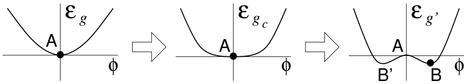
Gapped quantum matter, many-body quantum entanglement, and (symmetry-protected) topological orders

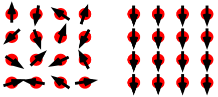
Xiao-Gang Wen, July, 2013

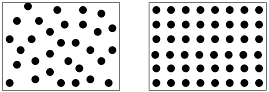
Can we understand/classify all strongly interacting gapped quantum phases systematically?

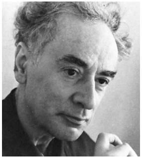
Symmetry breaking theory of phases (orders)

- For a long time, we believe that
  phase transition = change of symmetry
  the different phases = different symmetry $\rightarrow$ different materials

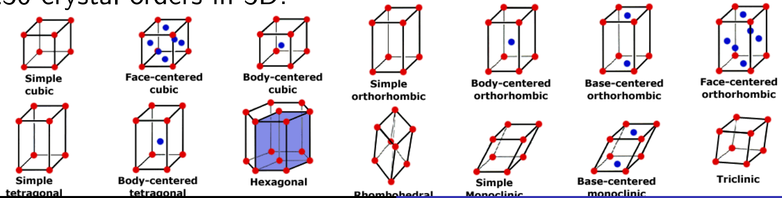






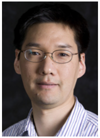


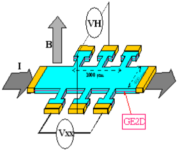
- **The math foundation is group theory:** classified by (G_H, G_Ψ)
  From 230 ways of translation symmetry breaking, we obtain the 230 crystal orders in 3D.

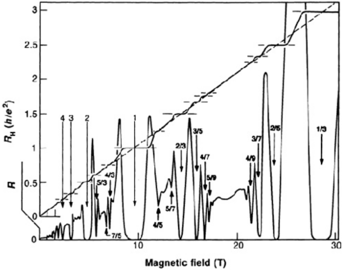


New quantum phases beyond symmetry-breaking

- Example:
  - Quantum Hall states $\sigma_{xy} = \frac{m}{n}\frac{e^2}{h}$
  - Spin liquid states







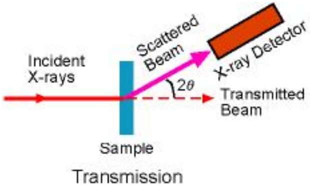
- FQH states and spin-liquid states have have different phases with no symmetry breaking, no crystal order, no spin order, ... so they must have a new order – **topological order** Wen 89

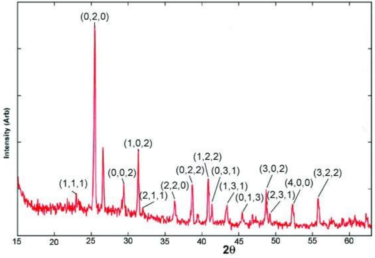
What IS topological order?

To define a physical concept, such as symmetry-breaking order or topological order, is to design a probe to measure it

For example,

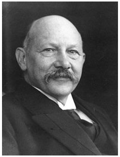
- crystal order is defined/probed by X-ray diffraction:

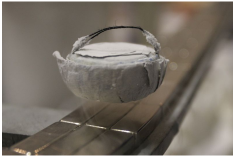




Symmetry-breaking orders through experiments

Order	Experiment
Crystal order	X-ray diffraction
Ferromagnetic order	Magnetization
Anti-ferromagnetic order	Neutron scattering
Superconducting order	Zero-resistance & Meissner effect
Topological order	???



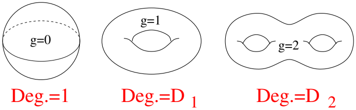


- All the above probes are linear responses. But topological order cannot be probed/defined through linear responses.

Topological orders through experiments (1990)

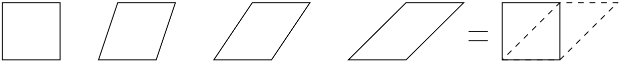
Topological order can be defined through two topological properties that are **robust against any local perturbations/impurities**

(1) **Topology-dependent ground state degeneracy** D_g Wen 89



(2) **Non-Abelian geometric's phases** of the degenerate ground state from deforming the torus: Wen 90

- Shear deformation T: $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = T_{\alpha\beta}|\Psi_\beta\rangle$



- 90° rotation S: $|\Psi_\alpha\rangle \to |\Psi''_\alpha\rangle = S_{\alpha\beta}|\Psi_\beta\rangle$

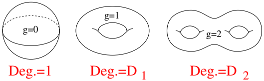
- T, S, define topological order "experimentally".
- T, S is a *universal probe* for any 2D topological orders, just like X-ray is a universal probe for any crystal orders.

