

Hidden simplicity of dependence on q

visible when theory is formulated in terms of currents

Classical equations:

$$\partial_+ J_- + (1+q)[J_+, J_-] = 0$$

$$\partial_- J_+ - (1-q)[J_+, J_-] = 0$$

Conformally invariant, e.g., $\sigma^\pm \to a_\pm^{-1}\sigma^\pm$, $J_\pm \to a_\pm J_\pm$

But can formally use either

$J_\pm \to (1\pm q)J_\pm$ or $\sigma^\pm \to (1\pm q)\sigma^\pm$

to generate $q \neq 0$ solutions from solutions of $q = 0$ theory

Faddeev-Reshetikhin model for the string on $R \times S^3$ with B-flux

$$\mathrm{tr} J_\pm^2 = -2\mu^2 \qquad J_\pm = i\mu S_\pm^k \hat{\sigma}^k \ , \qquad S_\pm^k S_\pm^k = 1$$

$$\partial_+ S_-^i + \mu(1+q)\epsilon^{ijk} S_+^j S_-^k = 0 \ , \qquad \partial_- S_+^i - \mu(1-q)\epsilon^{ijk} S_+^j S_-^k = 0$$

$$\mathrm{S} = \int d^2\sigma \, \Big[(1-q) C_+(S_-) + (1+q) C_-(S_+) - \tfrac{1}{2}(1-q^2)\mu^2 \, S_+^k S_-^k \Big]$$

$$C_\pm(S) \equiv -\tfrac{1}{2} \int_0^1 dx \ \epsilon^{ijk} S_i \partial_x S_j \partial_\pm S_k \ , \qquad \delta C_\pm = \tfrac{1}{2} \epsilon^{ijk} \delta S_i S_j \partial_\pm S_k$$

$$\tilde{\sigma}^+ = (1+q)\sigma^+ , \qquad \tilde{\sigma}^- = (1-q)\sigma^- , \qquad \sigma^\pm = \tfrac{1}{2}(\tau \pm \sigma)$$

$$\tilde{\mathrm{S}} = \int d^2\tilde{\sigma} \left[\tilde{C}_+(S_-) + \tilde{C}_-(S_+) - \tfrac{1}{2} S_+^k S_-^k\right]$$

$$\tilde{\tau} = \tau + q\sigma , \qquad \tilde{\sigma} = \sigma + q\tau$$

$$e = \tilde{e} + q\tilde{p} , \qquad p = \tilde{p} + q\tilde{e} , \qquad \tilde{e} = \frac{e - qp}{1-q^2} , \qquad \tilde{p} = \frac{p - qe}{1-q^2}$$

Suggests that q dependence of S-matrix should be simple

Limit of fast motion: Landau-Lifshits model ($\mu \gg 1$)

$$L = \mu(C_0 + qC_1) - \frac{1}{4}(1-q^2)(\partial_1 n_i)^2 , \qquad n_i n_i = 1$$

$$L = \mu \cos 2\theta \ (\dot{\beta} + q\beta') - \frac{1}{2}(1-q^2)(\theta'^2 + \sin^2 2\theta \ \beta'^2)$$

same coordinate transformation applies also in Pohlmeyer reduced theory which is Lorentz invariant, so simply $\mu \to \sqrt{1-q^2}\mu$

Bosonic S^3 part of action

$$L_S = \tfrac{1}{2}\Big[\partial_+\theta\partial_-\theta + \sin^2\theta\ \partial_+\phi_1\partial_-\phi_1 + \cos^2\theta\ \partial_+\phi_2\partial_-\phi_2$$
$$+ q\sin^2\theta\ (\partial_+\phi_1\partial_-\phi_2 - \partial_+\phi_2\partial_-\phi_1)\Big]$$

$$L_S = -\tfrac{1}{2}\Big[G(y)\partial^a\varphi\partial_a\varphi + F(y)\partial^a y_s\partial_a y_s + 2B_s(y)\epsilon^{ab}\partial_a y_s\partial_b\varphi\Big]\ ,$$
$$G = \frac{(1-\frac{1}{4}y^2)^2}{(1+\frac{1}{4}y^2)^2} = 1 - y^2 F\ , \qquad F = \frac{1}{(1+\frac{1}{4}y^2)^2}\ ,$$
$$B_s = qF(y)\ \hat{y}_s\ , \qquad \hat{y}_s \equiv \epsilon_{rs}y_r\ .$$

Dispersion relation

BMN geodesic $\quad t = \mathcal{J}\tau\ , \qquad \varphi = \mathcal{J}\tau\ , \qquad y_s = 0$

expand near big circle in $R \times S^3$: $\quad$ 2 transverse directions y_s

$$L = \tfrac{1}{2}(\dot{y}_r^2 - y_r'^2 - \mathcal{J}^2 y_r^2) + q\mathcal{J}\epsilon_{sr}y_s y_r' + O(y^4)$$
$$= \tfrac{1}{2}\Big[\dot{y}_r^2 - (y_r' - \mathcal{J}q\epsilon_{sr}y_s)^2 - \mathcal{J}^2(1-q^2)y_r^2\Big] + O(y^4)$$

Dispersion relation

$$e^2 - (p \pm \mathcal{J} q)^2 = (1 - q^2)\mathcal{J}^2 \ , \qquad e \equiv p_0 \ , \qquad p \equiv p_1$$
$$e = \pm\sqrt{p^2 \pm 2\mathcal{J} qp + \mathcal{J}^2} \ ,$$

rescaling $\mathcal{J}$: $\quad e = \sqrt{\hat{p}^2 + 1 - q^2} \ , \qquad \hat{p} = p \pm q$

on infinite line - continuous p - can shift
to get standard dispersion relation with $m = \sqrt{1 - q^2}$

2-particle S-matrix

Gauge fixing and expansion of the action to quartic order

$$t = u - b\varphi \ , \qquad b \equiv \frac{a}{1-a}$$

$$u = \mathrm{c}\,\tau \ , \qquad \tilde{\varphi} = \mathrm{c}\,\mathcal{J}\,\sigma \ , \qquad \mathrm{c} \equiv \frac{1}{1-a}$$

l.c. gauge and fix momentum along φ
or apply T-duality $\varphi \to \tilde{\varphi}$ and then fix static gauge

a=gauge parameter, $a = \frac{1}{2}$ is l.c. gauge

$$\tilde{L} = -\sqrt{h} + b\,\mathrm{c}P(\mathrm{c} - B_s y'_s) \ ,$$
$$h = \left[\mathrm{c}^2 Q - P(B_r \dot{y}_r)^2 - F\dot{y}_r^2\right]\left[P(\mathrm{c} - B_s y'_s)^2 + F y_s'^2\right] + \left[P B_s \dot{y}_s(\mathrm{c} - B_r y'_r) - F\dot{y}_r y'_r\right]^2$$
$$Q = 1 + b^2 P \ , \qquad\qquad P = (G - b^2)^{-1} \ .$$

$$\begin{aligned}
\tilde{L} &= L_2 + L_4 + \ldots \ , \\
L_2 &= \tfrac{1}{2}(\dot{y}_s^2 - y_s'^2 - y_s^2) + q\epsilon_{sp} y_s y'_p \ , \\
L_4 &= \tfrac{1}{2} y_s^2 y_r'^2 + \tfrac{1}{2} q\left[\dot{y}_r y'_r \epsilon_{sp} y_s \dot{y}_p - \tfrac{1}{2}(\dot{y}_r^2 + y_r'^2 + y_r^2)\epsilon_{sp} y_s y'_p\right] \\
&\qquad + (a - \tfrac{1}{2})\Big\{ \tfrac{1}{4}(y_s^2)^2 - \tfrac{1}{4}(\dot{y}_s^2 + y_s'^2)^2 + (y'_s \dot{y}_s)^2 \\
&\qquad\qquad + q\left[-\dot{y}_r y'_r \epsilon_{sp} y_s \dot{y}_p + \tfrac{1}{2}(\dot{y}_r^2 + y_r'^2 - y_r^2)\epsilon_{sp} y_s y'_p\right]\Big\}
\end{aligned}$$

$$y = y_1 + i y_2 = e^{iq\sigma} v \ , \qquad\qquad y^* = y_1 - i y_2 = e^{-iq\sigma} v^*$$

$$\begin{aligned}
L_2 &= \tfrac{1}{2}\left[\dot{y}\dot{y}^* - (y' - iqy)(y^{*\prime} + iqy^*) - (1 - q^2) y y^*\right] \\
&= \tfrac{1}{2}\left[\dot{v}\dot{v}^* - v' v'^* - (1 - q^2) v v^*\right] \ ,
\end{aligned}$$

$$\begin{aligned}
L_4 = \tfrac{1}{8}\Big\{ & 2q^2(1 - q^2) v^2 v^{*2} + 3iq(1 - q^2) v v^*(v v^{*\prime} - v^* v') + 4(1 - q^2) v v^* v' v'^* \\
& + q^2\left[v^2(v'^{*2} - \dot{v}^{*2}) + v^{*2}(v'^2 - \dot{v}^2)\right] - iq\left[v v'(v'^{*2} - \dot{v}^{*2}) - v^* v'^*(v'^2 - \dot{v}^2)\right]\Big\} \\
& + \tfrac{1}{4}(a - \tfrac{1}{2})\Big\{(1 - q^2)^2 v^2 v^{*2} - iq(1 - q^2) v v^*(v v^{*\prime} - v^* v') \\
& \qquad - iq\left[v v'(v'^{*2} - \dot{v}^{*2}) - v^* v'^*(v'^2 - \dot{v}^2)\right] - (v'^{*2} - \dot{v}^{*2})(v'^2 - \dot{v}^2)\Big\} \ .
\end{aligned}$$