Symmetry-breaking/topological orders through experiments

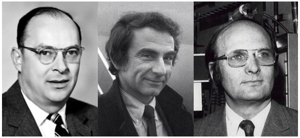
Order	Experiment
Crystal order	X-ray diffraction
Ferromagnetic order	Magnetization
Anti-ferromagnetic order	Neutron scattering
Superconducting order	Zero-resistance & Meissner effect
Topological order	Topological degeneracy, Non-Abelian geometric phases

- The linear-response probe **Zero-resistance** and **Meissner effect** define **superconducting order**. Treating the EM fields as non-dynamical fields
- The topological probe **Topological degeneracy** and **non-Abelian geometric phases** T, S define a completely new class of order – **topological order**.
- T, S determines the quasiparticle statistics. Keski-Vakkuri & Wen 93; Zhang-Grover-Turner-Oshikawa-Vishwanath 12; Cincio-Vidal 12

What is the microscopic picture of topological order?



- Why the macroscopic properties, D_g and $S\&T$, are indenpent of any local perturbations? What is the microscopic understanding?
- Zero-resistance and Meissner effect $\rightarrow$ macroscopic definition of superconducting order.
- It took 40 years to gain a microscopic picture of superconducting order: **electron-pair condensation**



Bardeen-Cooper-Schrieffer 57

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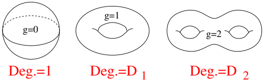

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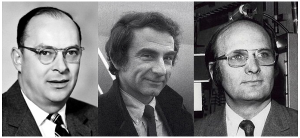

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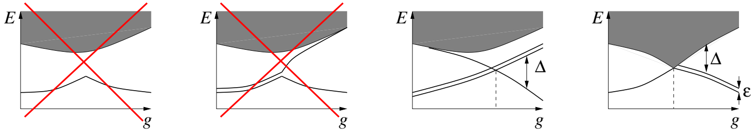
- It took 20 years to gain a microscopic understanding of topological order: **long-range entanglements** Chen-Gu-Wen 10 (defined by local unitary trans. and motivated by topological entanglement entropy). Kitaev-Preskill 06,Levin-Wen 06

Gapped quantum phases and local-unitary transformation



- Two gapped states, $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$, are in the same phase iff they are connected by a local unitary (LU) evolution

$$|\Psi(1)\rangle = P\left(e^{-\mathrm{i}\int_0^T dt\, H(t)}\right)|\Psi(0)\rangle$$

where $\tilde{H}(g) = \sum_{\mathbf{i}} O_{\mathbf{i}}(g)$ and $O_{\mathbf{i}}(g)$ are local hermitian operators.

Hastings, Wen 05;

Bravyi, Hastings, Michalakis 10

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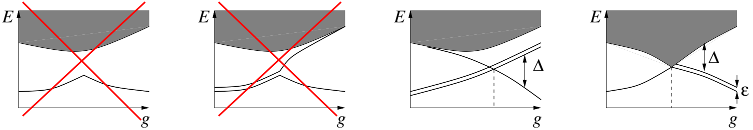

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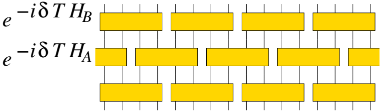

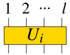

- Any LU evolution can be described by a finite-depth quantum circuit – LU transformation:

$$|\Psi(1)\rangle = P\left(e^{-\mathrm{i}\int_0^T dt\ H(t)}\right)|\Psi(0)\rangle = \prod_t \left(e^{-\mathrm{i}\delta t\ H(t)}\right)|\Psi(0)\rangle$$

$$= (\text{local unitary transformation})|\Psi(0)\rangle$$

Pattern of long-range entanglements = topological order

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break
  $\to$ all systems belong to one trivial phase

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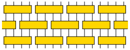
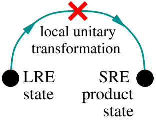
- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are **long range entangled (LRE) states**
  - There are **short range entangled (SRE) states**

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$



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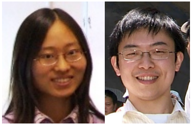

- According to Landau theory, no symmetry to break
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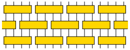
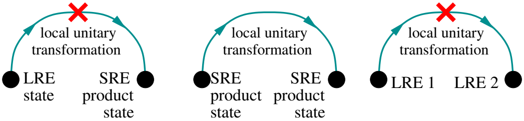
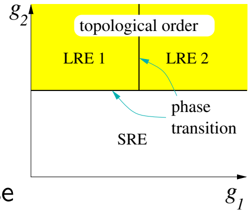
- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are **long range entangled (LRE) states** $\to$ many phases
  - There are **short range entangled (SRE) states** $\to$ one phase

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$





- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different **patterns of long-range entanglements** defined by the LU trans.
  = different **topological orders** Wen 1989
  $\to$ A classification by **tensor category theory** Levin-Wen 05, Chen-Gu-Wen 2010