Tree-level S-matrix of bosonic string on $R \times S^3$ with B-flux

simple structure: replace in $q = 0$ S-matrix
$e = \sqrt{p^2+1}$ by $e = \sqrt{(p \pm q)^2 + 1 - q^2}$

$$\mathbb{S} = \mathbb{I} + i\mathrm{h}^{-1}\,\mathbb{T}\,, \qquad \mathrm{h}^{-1} \equiv \frac{2\pi}{\sqrt{\lambda}}$$

$$\mathbb{T}\,|y_m(p) y_n(p')\rangle = T^{rs}_{mn}(p,p')\,|y_r(p) y_s(p')\rangle$$

q=0:

$$T^{rs}_{mn}(p,p') = \Big[\frac{p^2+p'^2}{2(e'p-ep')} + (a-\tfrac{1}{2})(ep'-e'p)\Big]\delta^r_m\delta^s_n - \frac{pp'}{e'p-ep'}\epsilon_m{}^r\epsilon_n{}^s\,,$$
$$e = \sqrt{p^2+1}\,, \qquad e' = \sqrt{p'^2+1}\,.$$

$(y, y^*) = y_1 \pm i y_2$ $\qquad\qquad$ $\pm$ labels

$$T^{++}_{++} = T^{--}_{--} \neq 0, \quad T^{+-}_{+-} = T^{-+}_{-+} \neq 0,$$

q dependence is actually very simple:

[not obvious from string Lagrangian]

$$T^{v\pm\pm}_{\pm\pm}(p,p') = \frac{p+p' \mp 2q}{2(p-p')}\left[e(p'\mp q)+e'(p\mp q)\right] + \left(a-\tfrac{1}{2}\right)\left[e(p'\mp q)-e'(p\mp q)\right] ,$$

$$T^{v\pm\mp}_{\pm\mp}(p,p') = \frac{p-p' \pm 2q}{2(p+p')}\left[e(p'\pm q)+e'(p\mp q)\right] + \left(a-\tfrac{1}{2}\right)\left[e(p'\mp q)-e'(p\pm q)\right] ,$$

$$T^{v\mp\pm}_{\pm\mp}(p,p') = 0 , \qquad e=\sqrt{p^2+1-q^2} , \qquad e'=\sqrt{p'^2+1-q^2} .$$

explicit q-dependence can be absorbed into
shift of momentum of particle v by q
and momentum of anti-particle v^* by $-q$

simple final expression for (y,y^*) S-matrix
and dispersion relation for $0\le q\le 1$

$$T^{\pm\pm}_{\pm\pm}(p,p') = \frac{p+p'}{2(p-p')}(e_\pm p'+e'_\pm p) + \left(a-\tfrac{1}{2}\right)(e_\pm p'-e'_\pm p) ,$$

$$T^{\pm\mp}_{\pm\mp}(p,p') = \frac{p-p'}{2(p+p')}(e_\pm p'+e'_\mp p) + \left(a-\tfrac{1}{2}\right)(e_\pm p'-e'_\mp p) ,$$

$$T^{\mp\pm}_{\pm\mp}(p,p') = 0 , \qquad e_\pm=\sqrt{(p\pm q)^2+1-q^2} , \qquad e'_\pm=\sqrt{(p'\pm q)^2+1-q^2}$$

Fermionic sector

$$L_2 = i(\eta^{ab}\delta_{IJ} - \epsilon^{ab}\rho_{3IJ})\partial_a x^m e_m^{\hat m}\,\bar\theta^I\Gamma_{\hat m}(\mathrm{D}_b)^{JK}\theta^K\ ,$$

$$\mathrm{D}_a = \partial_a + \tfrac{1}{4}\partial_a x^k e_k^{\hat k}\left[(\omega_{\hat m\hat n\hat k} - \tfrac{1}{2}\rho_3 H_{\hat m\hat n\hat k})\Gamma^{\hat m\hat n} - \tfrac{1}{3!}\rho_1 F_{\hat m\hat n\hat l}\Gamma^{\hat m\hat n\hat l}\Gamma_{\hat k}\right]$$

$$\rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$H_{\hat t\hat r'\hat s'} = -2q\epsilon_{\hat r'\hat s'}\ , \qquad H_{\hat\varphi\hat r\hat s} = -2q\epsilon_{\hat r\hat s}\ ,$$
$$F_{\hat t\hat r'\hat s'} = -2\sqrt{1-q^2}\,\epsilon_{\hat r'\hat s'}\ , \qquad F_{\hat\varphi\hat r\hat s} = -2\sqrt{1-q^2}\,\epsilon_{\hat r\hat s}$$

$$\begin{aligned}\mathcal{L}_2 = \quad & i\zeta_R^*(\partial_- + iq)\zeta_R + i\zeta_L^*(\partial_+ - iq)\zeta_L - \sqrt{1-q^2}\,(\zeta_R^*\zeta_L + \zeta_L^*\zeta_R)\\ & + i\chi_R^*(\partial_- + iq)\chi_R + i\chi_L^*(\partial_+ - iq)\chi_L - \sqrt{1-q^2}\,(\chi_R^*\chi_L + \chi_L^*\chi_R)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_4 = & -\tfrac{1}{2}\left[\sqrt{1-q^2}\ \zeta_L^*\zeta_R + \tfrac{q}{2}(\zeta_R^*\zeta_R - \zeta_L^*\zeta_L)\right]\partial_+ y^*\partial_- y\\ & -\tfrac{1}{2}\left[\sqrt{1-q^2}\ \zeta_R^*\zeta_L + \tfrac{q}{2}(\zeta_R^*\zeta_R - \zeta_L^*\zeta_L)\right]\partial_- y^*\partial_+ y\\ & +\left[\tfrac{i}{4}(\zeta_R^*\zeta_R - \zeta_L^*\zeta_L) + \tfrac{q}{2}\zeta_L^*\zeta_L'\right]y^*\partial_+ y + \left[-\tfrac{i}{4}(\zeta_R^*\zeta_R - \zeta_L^*\zeta_L) + \tfrac{q}{2}\zeta_L'^*\zeta_L\right]\partial_+ y^* y\\ & -\left[\tfrac{i}{4}(\zeta_R^*\zeta_R - \zeta_L^*\zeta_L) - \tfrac{q}{2}\zeta_R^*\zeta_R'\right]y^*\partial_- y - \left[-\tfrac{i}{4}(\zeta_R^*\zeta_R - \zeta_L^*\zeta_L) - \tfrac{q}{2}\zeta_R'^*\zeta_R\right]\partial_- y^* y\\ & +\tfrac{i}{4}(\zeta_R^*\partial_+\zeta_R - \partial_+\zeta_R^*\zeta_R + \zeta_L^*\partial_-\zeta_L - \partial_-\zeta_L^*\zeta_L)y^*y\ .\end{aligned}$$

Tree level superstring S-matrix: q=0 case is same as truncation of known AdS5 x S5 one

2 complex bosonic $(y_\pm, z_\pm)$ and 2 complex fermonic $(\zeta_\pm, \chi_\pm)$ fields

Boson-Boson

$$\mathbb{T}\,|y_\pm y_\pm{}'\rangle = (l_1+c)\,|y_\pm y_\pm{}'\rangle \qquad \mathbb{T}\,|y_\pm y_\mp{}'\rangle = (l_2+c)\,|y_\pm y_\mp{}'\rangle + l_4\,|\zeta_\pm\zeta_\mp{}'\rangle + l_4\,|\chi_\pm\chi_\mp{}'\rangle$$
$$\mathbb{T}\,|z_\pm z_\pm{}'\rangle = (-l_1+c)\,|z_\pm z_\pm{}'\rangle \qquad \mathbb{T}\,|z_\pm z_\mp{}'\rangle = (-l_2+c)\,|z_\pm z_\mp{}'\rangle - l_4\,|\chi_\pm\chi_\mp{}'\rangle - l_4\,|\zeta_\pm\zeta_\mp{}'\rangle$$
$$\mathbb{T}\,|y_\pm z_\pm{}'\rangle = (l_3+c)\,|y_\pm z_\pm{}'\rangle + l_5\,|\zeta_\pm\chi_\pm{}'\rangle - l_5\,|\chi_\pm\zeta_\pm{}'\rangle \qquad \mathbb{T}\,|y_\pm z_\mp{}'\rangle = (l_3+c)\,|y_\pm z_\mp{}'\rangle$$
$$\mathbb{T}\,|z_\pm y_\pm{}'\rangle = (-l_3+c)\,|z_\pm y_\pm{}'\rangle - l_5\,|\chi_\pm\zeta_\pm{}'\rangle + l_5\,|\zeta_\pm\chi_\pm{}'\rangle \qquad \mathbb{T}\,|z_\pm y_\mp{}'\rangle = (-l_3+c)\,|z_\pm y_\mp{}'\rangle$$

Fermion-Fermion

$$\mathbb{T}\,|\zeta_\pm\zeta_\pm{}'\rangle = c\,|\zeta_\pm\zeta_\pm{}'\rangle \qquad \mathbb{T}\,|\zeta_\pm\zeta_\mp{}'\rangle = l_4\,|y_\pm y_\mp{}'\rangle - l_4\,|z_\pm z_\mp{}'\rangle$$
$$\mathbb{T}\,|\chi_\pm\chi_\pm{}'\rangle = c\,|\chi_\pm\chi_\pm{}'\rangle \qquad \mathbb{T}\,|\chi_\pm\chi_\mp{}'\rangle = -l_4\,|z_\pm z_\mp{}'\rangle + l_4\,|y_\pm y_\mp{}'\rangle$$
$$\mathbb{T}\,|\zeta_\pm\chi_\pm{}'\rangle = l_5\,|y_\pm z_\pm{}'\rangle + l_5\,|z_\pm y_\pm{}'\rangle \qquad \mathbb{T}\,|\zeta_\pm\chi_\mp{}'\rangle = c\,|\zeta_\pm\chi_\mp{}'\rangle$$
$$\mathbb{T}\,|\chi_\pm\zeta_\pm{}'\rangle = -l_5\,|z_\pm y_\pm{}'\rangle - l_5\,|y_\pm z_\pm{}'\rangle \qquad \mathbb{T}\,|\chi_\pm\zeta_\mp{}'\rangle = c\,|\chi_\pm\zeta_\mp{}'\rangle$$

Boson-Fermion

$$\mathbb{T}\,|y_\pm\zeta_\pm{}'\rangle = (l_6+c)\,|y_\pm\zeta_\pm{}'\rangle - l_5\,|\zeta_\pm y_\pm{}'\rangle \qquad \mathbb{T}\,|y_\pm\zeta_\mp{}'\rangle = (l_7+c)\,|y_\pm\zeta_\mp{}'\rangle + l_4\,|\chi_\pm z_\mp{}'\rangle$$
$$\mathbb{T}\,|\zeta_\pm y_\pm{}'\rangle = (l_8+c)\,|\zeta_\pm y_\pm{}'\rangle - l_5\,|y_\pm\zeta_\pm{}'\rangle \qquad \mathbb{T}\,|\zeta_\pm y_\mp{}'\rangle = (l_9+c)\,|\zeta_\pm y_\mp{}'\rangle - l_4\,|z_\pm\chi_\mp{}'\rangle$$
$$\mathbb{T}\,|y_\pm\chi_\pm{}'\rangle = (l_6+c)\,|y_\pm\chi_\pm{}'\rangle - l_5\,|\chi_\pm y_\pm{}'\rangle \qquad \mathbb{T}\,|y_\pm\chi_\mp{}'\rangle = (l_7+c)\,|y_\pm\chi_\mp{}'\rangle - l_4\,|\zeta_\pm z_\mp{}'\rangle$$
$$\mathbb{T}\,|\chi_\pm y_\pm{}'\rangle = (l_8+c)\,|\chi_\pm y_\pm{}'\rangle - l_5\,|y_\pm\chi_\pm{}'\rangle \qquad \mathbb{T}\,|\chi_\pm y_\mp{}'\rangle = (l_9+c)\,|\chi_\pm y_\mp{}'\rangle + l_4\,|z_\pm\zeta_\mp{}'\rangle$$
$$\mathbb{T}\,|z_\pm\zeta_\pm{}'\rangle = (-l_6+c)\,|z_\pm\zeta_\pm{}'\rangle + l_5\,|\zeta_\pm z_\pm{}'\rangle \qquad \mathbb{T}\,|z_\pm\zeta_\mp{}'\rangle = (-l_7+c)\,|z_\pm\zeta_\mp{}'\rangle + l_4\,|\chi_\pm y_\mp{}'\rangle$$
$$\mathbb{T}\,|\zeta_\pm z_\pm{}'\rangle = (-l_8+c)\,|\zeta_\pm z_\pm{}'\rangle + l_5\,|z_\pm\zeta_\pm{}'\rangle \qquad \mathbb{T}\,|\zeta_\pm z_\mp{}'\rangle = (-l_9+c)\,|\zeta_\pm z_\mp{}'\rangle - l_4\,|y_\pm\chi_\mp{}'\rangle$$
$$\mathbb{T}\,|z_\pm\chi_\pm{}'\rangle = (-l_6+c)\,|z_\pm\chi_\pm{}'\rangle + l_5\,|\chi_\pm z_\pm{}'\rangle \qquad \mathbb{T}\,|z_\pm\chi_\mp{}'\rangle = (-l_7+c)\,|z_\pm\chi_\mp{}'\rangle - l_4\,|\zeta_\pm y_\mp{}'\rangle$$
$$\mathbb{T}\,|\chi_\pm z_\pm{}'\rangle = (-l_8+c)\,|\chi_\pm z_\pm{}'\rangle + l_5\,|z_\pm\chi_\pm{}'\rangle \qquad \mathbb{T}\,|\chi_\pm z_\mp{}'\rangle = (-l_9+c)\,|\chi_\pm z_\mp{}'\rangle + l_4\,|y_\pm\zeta_\mp{}'\rangle$$

$$l_1(p,p') = \frac{(p+p')(e'p+ep')}{2(p-p')} \ , \qquad l_2(p,p') = \frac{(p-p')(e'p+ep')}{2(p+p')} \ ,$$

$$l_3(p,p') = -\frac{1}{2}(e'p+ep') \ ,$$

$$l_4(p,p') = -\frac{pp'}{2(p+p')}\left[\sqrt{(e+p)(e'+p')} - \sqrt{(e-p)(e'-p')}\right] \ ,$$

$$l_5(p,p') = -\frac{pp'}{2(p-p')}\left[\sqrt{(e+p)(e'+p')} + \sqrt{(e-p)(e'-p')}\right] \ ,$$

$$l_6(p,p') = \frac{p'(e'p+ep')}{2(p-p')} \ , \qquad l_7(p,p') = -\frac{p'(e'p+ep')}{2(p+p')} \ ,$$

$$l_8(p,p') = \frac{p(e'p+ep')}{2(p-p')} \ , \qquad l_9(p,p') = \frac{p(e'p+ep')}{2(p+p')} \ .$$

$$e = \sqrt{p^2+1} \ , \qquad e' = \sqrt{p'^2+1} \ .$$

$q \neq 0$ generalization:

functions $l_{1,2,3,6,7,8,9}$: do replacement:

$$e \to e_\pm = \sqrt{(p \pm q)^2 + 1 - q^2} \ , \qquad e' \to e'_\pm = \sqrt{(p' \pm q)^2 + 1 - q^2} \ ,$$

symmetry of l.c. vacuum:
$psu(1|1) \times psu(1|1)$

Symmetry factorization property of S-matrix:

$$|y\rangle = |\phi\rangle \otimes |\phi\rangle \ , \qquad |z\rangle = |\psi\rangle \otimes |\psi\rangle \ ,$$
$$|\zeta\rangle = |\phi\rangle \otimes |\psi\rangle \ , \qquad |\chi\rangle = |\psi\rangle \otimes |\phi\rangle \ ,$$

$$\mathbb{T}\left|y_+\zeta'_+\right\rangle = (\mathbb{I}\otimes\hat{\mathbb{T}} + \hat{\mathbb{T}}\otimes\mathbb{I})\left(\left|\phi_+\phi'_+\right\rangle \otimes \left|\phi_+\psi'_+\right\rangle\right) \ ,$$
$$\mathbb{T}\left|z_+\chi'_-\right\rangle = -\,(\mathbb{I}\otimes\hat{\mathbb{T}} + \hat{\mathbb{T}}\otimes\mathbb{I})\left(\left|\psi_+\psi'_-\right\rangle \otimes \left|\psi_+\phi'_-\right\rangle\right)$$

$$l_6 = \frac{1}{2}(l_1 + l_3) \ , \qquad l_8 = \frac{1}{2}(l_1 - l_3) \ , \qquad l_7 = \frac{1}{2}(l_2 + l_3) \ , \qquad l_9 = \frac{1}{2}(l_2 - l_3)$$

factorization property does not constrain l_4 and l_5.

requirement of integrability – the classical *Yang-Baxter* equation :

$$l_4^{\pm\mp\to\pm\mp}(p,p') = -\frac{pp'}{2(p+p')}\left[\sqrt{(e_\pm + p \pm q)(e'_\mp + p' \mp q)} - \sqrt{(e_\pm - p \mp q)(e'_\mp - p' \pm q)}\right],$$
$$l_5^{\pm\pm\to\pm\pm}(p,p') = -\frac{pp'}{2(p-p')}\left[\sqrt{(e_\pm + p \pm q)(e'_\pm + p' \pm q)} + \sqrt{(e_\pm - p \mp q)(e'_\pm - p' \mp q)}\right].$$