Gapped phases w/ symmetry $\to$ SET and SPT phases

- **there are** LRE symmetric states $\to$ Symm. Enriched Topo. phases
  - 100s symm. spin liquid through the PSG of topo. excit. Wen 02
  - 8 trans. symm. enriched Z_2 topo. order in 2D, 256 in 3D Kou-Wen 09
  - Many symm. Z_2 spin liquid through $[\mathcal{H}^2(SG, Z_2)]^2 \times$ Hermele 12
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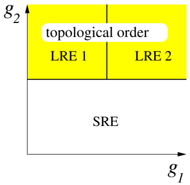
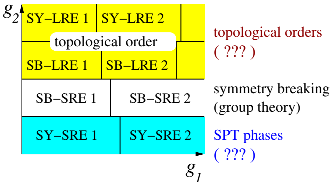
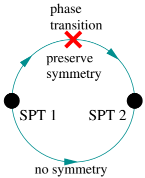
We may call them **symmetry protected trivial (SPT)** phase







- Haldane phase of 1D spin-1 chain w/ $SO(3)$ symm. Haldane 83

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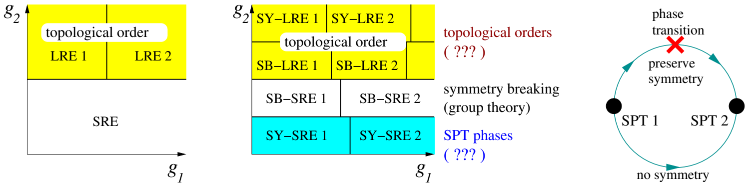

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- 1 topo. ins. w/ $U(1) \times T$ symm. in 2D, Kane-Mele 05; Bernevig-Zhang 06
  15 in 3D Moore-Balents 07; Fu-Kane-Mele 07

Free fermion SPT phases: A K-theory

q	$\pi_0(R_q)$	$d=1$	$d=2$	$d=3$
0	$\mathbb{Z}$		no symmetry (p_x+ip_y, e.g., SrRu)	T only (^{3}He-B)
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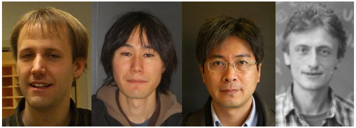
4	$\mathbb{Z}$			
5	0			
6	0			
7	0			no symmetry

	TRS	PHS	SLS	$d=1$	$d=2$	$d=3$
A (unitary)	0	0	0	-	$\mathbb{Z}$	-
AI (orthogonal)	+1	0	0	-	-	-
AII (symplectic)	−1	0	0	-	$\mathbb{Z}_2$	$\mathbb{Z}_2$
AIII (chiral unitary)	0	0	1	$\mathbb{Z}$	-	$\mathbb{Z}$
BDI (chiral orthogonal)	+1	+1	1	$\mathbb{Z}$	-	-
CII (chiral symplectic)	−1	−1	1	$\mathbb{Z}$	-	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	-
C	0	−1	0	-	$\mathbb{Z}$	-
DIII	−1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
CI	+1	−1	1	-	-	$\mathbb{Z}$

Kitaev 08



Schnyder-Ryu-Furusaki-Ludwig 08

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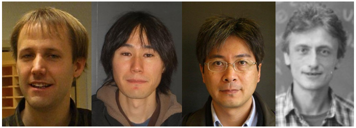

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- How to include strong interactions $\rightarrow$ **Mission impossible**?

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Kitaev 08

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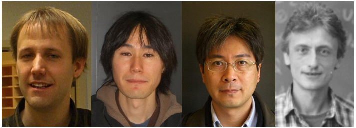

Schnyder-Ryu-Furusaki-Ludwig 08

- How to include strong interactions → **Mission impossible**?
- SPT phases are 'trivial' (short-range entangled) → **Mission possible**

Can we understand/classify all interacting SPT phases?

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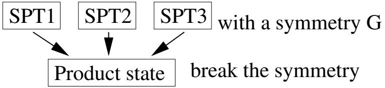

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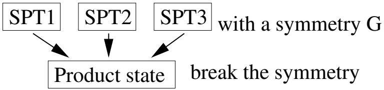

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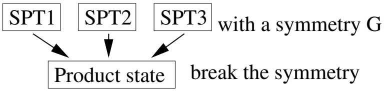

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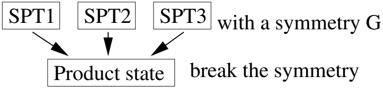

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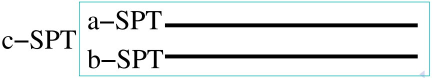
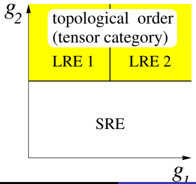
- $\mathcal{H}^{d+1}[G, U(1)]$ form an Abelian group: $a+b=c$,
  - Stacking a-SPT state and b-SPT state give us a c-SPT state.