Summary of tree-level S-matrix in factorized form:

$$q^2 + \hat{q}^2 = 1 \,, \qquad \hat{q} = \sqrt{1-q^2}\,. \qquad \mathrm{h} = \frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'}\,.$$

$$q\sqrt{\lambda} = k \,, \qquad \text{i.e.} \qquad k = 2\pi\,\mathrm{h}\,q\,.$$

$$|y\rangle = |\phi\rangle \otimes |\phi\rangle \,, \qquad |z\rangle = |\psi\rangle \otimes |\psi\rangle$$

$$|\zeta\rangle = |\phi\rangle \otimes |\psi\rangle \,, \qquad |\chi\rangle = |\psi\rangle \otimes |\phi\rangle$$

$$e_\pm^2 = 1 + p^2 \pm 2q\,p$$

where $|\phi\rangle$ is bosonic and $|\psi\rangle$ is fermionic. The factorization property means that the S-matrix for $\{y, z, \zeta, \chi\}$ can be constructed from an S-matrix for $\{\phi, \psi\}$, which takes the following form

$$\mathbb{S}\,|\phi_\pm\phi_\pm'\rangle = A_\pm\,L_{1\pm}\,|\phi_\pm\phi_\pm'\rangle \,, \qquad \mathbb{S}\,|\phi_\pm\psi_\pm'\rangle = A_\pm\,L_{3\pm}\,|\phi_\pm\psi_\pm'\rangle + A_\pm\,L_{5\pm}\,|\psi_\pm\phi_\pm'\rangle$$

$$\mathbb{S}\,|\psi_\pm\psi_\pm'\rangle = A_\pm\,\Lambda_{1\pm}\,|\psi_\pm\psi_\pm'\rangle \,, \qquad \mathbb{S}\,|\psi_\pm\phi_\pm'\rangle = A_\pm\,\Lambda_{3\pm}\,|\psi_\pm\phi_\pm'\rangle + A_\pm\,\Lambda_{5\pm}\,|\phi_\pm\psi_\pm'\rangle$$

$$\mathbb{S}\,|\phi_\pm\psi_\mp'\rangle = \bar{A}_\pm\,L_{6\pm}\,|\phi_\pm\psi_\mp'\rangle \,, \qquad \mathbb{S}\,|\phi_\pm\phi_\mp'\rangle = \bar{A}_\pm\,L_{2\pm}\,|\phi_\pm\phi_\mp'\rangle + \bar{A}_\pm\,L_{4\pm}\,|\psi_\pm\psi_\mp'\rangle$$

$$\mathbb{S}\,|\psi_\pm\phi_\mp'\rangle = \bar{A}_\pm\,\Lambda_{6\pm}\,|\psi_\pm\phi_\mp'\rangle \,, \qquad \mathbb{S}\,|\psi_\pm\psi_\mp'\rangle = \bar{A}_\pm\,\Lambda_{2\pm}\,|\psi_\pm\psi_\mp'\rangle + \bar{A}_\pm\,\Lambda_{4\pm}\,|\phi_\pm\phi_\mp'\rangle$$

$$e_\pm = \sqrt{\hat{q}^2 + (p \pm q)^2}\,, \qquad e'_\pm = \sqrt{\hat{q}^2 + (p' \pm q)^2}$$

$$A_\pm = 1 - \frac{i}{2\mathrm{h}}\left(\mathrm{a} - \tfrac{1}{2}\right)(e'_\pm p - e_\pm p') + \mathcal{O}(\mathrm{h}^{-2}) , \qquad \bar{A}_\pm = 1 - \frac{i}{2\mathrm{h}}\left(\mathrm{a} - \tfrac{1}{2}\right)(e'_\mp p - e_\pm p') + \mathcal{O}(\mathrm{h}^{-2})$$

other non-trivial functions of the momenta p, p' and energies $e_\pm, e'_\pm$ are

$$\begin{aligned}
L_{1\pm} &= 1 + \frac{i}{2\mathrm{h}} l_{1\pm} + \mathcal{O}(\mathrm{h}^{-2}) , & \Lambda_{1\pm} &= 1 - \frac{i}{2\mathrm{h}} l_{1\pm} + \mathcal{O}(\mathrm{h}^{-2}) , \\
L_{3\pm} &= 1 + \frac{i}{2\mathrm{h}} l_{3\pm} + \mathcal{O}(\mathrm{h}^{-2}) , & \Lambda_{3\pm} &= 1 - \frac{i}{2\mathrm{h}} l_{3\pm} + \mathcal{O}(\mathrm{h}^{-2}) , \\
L_{6\pm} &= 1 + \frac{i}{2\mathrm{h}} l_{3\pm} + \mathcal{O}(\mathrm{h}^{-2}) , & \Lambda_{6\pm} &= 1 - \frac{i}{2\mathrm{h}} l_{3\pm} + \mathcal{O}(\mathrm{h}^{-2}) , \\
L_{2\pm} &= 1 + \frac{i}{2\mathrm{h}} l_{2\pm} + \mathcal{O}(\mathrm{h}^{-2}) , & \Lambda_{2\pm} &= 1 - \frac{i}{2\mathrm{h}} l_{2\pm} + \mathcal{O}(\mathrm{h}^{-2}) , \\
L_{5\pm} &= -\frac{i}{\mathrm{h}} l_{5\pm} + \mathcal{O}(\mathrm{h}^{-2}) , & \Lambda_{5\pm} &= -\frac{i}{\mathrm{h}} l_{5\pm} + \mathcal{O}(\mathrm{h}^{-2}) , \\
L_{4\pm} &= \frac{i}{\mathrm{h}} l_{4\pm} + \mathcal{O}(\mathrm{h}^{-2}) , & \Lambda_{4\pm} &= \frac{i}{\mathrm{h}} l_{4\pm} + \mathcal{O}(\mathrm{h}^{-2}) ,
\end{aligned}$$

$$l_{1\pm} = \frac{(p+p')(e'_\pm p + e_\pm p')}{2(p-p')} , \qquad l_{2\pm} = \frac{(p-p')(e'_\mp p + e_\pm p')}{2(p+p')} ,$$

$$l_{3\pm} = -\frac{1}{2}(e'_\pm p + e_\pm p') ,$$

$$l_{4\pm} = -\frac{pp'}{2(p+p')}\left[\sqrt{(e_\pm + p \pm q)(e'_\mp + p' \mp q)} - \sqrt{(e_\pm - p \mp q)(e'_\mp - p' \pm q)}\right]$$

$$l_{5\pm} = -\frac{pp'}{2(p-p')}\left[\sqrt{(e_\pm + p \pm q)(e'_\pm + p' \pm q)} + \sqrt{(e_\pm - p \mp q)(e'_\pm - p' \mp q)}\right]$$

Exact S-matrix

to all orders in inverse string tension ?

Use constructive approach

(symmetry + general properties)

as in AdS5 x S5 case

Strategy:

(i) symmetry algebra and its reduction in l.c. gauge

(ii) represent l.c. symmetry on particle states and find exact q-dependent dispersion relation

(iii) construct 2-particle S-matrix consistent with symmetry, YBE, unitarity, crossing;

symmetry of S-matrix realized as Hopf algebra (non-trivial p-dependent coproduct as in AdS5 x S5 case)

(iv) check against perturbative S-matrix found directly from superstring action

Remarkably, resulting exact S-matrix has same form as in q=0 (pure RR) case but in terms of q-dependent kinematic (Zhukovsky) variables x(e,p; q)

S-matrix symmetry algebra and its representation

supercoset $\frac{PSU(1,1|2)\times PSU(1,1|2)}{SU(1,1)\times SU(2)}$

supersymmetry $PSU(1,1|2)\times PSU(1,1|2)$ same for any q

q like tension h is parameter in string action

supercoset symm. preserved by l.c. gauge (BMN vacuum) same for any q
but form of its representation on 1-particle states may depend on q

$U(1)$ generators $\mathfrak{R}$ and $\mathfrak{L}$ four supercharges $\mathfrak{Q}_{\pm\mp}$ and $\mathfrak{S}_{\pm\mp}$

three central extension generators $\mathfrak{C}$, $\mathfrak{P}$ and $\mathfrak{K}$

$$\mathfrak{M}=\frac{1}{2}(\mathfrak{R}+\mathfrak{L})\ , \qquad \mathfrak{B}=\frac{1}{2}(\mathfrak{R}-\mathfrak{L})$$

$$[\mathfrak{B},\,\mathfrak{Q}_{\pm\mp}]=\pm i\mathfrak{Q}_{\pm\mp}\ , \qquad [\mathfrak{B},\,\mathfrak{S}_{\pm\mp}]=\pm i\mathfrak{S}_{\pm\mp}\ ,$$

$$\{\mathfrak{Q}_{\pm\mp},\,\mathfrak{Q}_{\mp\pm}\}=\mathfrak{P}\ , \qquad \{\mathfrak{S}_{\pm\mp},\,\mathfrak{S}_{\mp\pm}\}=\mathfrak{K}\ , \qquad \{\mathfrak{Q}_{\pm\mp},\,\mathfrak{S}_{\mp\pm}\}=\pm i\,\mathfrak{M}+\mathfrak{C}$$