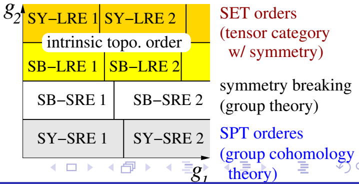


Bosonic SPT phases in any dim. and for any symmetry

Symmetry G	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \rtimes Z_2^T$ (top. ins.)	$\mathbb{Z}$	$\mathbb{Z}_2$ (0)	$\mathbb{Z}_2$ $(\mathbb{Z}_2)$	$\mathbb{Z}_2^2$ $(\mathbb{Z}_2)$
$U(1) \rtimes Z_2^T \times$ trans	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^8$
$U(1) \times Z_2^T$ (spin sys.)	0	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^3$
$U(1) \times Z_2^T \times$ trans	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
Z_2^T (top. SC)	0	$\mathbb{Z}_2$ $(\mathbb{Z})$	0 (0)	$\mathbb{Z}_2$ (0)
$Z_2^T \times$ trans	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
$U(1)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$U(1) \times$ trans	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$
Z_n	$\mathbb{Z}_n$	0	$\mathbb{Z}_n$	0
$Z_n \times$ trans	$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_n^2$	$\mathbb{Z}_n^4$
$D_{2h} = Z_2 \times Z_2 \times Z_2^T$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$
$SO(3)$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$SO(3) \times Z_2^T$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$

Table of $\mathcal{H}^{d+1}[G, U_T(1)]$
"Z_2^T": time reversal,
"trans": translation,
others: on-site symm.
$0 \to$ only trivial phase.
$(\mathbb{Z}_2) \to$ free fermion result





Understand group cohomology through topological terms

- Consider an $d+1$D system with symmetry G:
  $S = \int d^dxdt \frac{1}{2\lambda}(\partial g(x^i,t))^2$, symmetry $g(x) \to hg(x), \quad h,g \in G$
  If under RG, $\lambda \to \infty \to$ symmetric ground state described by a fixed point theory $S_{\text{fixed}} = 0$ or $\mathrm{e}^{-S_{\text{fixed}}} = 1$.

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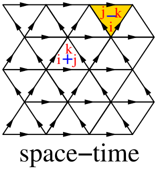
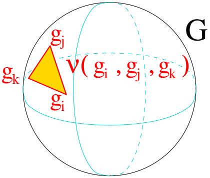
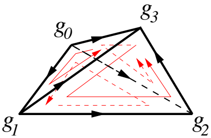
- Can we define topological terms and topological non-linear σ-models when space-time is a discrete lattice?

Lattice topological non-linear σ-model in 1+1D

- Generalize to space-time lattice $e^{-S} = \prod \nu(g_i, g_j, g_k)$,
  where $\nu(g_i, g_j, g_k) = e^{-\int_{\triangle} L}$, with branched structure







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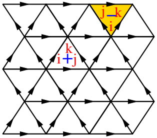

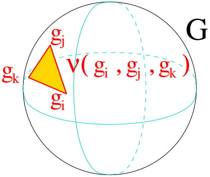

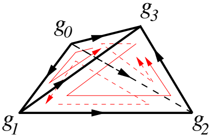

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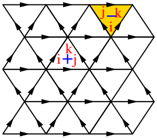
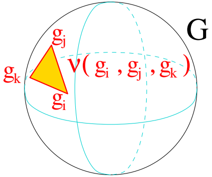
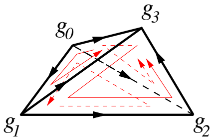


space–time





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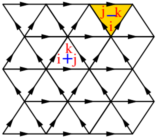

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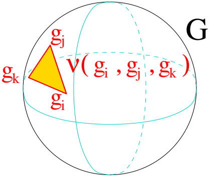

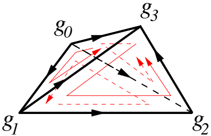

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The solutions of the above equation are called **group cocycle**.

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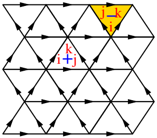

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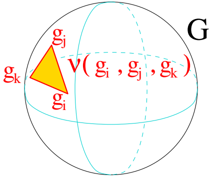

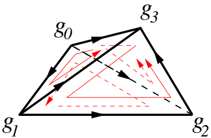

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  The solutions of the above equation are called **group cocycle**.
- $\nu_2(g_0, g_1, g_2)$ and $\tilde\nu_2(g_0, g_1, g_2) = \nu_2(g_0, g_1, g_2)\frac{\beta_1(g_1,g_2)\beta_1(g_0,g_1)}{\beta_1(g_0,g_2)}$ are both cocycles. We say $\nu_2 \sim \tilde\nu_2$ (equivalent).
  The set of the equivalent classes of ν_2 is denoted as $\mathcal{H}^2[G, U(1)]$.

Group cohomology $\mathcal{H}^d[G, U(1)]$ in any dimensions

- d-Cochain: $U(1)$ valued function of $d+1$ variables
  $\nu_d(g_0, ..., g_d) = \nu_d(gg_0, ..., gg_d) \in U(1)$, $\to$ on-site G -symmetry

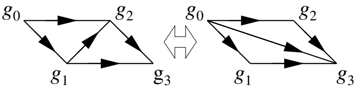
- δ-map: $d+1$ variable function $\nu_d \to d+2$ variable function $(\delta\nu_d)$
  $$(\delta\nu_d)(g_0, ..., g_{d+1}) = \prod_i \nu_d^{(-)^i}(g_0, ..., \hat{g}_i, ..., g_{d+1})$$

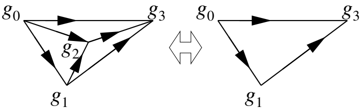
- Cocycles = cochains that satisfy
  $$(\delta\nu_d)(g_0, ..., g_{d+1}) = 1.$$

- Equivalence relation generated by any $d-1$-cochain:
  $$\nu_d(g_0, ..., g_d) \sim \nu_d(g_0, ..., g_d)(\delta\beta_{d-1})(g_0, ..., g_d)$$

- $\mathcal{H}^{d+1}[G, U(1)]$ is the equivalence class of cocycles ν_d.
  Lattice topological non-linear σ-models with symmetry G in d-spatial dimensions are classified by $\mathcal{H}^{d+1}[G, U(1)]$:
  $e^{-S} = \prod_{M_{1+d}} \nu_{d+1}^{s(i,j,...)}(g_i, g_j, ...)$, $\quad \nu_{d+1}(g_0, g_1, ..., g_{d+1}) \in \mathcal{H}^{d+1}[G, U(1)]$

Topological invariance in topological non-linear σ-models





As we change the lattice, the action amplitude e^{-S} does not change:

$$\nu_2(g_0, g_1, g_2)\nu_2^{-1}(g_1, g_2, g_3) = \nu_2(g_0, g_1, g_3)\nu_2^{-1}(g_0, g_2, g_3)$$

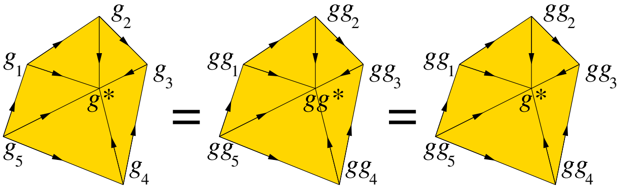
$$\nu_2(g_0, g_1, g_2)\nu_2^{-1}(g_1, g_2, g_3)\nu_2(g_0, g_2, g_3) = \nu_2(g_0, g_1, g_3)$$

as implied by the cocycle condition:

$$\nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)\nu_2^{-1}(g_0, g_2, g_3)\nu_2^{-1}(g_0, g_1, g_2) = 1$$

The topological non-linear σ-model is a RG fixed-point.

The ground state of the topological non-linear σ-model



The ground state wave function $\Psi(\{g_i\}) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$

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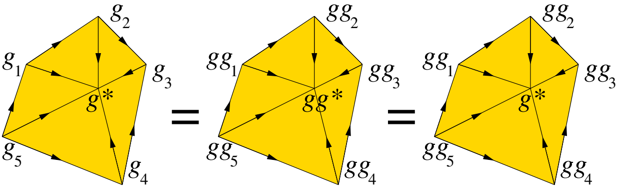

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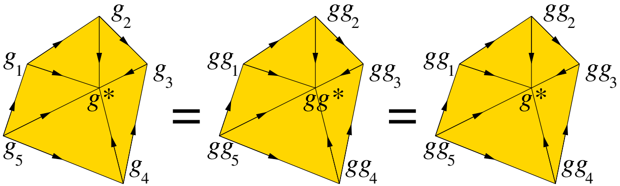

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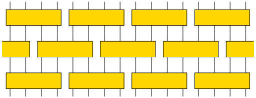
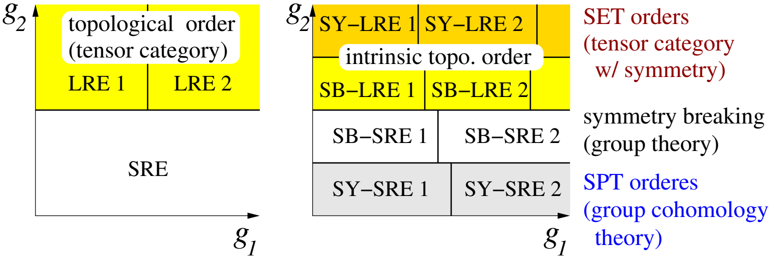
- **is symmetric** under the G-transformation $\Psi(\{g_i\}) = \Psi(\{gg_i\})$
- **is equivalent to a product state** $|\Psi_0\rangle = \otimes_i \sum_{g_i} |g_i\rangle$ under a LU transformation (note that $\Psi_0(\{g_i\}) = 1$)

$$\Psi(\{g_i\}) = \prod_{i=\text{even}} \nu_2(g_i, g_{i+1}, g^*) \prod_{i=\text{odd}} \nu_2(g_i, g_{i+1}, g^*) \Psi_0(\{g_i\})$$

$=$  $\Psi_0(\{g_i\})$

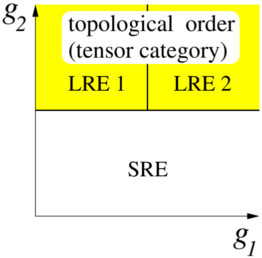
The ground state is symmetric with a trivial topological order

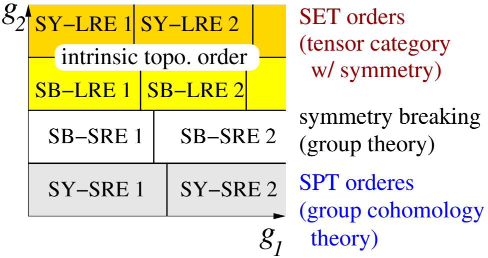
Summary: group cohomology → SPT states



How to probe the topological order and SPT order?