This superalgebra is a centrally-extended semi-direct sum of $\mathfrak{u}(1)$ (generated by $\mathfrak{B}$) with copies of the superalgebra $\mathfrak{psu}(1|1)$, i.e.

$$[\mathfrak{u}(1) \Subset \mathfrak{psu}(1|1)^2] \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3 \ .$$

subalgebra of the familiar $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$

symmetry of the full S-matrix $[\mathfrak{u}(1) \Subset \mathfrak{psu}(1|1)^2]^2 \ltimes \mathfrak{u}(1) \ltimes \mathbb{R}^3$

particular representation of this symmetry algebra

one complex boson ϕ and one complex fermion ψ. $\phi_+ = \phi,\ \phi_- = \phi^*;\ \psi_+ = \psi,\ \phi_- = \psi^*$

$$\begin{aligned}
&\mathfrak{B}\,|\phi_\pm\rangle = \pm i\,|\phi_\pm\rangle\ , && \mathfrak{B}\,|\psi_\pm\rangle = \mp i\,|\psi_\pm\rangle\ ,\\
&\mathfrak{Q}_{\pm\mp}\,|\phi_\pm\rangle = 0\ , && \mathfrak{Q}_{\pm\mp}\,|\psi_\pm\rangle = b_\pm\,|\phi_\pm\rangle\ ,\\
&\mathfrak{Q}_{\mp\pm}\,|\phi_\pm\rangle = a_\pm\,|\psi_\pm\rangle\ , && \mathfrak{Q}_{\mp\pm}\,|\psi_\pm\rangle = 0\ ,\\
&\mathfrak{S}_{\pm\mp}\,|\phi_\pm\rangle = 0\ , && \mathfrak{S}_{\pm\mp}\,|\psi_\pm\rangle = d_\pm\,|\phi_\pm\rangle\ ,\\
&\mathfrak{S}_{\mp\pm}\,|\phi_\pm\rangle = c_\pm\,|\psi_\pm\rangle\ , && \mathfrak{S}_{\mp\pm}\,|\psi_\pm\rangle = 0\ ,\\
&\mathfrak{M}\,|\phi_\pm\rangle = \pm\tfrac{i}{2}M_\pm\,|\phi_\pm\rangle\ , && \mathfrak{M}\,|\psi_\pm\rangle = \pm\tfrac{i}{2}M_\pm\,|\psi_\pm\rangle\\
&\mathfrak{C}\,|\phi_\pm\rangle = C_\pm\,|\phi_\pm\rangle\ , && \mathfrak{C}\,|\psi_\pm\rangle = C_\pm\,|\psi_\pm\rangle\ ,\\
&\mathfrak{P}\,|\phi_\pm\rangle = P_\pm\,|\phi_\pm\rangle\ , && \mathfrak{P}\,|\psi_\pm\rangle = P_\pm\,|\psi_\pm\rangle\ ,\\
&\mathfrak{K}\,|\phi_\pm\rangle = K_\pm\,|\phi_\pm\rangle\ , && \mathfrak{K}\,|\psi_\pm\rangle = K_\pm\,|\psi_\pm\rangle\ .
\end{aligned}$$

$a_\pm, b_\pm, c_\pm, d_\pm, C_\pm, P_\pm$ and $K_\pm$ are the representation parameters

functions of the energy and momentum of the state.

$\{\phi_+, \psi_+\}$ and $\{\phi_-, \psi_-\}$ are two irreducible representations related by conjugation

For the supersymmetry algebra to close

$$a_\pm b_\pm = P_\pm \ , \qquad c_\pm d_\pm = K_\pm \ , \qquad a_\pm d_\pm = C_\pm + \frac{M_\pm}{2} \ , \qquad b_\pm c_\pm = C_\pm - \frac{M_\pm}{2}$$

$$C_\pm^2 = \frac{M_\pm^2}{4} + P_\pm K_\pm$$

shortening conditions for the two irreducible atypical representation

will be interpreted as dispersion relations:

Can get exact dispersion relation from the algebra

To define action of this symmetry on 2-particle states need to introduce coproduct:

Hopf algebra structure

$$\Delta(\mathfrak{B}) = \mathfrak{B} \otimes \mathbb{I} + \mathbb{I} \otimes \mathfrak{B}\ , \qquad \Delta(\mathfrak{M}) = \mathfrak{M} \otimes \mathbb{I} + \mathbb{I} \otimes \mathfrak{M}\ , \qquad \Delta(\mathfrak{C}) = \mathfrak{C} \otimes \mathbb{I} + \mathbb{I} \otimes \mathfrak{C}\ ,$$

$$\Delta(\mathfrak{Q}) = \mathfrak{Q} \otimes \mathbb{I} + \mathfrak{U} \otimes \mathfrak{Q}\ , \qquad \Delta(\mathfrak{S}) = \mathfrak{S} \otimes \mathbb{I} + \mathfrak{U}^{-1} \otimes \mathfrak{S}$$

$$\Delta(\mathfrak{P}) = \mathfrak{P} \otimes \mathbb{I} + \mathfrak{U}^2 \otimes \mathfrak{P}\ , \qquad \Delta(\mathfrak{K}) = \mathfrak{K} \otimes \mathbb{I} + \mathfrak{U}^{-2} \otimes \mathfrak{K}\ ,$$

and opposite coproduct $\quad \Delta^{op}(\mathfrak{J}) = \mathcal{P}(\Delta(\mathfrak{J}))$

$\mathfrak{J}$ is an arbitrary generator and $\mathcal{P}$ defines the graded permutation of the tensor product

coproduct differs from the usual product by the introduction of a new abelian generator

$$\mathfrak{U}, \text{ with } \Delta(\mathfrak{U}) = \mathfrak{U} \otimes \mathfrak{U}$$

$$\mathfrak{U}\,|\phi_\pm\rangle = U_\pm\,|\phi_\pm\rangle\ , \qquad \mathfrak{U}\,|\psi_\pm\rangle = U_\pm\,|\psi_\pm\rangle$$

This braiding allows for the existence of a non-trivial S-matrix

[psu(2|2): Gomez,Hernandez 06; Plefka,Spill,Torrielli 06; for this case in PR context: Hoare, AT 11]

The factorized tree-level S-matrix of the theory with mixed 3-form flux ($q \neq 0$) co-commutes ($\Delta^{op}(\mathfrak{J})\,\mathbb{S} = \mathbb{S}\,\Delta(\mathfrak{J})$) with the supersymmetry algebra if at the leading order in the large $\mathrm{h} \to \infty$, $\mathrm{p} \to 0$, $p \equiv \mathrm{h}\,\mathrm{p} = \text{fixed}$

$$a_\pm = \frac{e^{-\frac{i\pi}{4}}}{\sqrt{2}}\sqrt{e_\pm + 1 \pm q\,p}\ , \qquad b_\pm = -\frac{ie^{\frac{i\pi}{4}}}{\sqrt{2}}\frac{\hat{q}\,p}{\sqrt{e_\pm + 1 \pm q\,p}}\ ,$$

$$c_\pm = \frac{ie^{-\frac{i\pi}{4}}}{\sqrt{2}}\frac{\hat{q}\,p}{\sqrt{e_\pm + 1 \pm q\,p}}\ , \qquad d_\pm = \frac{e^{\frac{i\pi}{4}}}{\sqrt{2}}\sqrt{e_\pm + 1 \pm q\,p}\ ,$$

$$U_\pm = 1 + \frac{ip}{2\mathrm{h}}\ , \quad M_\pm = 1 \pm q\,p\ , \quad C_\pm = \frac{e_\pm}{2}\ , \quad P_\pm = -\frac{i}{2}\,\hat{q}\,p\ , \quad K_\pm = \frac{i}{2}\,\hat{q}\,p$$

Leads to exact expressions for representation parameters:

$$U_\pm = e^{\frac{i}{2}\mathrm{p}}\ , \qquad M_\pm = 1 \pm 2\mathrm{h}\,q\,\sin\frac{\mathrm{p}}{2}\ , \qquad C_\pm = \frac{e_\pm}{2}$$

$$P_\pm = \frac{\mathrm{h}\,\hat{q}}{2}\,(1 - e^{i\mathrm{p}})\ , \qquad K_\pm = \frac{\mathrm{h}\,\hat{q}}{2}\,(1 - e^{-i\mathrm{p}})$$

e is energy, $\quad$ p is momentum $\qquad \hat{q} = \sqrt{1 - q^2}$

$$a_\pm = \frac{\alpha_\pm\, e^{-\frac{i\pi}{4}}}{\sqrt{2}} \sqrt{e_\pm + 1 \pm 2\mathrm{h}\, q\, \sin\tfrac{\mathrm{p}}{2}}\ , \qquad b_\pm = \frac{\alpha_\pm^{-1} e^{\frac{i\pi}{4}}}{\sqrt{2}} \frac{\mathrm{h}\,\hat{q}\,(1 - e^{i\mathrm{p}})}{\sqrt{e_\pm + 1 \pm 2\mathrm{h}\, q\, \sin\frac{\mathrm{p}}{2}}}$$

$$c_\pm = \frac{\alpha_\pm\, e^{-\frac{i\pi}{4}}}{\sqrt{2}} \frac{\mathrm{h}\,\hat{q}\,(1 - e^{-i\mathrm{p}})}{\sqrt{e_\pm + 1 \pm 2\mathrm{h}\, q\, \sin\frac{\mathrm{p}}{2}}}\ , \qquad d_\pm = \frac{\alpha_\pm^{-1} e^{\frac{i\pi}{4}}}{\sqrt{2}} \sqrt{e_\pm + 1 \pm 2\mathrm{h}\, q\, \sin\tfrac{\mathrm{p}}{2}}$$

$\alpha_\pm$ are free phases

leads to exact dispersion relation

$$e_\pm = \sqrt{1 \pm 4hq \sin\frac{\mathrm{p}}{2} + 4h^2 \sin^2\frac{\mathrm{p}}{2}}$$

Comments on exact dispersion relation

$$e_\pm = \sqrt{\hat{q}^2 + (2\,\mathrm{h}\,\sin\frac{\mathrm{p}}{2} \pm q)^2} \qquad = \sqrt{1 \pm \frac{2k}{\pi}\sin\frac{\mathrm{p}}{2} + \ 4\,\mathrm{h}^2\sin^2\frac{\mathrm{p}}{2}}$$

interpolates between the RR case $(\hat{q} = 1,\ q = 0)$ and the NSNS case $(\hat{q} = 0,\ q = 1)$

$q = 1$: superstring generalization of the $SU(1,1) \times SU(2)$ WZW theory

$$e_\pm = \left|1 \pm 2\mathrm{h}\,\sin\frac{\mathrm{p}}{2}\right| , \qquad \mathrm{h} = \frac{k}{2\pi} \qquad (\text{“massless” at small p})$$

Interpretation: dispersion relation of “dressed” elementary string excitation (cf. “giant magnon”)

perturbative string (BMN) limit $\quad \mathrm{h} \to \infty\,, \qquad \mathrm{p} \to 0\,, \qquad p \equiv \mathrm{h}\,\mathrm{p} = \text{fixed}$

$$e_\pm = \sqrt{\hat{q}^2 + (p \pm q)^2} + \mathcal{O}(\mathrm{h}^{-2})$$

different strong coupling limit – semiclassical or “giant magnon”

$$e_\pm = 2\mathrm{h}\,\sin\frac{\mathrm{p}}{2} \pm q \ + \mathcal{O}(\mathrm{h}^{-1})$$

compare to $q = 0$ case:

dispersion relation

$$e = \sqrt{1 + 4h^2 \sin^2 \tfrac{\mathrm{p}}{2}}\ , \qquad h = \tfrac{\sqrt{\lambda}}{2\pi}$$

$$e = \sqrt{1 + p^2} - \frac{p^4}{24h^2\sqrt{1+p^2}} + O(h^{-3})\ , \qquad \mathrm{p} = h^{-1} p$$

$q \neq 0$:

$$e = \sqrt{1 - q^2 + (p+q)^2} - \frac{p^3(p+q)}{24h^2\sqrt{1 - q^2 + (p+q)^2}} + O(h^{-3})$$

Another interpretation of exact dispersion relation:
discretization of spatial world-sheet direction

quadratic term in bosonic string action (near BMN geodesic)

$$I = \frac{1}{2}\mathrm{h} \int d\tau d\sigma \Big(- \partial^a y_r \partial_a y_r - y_r y_r + \ q \, \epsilon_{rs} y_r \partial_1 y_s \Big)$$

$$e_\pm = \sqrt{\hat{q}^2 + (p \pm q)^2} \, .$$

now assume that the spatial direction σ is compact with length $\ell = 2\pi \mathcal{J}$

discretize σ into J points with step ε

$$\varepsilon = \frac{\ell}{J} = 2\pi\frac{\mathcal{J}}{J}\,, \qquad y_{r(n)}(\tau) = y_r(\tau, n\varepsilon), \qquad n = 0, ..., J-1, \qquad y_{r(J)} = y_{r(0)}$$

$$I = \frac{1}{2}\mathrm{h}\varepsilon\sum_{n=1}^{J}\int d\tau\Big[\dot{y}^2_{r(n)} - \varepsilon^{-2}(y_{r(n+1)} - y_{r(n)})^2 - y^2_{r(n)} + 2q\, y_{1(n)}(y_{2(n+1)} - y_{2(n)})\Big]$$

$$y_{r(n)}(\tau) \;\rightarrow\; \mathrm{y}_r e^{-ie\tau} e^{-ipn}\,, \qquad \mathrm{p} = \frac{2\pi\tilde{n}}{J}\,, \qquad \tilde{n} = 0, 1, ..., J-1$$

The corresponding dispersion relation

$$e^2_{\pm} = 1 + 4\varepsilon^{-2}\sin^2\frac{\mathrm{p}}{2} \pm 4q\varepsilon^{-1}\sin\frac{\mathrm{p}}{2}$$

equivalent to the exact dispersion relation if $\varepsilon^{-1} = \mathrm{h}$

step of the lattice ε has the interpretation of the inverse of string tension.

$q = 1$: massless operators $\partial_0 \pm \partial_1$ on a 1-d spatial lattice.

$$L = \psi^*(\partial_0 \pm \partial_1 - i)\psi \qquad\qquad e_{\pm} = \Big|1 \pm 2\varepsilon^{-1}\sin\frac{\mathrm{p}}{2}\Big|$$

Lessons about weakly-coupled dual theory?

Exact S-matrix: q=0 case

formal symmetry under interchange of the two irreducible atypical representations

drop the subscripts $_\pm$ on both the representation parameters

Zhukovsky variables $x^{\pm} = x^{\pm}(\mathrm{p})$

$$e^{i\mathrm{p}} = \frac{x^+}{x^-}\,, \qquad e + 1 = i\mathrm{h}(x^- - x^+) \qquad \eta \equiv \sqrt{i(x^- - x^+)} = \sqrt{\frac{e+1}{\mathrm{h}}}$$

functions parametrizing the exact S-matrix [Borsato,OhlssonSax,Sfondrini,Stefanski,Torrielli 13]

$$L_1 = \mathrm{S}_1\,, \qquad \Lambda_1 = \mathrm{S}_1\sqrt{\frac{x^-x'^+}{x^+x'^-}}\,\frac{x^+ - x'^-}{x^- - x'^+}\,,$$

$$L_3 = \mathrm{S}_1\sqrt{\frac{x^-}{x^+}}\,\frac{x^+ - x'^+}{x^- - x'^+}\,, \qquad \Lambda_3 = \mathrm{S}_1\sqrt{\frac{x'^+}{x'^-}}\,\frac{x^- - x'^-}{x^- - x'^+}\,,$$

$$L_5 = -i\,\frac{\alpha}{\alpha'}\,\mathrm{S}_1\sqrt{\frac{x^-x'^+}{x^+x'^-}}\,\frac{\eta\eta'}{x^- - x'^+}\,, \qquad \Lambda_5 = -i\,\frac{\alpha'}{\alpha}\,\mathrm{S}_1\,\frac{\eta\eta'}{x^- - x'^+}\,,$$

$$L_6 = \mathrm{S}_2\,, \qquad \Lambda_6 = \mathrm{S}_2\sqrt{\frac{x^-x'^-}{x^+x'^+}}\,\frac{1 - x^+x'^+}{1 - x^-x'^-}\,,$$

$$L_2 = \mathrm{S}_2\sqrt{\frac{x^-}{x^+}}\,\frac{1 - x^+x'^-}{1 - x^-x'^-}\,, \qquad \Lambda_2 = \mathrm{S}_2\sqrt{\frac{x'^-}{x'^+}}\,\frac{1 - x^-x'^+}{1 - x^-x'^-}\,,$$

$$L_4 = i\,\alpha\alpha'\,\mathrm{S}_2\sqrt{\frac{x^-x'^-}{x^+x'^+}}\,\frac{\eta\eta'}{1 - x^-x'^-}\,, \qquad \Lambda_4 = i\,\frac{1}{\alpha\alpha'}\,\mathrm{S}_2\,\frac{\eta\eta'}{1 - x^-x'^-}\,.$$

S-matrix is completely fixed, up to the two phases S_1 and S_2, just by demanding invariance under the four supercharges.

this S-matrix satisfies the Yang- Baxter equation, QFT unitarity, and also braiding unitarity if

$$S_1(x^+,x^-;x'^+,x'^-)\,S_1(x'^+,x'^-;x^+,x^-)=1\ ,$$

$$S_2(x^+,x^-;x'^+,x'^-)\,S_2(x'^+,x'^-;x^+,x^-)=\sqrt{\frac{x^+x'^+}{x^-x'^-}\,\frac{1-x^-x'^-}{1-x^+x'^+}}$$

This suggests that a natural strategy to generalize to the case of $q\neq 0$

the corresponding Zhukovsky variables $\qquad x^{\pm}_{\pm}$

S-matrix will remain unchanged up to the introduction of

the $_{\pm}$ subscripts on $x^{\pm}$ and $x'^{\pm}$.