Summary: group cohomology → SPT states





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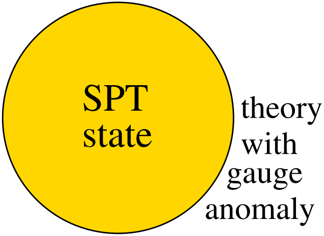
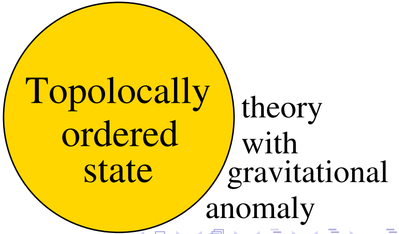
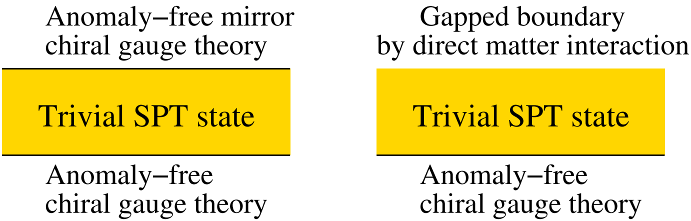
Bulk topological phases ↔ Boundary anomalous theories





An application of SPT classification of gauge anomaly

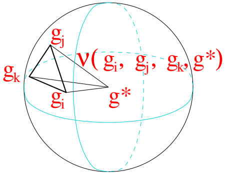
Solved the chiral-fermion/chiral-gauge theory problem:
Any anomaly-free chiral gauge theory can be defined as the ordinary lattice gauge theory in the same dimension, if we include direct interactions between the matter fields.



The key and the hard part is to show that a chiral gauge theory is really free of ALL anomalies.

Boundary excitations of SPT phases

- SPT boundary excitations are describe by a lattice non-linear σ-model at the boundary with a non-local Lagrangian term (a generalization of the WZW term for continuous σ-model):



2+1D space–time

$$e^{-\int_{\partial M_{1+d}} \mathcal{L}_{NLL}} = \prod_{\partial M_{1+d}} \nu_{d+1}^{s(i,j,k)}(g_i, g_j, g_k, g^*) \neq \prod_{\partial M_{1+d}} \mu_{\text{symm}}(g_i, g_j, g_k)$$

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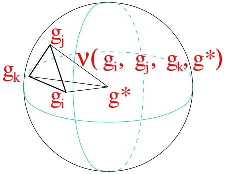

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either symmetric in one higher dimension or "non-symmetric" in the same dimension $\to$ discretized WZW term.

- **Conjecture** (proved for 1+1D boundary Chen-Liu-Wen 11): The boundary are gapless or degenerate: Chen-Liu-Wen 11; Xu 12; Senthil-Vishwanath 12; ...
(a) if the boundary does not break the symmetry $\to$ gapless or topologically ordered (degenerate)
(b) if the boundary break the symmetry $\to$ gapless or degenerate.
Generalize the result for WZW model in (1+1)D Witten 89

Symmetry of the effective boundary theory

- Effective boundary symmetry in path-integral formalism:

$$e^{-\int_{\partial M_{1+d}} \mathcal{L}_{Bnd}} = \prod_{\partial M_{1+d}} \nu_{d+1}^{s(i,j,k)}(g_i, g_j, g_k, g^*) = \prod_{\partial M_{1+d}} \nu_{d+1}^{s(i,j,k)}(gg_i, gg_j, gg_k, g^*)$$

but locally $\nu_{d+1}(gg_i, gg_j, gg_k, g^*) \neq \nu_{d+1}(g_i, g_j, g_k, g^*)$

Under the symmetry transformation
$\mathcal{L}_{Bnd}[gg(x)] = \mathcal{L}_{Bnd}[g(x)] + \mathrm{d}f[g(x)]$.
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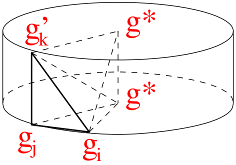

- Effective boundary symmetry in Hamiltonian formalism $\to$ **non-on-site symmetry**

$$(\mathrm{e}^{-\hat{H}_{\text{Bnd}}})_{\{g_i',\dots\},\{g_i,\dots\}} \neq (\mathrm{e}^{-\hat{H}_{\text{Bnd}}})_{\{gg_i',\dots\},\{gg_i,\dots\}}$$

$$= U^\dagger_{\{g_i',\dots\}} (\mathrm{e}^{-\hat{H}_{\text{Bnd}}})_{\{g_i',\dots\},\{g_i,\dots\}} U_{\{g_i,\dots\}}$$

where $U_{\{g_i,\dots\}} = \prod_{\langle ij\rangle} \nu_{d+1}(g_i, g_j, g^*, g^{-1}g^*)$

$$\hat{U}(g) = \prod_i \hat{U}_0(g) \prod_{\langle ij\rangle} \nu_{d+1}(g_i, g_j, g^*, g^{-1}g^*)$$