Exact S-matrix: generic q

$$e^{i\mathrm{p}} = \frac{x_\pm^+}{x_\pm^-}\,, \qquad e_\pm + 1 \pm 2\,\mathrm{h}\,q\,\sin\frac{\mathrm{p}}{2} = i\mathrm{h}\,\hat{q}\,(x_\pm^- - x_\pm^+)$$

$$\hat{q}\left(x_\pm^+ + \frac{1}{x_\pm^+} - x_\pm^- - \frac{1}{x_\pm^-}\right) \pm 2q\left(\sqrt{\frac{x_\pm^-}{x_\pm^+}} - \sqrt{\frac{x_\pm^+}{x_\pm^-}}\right) = \frac{2i}{\mathrm{h}} \qquad \hat{q} = \sqrt{1-q^2}$$

$$x_\pm^\pm = r_\pm\, e^{\pm\frac{i\mathrm{p}}{2}}\,, \qquad r_\pm = \frac{e_\pm + 1 \pm 2\mathrm{h}\,q\,\sin\frac{\mathrm{p}}{2}}{2\mathrm{h}\,\hat{q}\,\sin\frac{\mathrm{p}}{2}}$$

$$L_{1\pm} = \mathrm{S}_{1\pm}\,, \qquad \Lambda_{1\pm} = \mathrm{S}_{1\pm}\sqrt{\frac{x_\pm^- x_\pm'^+}{x_\pm^+ x_\pm'^-}}\,\frac{x_\pm^+ - x_\pm'^-}{x_\pm^- - x_\pm'^+}\,,$$

$$L_{3\pm} = \mathrm{S}_{1\pm}\sqrt{\frac{x_\pm^-}{x_\pm^+}}\,\frac{x_\pm^+ - x_\pm'^+}{x_\pm^- - x_\pm'^+}\,, \qquad \Lambda_{3\pm} = \mathrm{S}_{1\pm}\sqrt{\frac{x_\pm'^+}{x_\pm'^-}}\,\frac{x_\pm^- - x_\pm'^-}{x_\pm^- - x_\pm'^+}\,,$$

$$L_{5\pm} = -\,i\,\frac{\alpha_\pm}{\alpha'_\pm}\,\mathrm{S}_{1\pm}\sqrt{\frac{x_\pm^- x_\pm'^+}{x_\pm^+ x_\pm'^-}}\,\frac{\eta_\pm \eta'_\pm}{x_\pm^- - x_\pm'^+}\,, \qquad \Lambda_{5\pm} = -\,i\,\frac{\alpha'_\pm}{\alpha_\pm}\,\mathrm{S}_{1\pm}\,\frac{\eta_\pm\eta'_\pm}{x_\pm^- - x_\pm'^+}\,,$$

$$L_{6\pm} = \mathrm{S}_{2\pm}\,, \qquad \Lambda_{6\pm} = \mathrm{S}_{2\pm}\sqrt{\frac{x_\pm^- x_\mp'^-}{x_\pm^+ x_\mp'^+}}\,\frac{1 - x_\pm^+ x_\mp'^+}{1 - x_\pm^- x_\mp'^-}\,,$$

$$L_{2\pm} = \mathrm{S}_{2\pm}\sqrt{\frac{x_\pm^-}{x_\pm^+}}\,\frac{1 - x_\pm^+ x_\mp'^-}{1 - x_\pm^- x_\mp'^-}\,, \qquad \Lambda_{2\pm} = \mathrm{S}_{2\pm}\sqrt{\frac{x_\mp'^-}{x_\mp'^+}}\,\frac{1 - x_\pm^- x_\mp'^+}{1 - x_\pm^- x_\mp'^-}\,,$$

$$L_{4\pm} = i\,\alpha_\pm\alpha'_\mp\,\mathrm{S}_{2\pm}\sqrt{\frac{x_\pm^- x_\mp'^-}{x_\pm^+ x_\mp'^+}}\,\frac{\eta_\pm\eta'_\mp}{1 - x_\pm^- x_\mp'^-}\,, \qquad \Lambda_{4\pm} = i\,\frac{1}{\alpha_\pm\alpha'_\mp}\,\mathrm{S}_{2\pm}\,\frac{\eta_\pm\eta'_\mp}{1 - x_\pm^- x_\mp'^-}\,.$$

four phases $S_{1\pm}$, $S_{2\pm}$ are not fixed by symmetry or the Yang-Baxter equation

S-matrix in the $q \neq 0$ case is QFT unitary, while for braiding unitarity

$$S_{1\pm}(x^+_\pm, x^-_\pm; x'^+_\pm, x'^-_\pm)\, S_{1\pm}(x'^+_\pm, x'^-_\pm; x^+_\pm, x^-_\pm) = 1\,, \qquad (x^\pm_\pm)^* = x^\mp_\pm$$

$$S_{2\pm}(x^+_\pm, x^-_\pm; x'^+_\mp, x'^-_\mp)\, S_{2\mp}(x'^+_\mp, x'^-_\mp; x^+_\pm, x^-_\pm) = \sqrt{\frac{x^+_\pm x'^+_\mp}{x^-_\pm x'^-_\mp}\,\frac{1 - x^-_\pm x'^-_\mp}{1 - x^+_\pm x'^+_\mp}}$$

crossing symmetry $\qquad \bar{x}^\pm_\pm = \dfrac{1}{x^\pm_\mp} \qquad \bar{e}_\pm = -e_\mp\,, \qquad \bar{\mathrm{p}} = -\mathrm{p}$

$$S^c_{1\pm} = S_{2\pm}\sqrt{\frac{x^-_\pm}{x^+_\pm}\,\frac{1 - x^+_\pm x'^-_\mp}{1 - x^-_\pm x'^-_\mp}}\,, \qquad S^c_{2\pm} = S_{1\mp}\sqrt{\frac{x'^+_\mp}{x'^-_\mp}\,\frac{x^-_\mp - x'^-_\mp}{x^-_\mp - x'^+_\mp}}$$

natural to conjecture that the pattern of the generalization to the $q \neq 0$ case may also apply to the phases

$$S_{1\pm} \stackrel{?}{=} S_1(x^+_\pm, x^-_\pm; x'^+_\pm, x'^-_\pm)\,, \qquad S_{2\pm} \stackrel{?}{=} S_2(x^+_\pm, x^-_\pm; x'^+_\mp, x'^-_\mp)$$

However, this prescription is ambiguous

modification that resolves this ambiguity. for the strong coupling limit

$$\mathrm{S}_1 A\Big|_{a=0} = \sqrt{\frac{x^+ x'^-}{x^- x'^+}\,\frac{x^- - x'^+}{x^+ - x'^-}\,\frac{1-\frac{1}{x^- x'^+}}{1-\frac{1}{x^+ x'^-}}}\ \sigma^{-1}_{\text{AFS}}\ ,$$

$$\sigma_{\text{AFS}}(x^+, x^-; x'^+, x'^-) = B\, e^{i\mathrm{h}\,\vartheta_0}\ , \qquad B = \frac{1-\frac{1}{x^- x'^+}}{1-\frac{1}{x^+ x'^-}}\ ,$$

$$\vartheta_0 = \frac{1}{4}\Big(x^+ + \frac{1}{x^+} + x^- + \frac{1}{x^-} - x'^+ - \frac{1}{x'^+} - x'^- - \frac{1}{x'^-}\Big) \log\Big[\frac{1-\frac{1}{x^+ x'^-}}{1-\frac{1}{x^+ x'^+}}\,\frac{1-\frac{1}{x^- x'^+}}{1-\frac{1}{x^- x'^-}}\Big]$$

$$\sigma_{\text{AFS}} \ \rightarrow\ \sigma_{\text{AFS}_\text{q}} = B\, e^{i\mathrm{h}\,\hat{q}\,\vartheta_0} \qquad \hat{q} = \sqrt{1-q^2}$$

recover the tree-level string-theory S-matrix for the elementary massive excitations

"giant magnon" limit: $\mathrm{h} \to \infty\ , \quad \mathrm{p} = \text{fixed}\ , \quad q = \text{fixed}$

$$x^\pm_\pm = r_\pm\, e^{\pm\frac{i\mathrm{p}}{2}}\ , \qquad r_\pm = \frac{1}{\hat{q}}\Big[1 \pm q + \frac{\mathrm{h}^{-1}}{2\sin\frac{\mathrm{p}}{2}} + O(\mathrm{h}^{-2})\Big] = \frac{1}{\sqrt{1\mp q}} + O(\mathrm{h}^{-1})$$

$$\mathrm{S}_{1\pm} \sim \mathrm{S}_{2\pm} \sim\ e^{-i\mathrm{h}\vartheta_0}\ , \qquad \vartheta_0 = \Big(\cos\frac{\mathrm{p}}{2} - \cos\frac{\mathrm{p}'}{2}\Big)\log\frac{1-\hat{q}^2\cos^2\frac{\mathrm{p}-\mathrm{p}'}{4}}{1-\hat{q}^2\cos^2\frac{\mathrm{p}+\mathrm{p}'}{4}} + \mathcal{O}(\mathrm{h}^{-1})$$

$q = 1$: phase is trivial relation to WZW theory?