An example: $SU(2)$ SPT state in 2+1D Liu & Wen 12

For $SU(2)$, $\text{Hom}[\pi_3(SU(2)), \mathbb{Z}] = \mathcal{H}^3[SU(2), U(1)] = \mathbb{Z}$
$\to$ we can use the topological terms in field theory to classify the $SU(2)$-SPT states:

$$S_{\text{top}} = -i\frac{k}{12\pi}\int_M \text{Tr}(g^{-1}dg)^3, \quad k \in \mathbb{Z}, \quad g \in SU(2)$$

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- The edge excitations are gapless (described by fixed-point WZW):

$$S_{\text{Bnd}} = \int_{\partial M}\frac{k}{8\pi}\text{Tr}(\partial g^{-1}\partial g) - i\int_M \frac{k}{12\pi}\text{Tr}(g^{-1}dg)^3,$$

- At the fixed point, we have a equation of motion

$$\partial_{\bar z}[(\partial_z g)g^{-1}] = 0, \quad \partial_z[(\partial_{\bar z} g^{-1})g] = 0, \quad z = x + \mathrm{i}\,t.$$

Right movers $[(\partial_z g)g^{-1}](z) \to SU(2)$-charges
Left movers $[(\partial_{\bar z} g^{-1})g](\bar z) \to SU_L(2)$-charges, $g(x)\to g(x)h_L$

Level-k Kac-Moody algebra Witten 84

A classification of gauge anomalies

- The edge $SU(2)$ symmetry is anomalous (non-on-site): The above edge excitations cannot be described by a pure 1+1D lattice model with an on-site $SU(2)$ symmetry $U(g) = \otimes_i U_i(g)$.
- If we gauge the $SU(2)$ symmetry, we will get an anomalous chiral gauge theory on the edge, and an $SU(2)$ Chern-Simons theory of level-k in the bulk.
- 2+1D $SU(2)$-SPT phases are classified by $\mathcal{H}^3(SU(2), U(1)) = \mathbb{Z}$
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  In general and roughly speaking (after we gauge the symmetry), **$d+1$D G-gauge-anomalies are classified $d+2$D SPT phases, which is, in turn, classified by $\mathcal{H}^{d+2}(G, U(1))$**
- $\mathcal{H}^{d+2}(G, U(1)) = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \to$ Adler-Bell-Jackiw anomalies.
- $\mathcal{H}^{d+2}(G, U(1)) = \mathbb{Z}_m \oplus \mathbb{Z}_n \oplus \cdots \to$ new global anomalies, for bosonic systems.

Two definitions of the gauge anomaly

First definition: non gauge invariance

- $S = \int \mathrm{d}^d x \mathcal{L}(a_\mu, \psi) = \int \mathrm{d}^d x \mathcal{L}(a_\mu + \partial_\mu f, \mathrm{e}^{\mathrm{i} f}\psi)$, but
  $Z = \int D\psi D A_\mu \ \mathrm{e}^{\mathrm{i} \int \mathrm{d}^d x \mathcal{L}(a_\mu, \psi)} \neq \int D\psi D a_\mu \ \mathrm{e}^{\mathrm{i} \int \mathrm{d}^d x \mathcal{L}(a_\mu + \partial_\mu f, \mathrm{e}^{\mathrm{i} f}\psi)}$
- Example: 1+1D chiral $SU(2)$ gauge theory:
  $S = \int \mathrm{d}^2 x \left[\psi_R^\dagger (i\partial_t - a_0 - i\partial_x + a_x)\psi_R + \psi_L^\dagger (i\partial_t + i\partial_x)\psi_L + \frac{1}{\lambda}(F_{\mu\nu})^2\right]$

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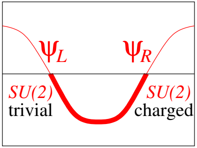
Second' definition

- Take $\lambda \to 0$ limit $\to$ a theory with chiral $SU(2)$ symmetry
  $S = \int \mathrm{d}^2 x \left[\psi_R^\dagger (i\partial_t - i\partial_x)\psi_R + \psi_L^\dagger (i\partial_t + i\partial_x)\psi_L\right]$
  $\psi_R \to U\psi_R, \quad \psi_L \to \psi_L.$
- The above theory has no non-perturbative definition, say on lattice, without breaking the symmetry.

Lattice models and non-perturbative definition

$$H = \sum_i \psi_i^\dagger \psi_{i+1} + h.c.$$

We can have a non-perturbative definition only if we break the $SU(2)$ symmetry.



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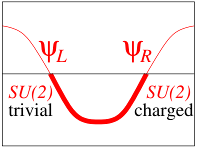

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- Go to one higher dimension:
  $\nu = 1$ QH state state for spin-up and spin-down fermions +
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  which is a non-trivial 2+1D symmetry protected topological (SPT) state protected by $SU(2)$ symmetry.