Summary

Full S-matrix is product of two factor S-matrices:
same structure as in q=0 case
but with $\pm$ subscripts added

Remains to check against semiclassical finite gap eqs;
derive classical and one-loop phase from
giant magnon scattering, etc.
Check against perturbative string S-matrix at loop level
using unitarity approach [Bianchi,Forini,Hoare 13; Engelund,McKeown,Roiban 13]

Exact disp. relation and exact S-matrix - starting point for
construction of corresponding Bethe Ansatz
Should have again same structure as in q=0 case
as full symmetry algebra is same: $psu(1,1|2)^2$

Open problems:

1. fix 4 phases from crossing conditions
2. construct corresponding asymptotic BA
3. clarify role of massless modes and find corresponding TBA / Y-system

Similar construction of l.c. S-matrix for AdS2 x S2xT6

From exact S-matrix to Asymptotic Bethe Ansatz: q=0

[Borsato,OhlssonSax,Sfondrini,Stefanski,Torrielli 13]

$$1=\prod_{j=1}^{K_2}\frac{y_{1,k}-x_j^+}{y_{1,k}-x_j^-}\prod_{j=1}^{K_{\bar 2}}\frac{1-\frac{1}{y_{1,k}\bar x_j^-}}{1-\frac{1}{y_{1,k}\bar x_j^+}},$$

$$\left(\frac{x_k^+}{x_k^-}\right)^L=\prod_{\substack{j=1\\ j\neq k}}^{K_2}\frac{x_k^+-x_j^-}{x_k^--x_j^+}\frac{1-\frac{1}{x_k^+x_j^-}}{1-\frac{1}{x_k^-x_j^+}}\sigma^2(x_k,x_j)\prod_{j=1}^{K_1}\frac{x_k^--y_{1,j}}{x_k^+-y_{1,j}}\prod_{j=1}^{K_3}\frac{x_k^--y_{3,j}}{x_k^+-y_{3,j}}$$

$$\times\prod_{j=1}^{K_{\bar 2}}\frac{1-\frac{1}{x_k^+\bar x_j^+}}{1-\frac{1}{x_k^-\bar x_j^-}}\frac{1-\frac{1}{x_k^+\bar x_j^-}}{1-\frac{1}{x_k^-\bar x_j^+}}\tilde\sigma^2(x_k,\bar x_j)\prod_{j=1}^{K_{\bar 1}}\frac{1-\frac{1}{x_k^-y_{\bar 1,j}}}{1-\frac{1}{x_k^+y_{\bar 1,j}}}\prod_{j=1}^{K_{\bar 3}}\frac{1-\frac{1}{x_k^-y_{\bar 3,j}}}{1-\frac{1}{x_k^+y_{\bar 3,j}}},$$

$$1=\prod_{j=1}^{K_2}\frac{y_{3,k}-x_j^+}{y_{3,k}-x_j^-}\prod_{j=1}^{K_{\bar 2}}\frac{1-\frac{1}{y_{3,k}\bar x_j^-}}{1-\frac{1}{y_{3,k}\bar x_j^+}},$$

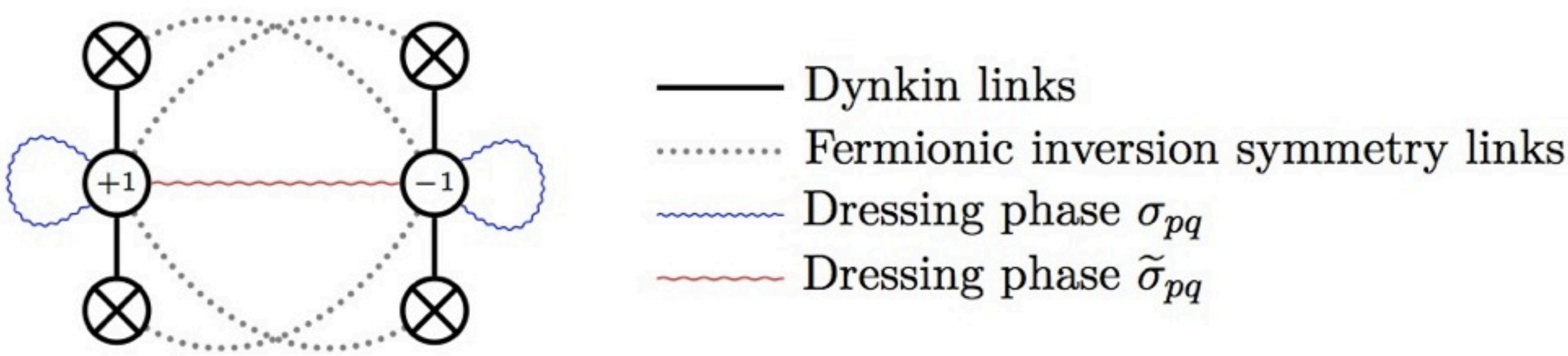
$$1=\prod_{j=1}^{K_{\bar 2}}\frac{y_{\bar 1,k}-\bar x_j^-}{y_{\bar 1,k}-\bar x_j^+}\prod_{j=1}^{K_2}\frac{1-\frac{1}{y_{\bar 1,k}x_j^+}}{1-\frac{1}{y_{\bar 1,k}x_j^-}},$$

$$\left(\frac{\bar x_k^+}{\bar x_k^-}\right)^L=\prod_{\substack{j=1\\ j\neq k}}^{K_{\bar 2}}\frac{\bar x_k^--\bar x_j^+}{\bar x_k^+-\bar x_j^-}\frac{1-\frac{1}{\bar x_k^+\bar x_j^-}}{1-\frac{1}{\bar x_k^-\bar x_j^+}}\sigma^2(\bar x_k,\bar x_j)\prod_{j=1}^{K_{\bar 1}}\frac{\bar x_k^+-y_{\bar 1,j}}{\bar x_k^--y_{\bar 1,j}}\prod_{j=1}^{K_{\bar 3}}\frac{\bar x_k^+-y_{\bar 3,j}}{\bar x_k^--y_{\bar 3,j}}$$

$$\times\prod_{j=1}^{K_2}\frac{1-\frac{1}{\bar x_k^-x_j^-}}{1-\frac{1}{\bar x_k^+x_j^+}}\frac{1-\frac{1}{\bar x_k^+x_j^-}}{1-\frac{1}{\bar x_k^-x_j^+}}\tilde\sigma^2(\bar x_k,x_j)\prod_{j=1}^{K_1}\frac{1-\frac{1}{\bar x_k^+y_{1,j}}}{1-\frac{1}{\bar x_k^-y_{1,j}}}\prod_{j=1}^{K_3}\frac{1-\frac{1}{\bar x_k^+y_{3,j}}}{1-\frac{1}{\bar x_k^-y_{3,j}}},$$

$$1=\prod_{j=1}^{K_{\bar 2}}\frac{y_{\bar 3,k}-\bar x_j^-}{y_{\bar 3,k}-\bar x_j^+}\prod_{j=1}^{K_2}\frac{1-\frac{1}{y_{\bar 3,k}x_j^+}}{1-\frac{1}{y_{\bar 3,k}x_j^-}}.$$

Dynkin diagram for $\mathfrak{psu}(1,1|2)^2$ with the various interaction terms



cf. BA for $su(4) \oplus su(4)$ spin chain

$$E = E_2 + E_{\bar{2}}, \qquad E_j = \sum_{k=1}^{K_j} \sqrt{1 + 16h^2 \sin^2 \frac{p_k}{2}}.$$

Dressing phases: q=0

[Borsato,OhlssonSax,Sfondrini,Stefanski,Torrielli 13]

$$\sigma(p_1,p_2) = e^{i\,\theta(p_1,p_2)}, \qquad \tilde{\sigma}(p_1,p_2) = e^{i\,\tilde{\theta}(p_1,p_2)}$$

$$\sigma^+(p_1,p_2) \equiv \sigma(p_1,p_2)\,\tilde{\sigma}(p_1,p_2)\,, \qquad \sigma^-(p_1,p_2) \equiv \frac{\sigma(p_1,p_2)}{\tilde{\sigma}(p_1,p_2)}$$

$$\theta(p_1,p_2) = \chi(x_1^+,x_2^+) + \chi(x_1^-,x_2^-) - \chi(x_1^+,x_2^-) - \chi(x_1^-,x_2^+)$$

$AdS_5 \times S^5$: BES phase [Beisert,Eden,Staudacher; Dorey,Hofman,Maldacena]

$$\chi^{\rm BES}(x,y) = i \oint \frac{dw}{2\pi i} \oint \frac{dw'}{2\pi i} \frac{1}{x-w} \frac{1}{y-w'} \log \frac{\Gamma[1+ih(w+1/w-w'-1/w')]}{\Gamma[1-ih(w+1/w-w'-1/w')]}$$

$$\chi^{\rm HL}(x,y) = \frac{\pi}{2} \oint \frac{dw}{2\pi i} \oint \frac{dw'}{2\pi i} \frac{1}{x-w} \frac{1}{y-w'} \,{\rm sign}(w'+1/w'-w-1/w)$$

$AdS_3 \times S^3$: $\qquad \chi^+(x,y) = 2\,\chi^{\rm BES}(x,y) - \chi^{\rm HL}(x,y)$

$$\chi^-(x,y) = \oint \frac{dw}{8\pi} \frac{1}{x-w} \log\left[(y-w)\left(1-\frac{1}{yw}\right)\right] {\rm sign}((w-1/w)/i) \quad - x \leftrightarrow y$$