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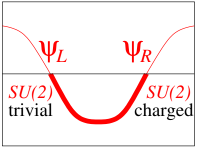

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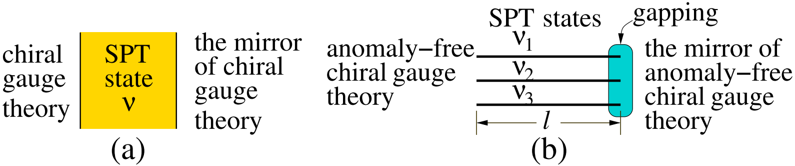
Summary

(fermionic) gauge anomaly
$\leftrightarrow$ (fermionic) anomalous symmetry (non-on-site symmetry)
$\leftrightarrow$ (fermionic) SPT state in one higher dimension
$\leftrightarrow$ group (super)-cohomology

A lattice definition of any anomaly-free gauge theories



- For anomaly-free gauge theories, the corresponding SPT state in one-higher dimensions is trivial $\sum \nu_i = 0$ $\to$
- the edge states can be gapped with no ground state degeneracy and symmetry breaking $\to$
- any anomaly-free gauge theories can be defined on lattices by considering **interacting** boson/fermion.

Try to define the $U(1) \times SU(2) \times SU(3)$ standard model

- After so many years of study, $U(1) \times SU(2) \times SU(3)$ standard model is not even a proper quantum theory, since we still do not have a non-perturbative definition of the model.
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Try to define a $U(1)$ chiral fermion model

- *We cannot define the chiral fermion as a 3D inter. lattice model*
  - We can define the chiral fermion model as a boundary of a 4D gapped $U(1)$ symmetric free/inter. fermion lattice model
  - The 4D fermion lattice model is a non-trivial inter. fermionic $U(1)$ SPT state (which induces $A\mathrm{d}A\mathrm{d}A$ CS-term).
  - We cannot gap out the mirror sector without breaking the $U(1)$ symmetry, even with interactions. (Can be proved)

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- If we view the massless $U(1)$ chiral fermion as the boundary fermion of a 4+1D lattice, after we turn on the $U(1)$ gauge field, the massless gauge boson will live in the 4+1D bulk.

Try to define the $SO(10)$ chiral fermion model

- Try to put 16 chiral fermion in 3+1D
  $\hat{H} = \psi_\alpha^\dagger \mathrm{i}\partial_i \sigma^i \psi_\alpha, \quad \psi_\alpha$ form the 16-dim. spinor rep. of $SO(10)$
  on a 3D spatial lattice.

- *We cannot define the $SO(10)$ chiral fermion as a 3D* **free** *model*
  - We can define the $SO(10)$ chiral fermion model as a boundary of a 4D gapped $SO(10)$ symmetric free fermion lattice model
  - The 4D free fermion lattice model is a non-trivial free fermionic $SO(10)$ SPT state.
  - We cannot gap out the mirror sector without breaking the $SO(10)$ symm. But can gap out the mirror sector by breaking the $SO(10)$ symm. $\delta H = \psi_\alpha^T \epsilon n_a \Gamma^a_{\alpha\beta} \psi_\beta + h.c., \quad n_a$ form the 10-dim. rep. of $SO(10)$ Γ^a has 8 $+1$ eigenvalues and 8 -1 eigenvalues.

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$\delta H = \psi_\alpha^T \epsilon n_a \Gamma^a_{\alpha\beta} \psi_\beta + h.c.$ n_a form the 10-dim. rep. of $SO(10)$

(2) we let n_a to have long-wave length fluctuations, to restore the $SO(10)$ symmetry, hopefully, the fluctuations do not kill the gap.

Try to define the $SO(10)$ chiral fermion model

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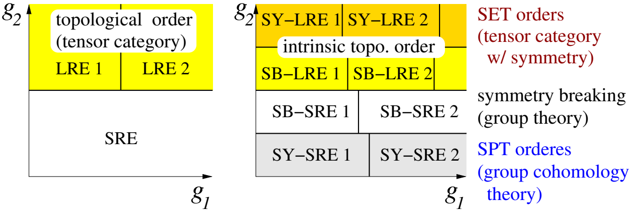
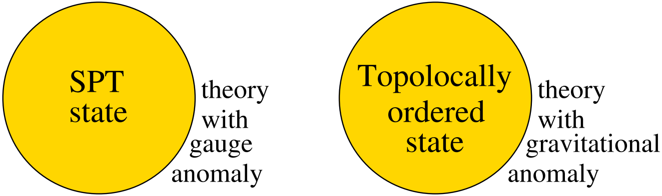
- **We can define the $U(1) \times SU(2) \times SU(3)$ standard model as a 3D interacting fermionic lattice model with continuous time.**

Topological states, anomalies, and the standard model

- **A classification of gapped quantum pahses**



- **Bulk topological phases $\leftrightarrow$ Boundary anomalous theories**



- **We can put the $U(1) \times SU(2) \times SU(3)$ standard model on lattice by simply allow fermions to interact**