Anomalous hydrodynamics of FQH-States

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Based on two papers

Phys. Rev. Lett. 108, 206810 (2012), arXiv:1211.5132

Discussions with friends: A.G. Abanov, E. Bettelheim

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1. Fractional Quantum Hall Effect;
2. Quantum Hydrodynamics of Incompressible Fluid;

Search for Conformal Symmetry in Hydrodynamics

Peculiar fluid of FQH states: $\nu = 1/3$

Particles on a plane in a quantized magnetic field with a strong Coulomb interaction at a fractionally filled $\nu = 1/3$ Landau level form a quantum fluid

SCALES

* Energies
  - Cyclotron energy - distance between Landau levels;

$$\hbar\omega_c = \frac{e\hbar B}{mc} \sim 25K$$

  - Coulomb interaction → a gap at fractional filling: number of electrons per flux quantum $\nu = 1/3$,

$$\Delta \sim \frac{e^2}{\ell} \sim 10K$$

* Length scale: $\ell = \sqrt{\frac{\hbar c}{eB}} \sim 10nm$
* Size of the device $\sim 10 - 100\mu m$
* Number of electrons $N \sim 10^6$

Fractional Quantum Hall States Exist only because

$$\boxed{\hbar\omega_c \gg \Delta}$$

Holomorpic States

At

$$\hbar\omega_c \to \infty$$

All states are contained within the first Landau level are holomorphic:

$$\Psi(z_1, z_2, \ldots, z_N), \quad z_i = x_i + iy_i$$

If the only remaining scale, the gap

$$\Delta \to \infty$$

States below the gap is "*topological sector*":

boundaries -state correspondence

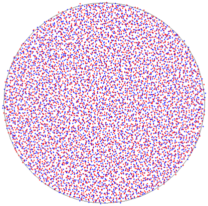
Characteristic features of FQH liquids

- States
  - Liquid;
  - Incompressibile;
  - Fractionally quantized vortices;
- Energy and Forces
  - Dissipation-free;
  - Fractionally quantized Lorenz force (Hall conductance)
  - Fractionally quantized Lorenz shear force (aka odd viscosity or Hall viscosity)
    J. Avron, R. Seiler, P. Zograf, Phys. Rev. Lett. 75, 697 (1995);

LAUGHLIN STATE(S)

All these features are encompassed by the Laughlin w.f.
(interesting physics occurs only at $N \to \infty$)

$$\Psi_0(z_1, \ldots, z_N) = \left[\prod_{i \neq j}^{N} (z_i - z_j) \right]^{\beta} e^{-\sum_i |z_i|^2 / 4\ell^2}$$



after N. Kang

- ℓ -magnetic length;
- $\nu = 1/\beta$ - is a filling fraction;
- $\beta = 1$ - IQHE; $\beta = 3$- FQHE.

Important features:

- Wave-function is holomorphic;
- Degree of zero at $z_i \to z_j$ is larger than 1;

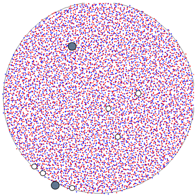
VORTICES

Each state ⟺ holomorphic symmetric polynomial:

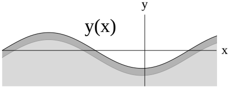
$$|\text{States}\rangle = Q(z_i) \cdot \prod_{i \neq j}^{N} (z_i - z_j)^{\beta}$$

Quasi-holes - punctures in the bulk

$$\Psi_h(z) = \prod_i (z - z_i) \cdot \Psi_0$$



* How do they move?



Quantum Hydrodynamics

Reducing all complexity of Quantum states to just one pair of canonical fields: density ρ and velocity v

$$[\mathrm{v}(r,t),\, \rho(r',t)] = -i\hbar\nabla\delta(r-r')$$

Classical case: local equilibrium

principal of local equilibrium allows to reduce the Boltzmann kinetic equation for the distribution function to hydrodynamics equations for density and velocity.

Quantum case:

A strong coherence of flows (?), holomorphic states.

Known Quantum Fluids

Superfluid: Landau (1946), Feynman (1956); Khalatnikov (1965)

Electronic liquids in 1D: Luttinger (1964);

FQHE: Girvin, MacDonald, Platzmann (1984);

Also:

N. Read (1989) and M. Stone (1990), I. Tokatly (2007), D. T. Son (2007-2012), N. Read (2007-2012).

Minimal set of assumptions:

- flow is incompressible;
- inviscid;

All states are in the form of

$$\text{States} = \left[\text{symm.holomorphic pol}\right] \times \prod_{i \neq j}^{N} (z_i - z_j)^{\beta}$$

$$N \to \infty$$

Incompressible rotating 2D fluid

* Vortices are only degrees freedom;
* Turbulent flow: state with many vortices;
* Quantum fluid: circulation of vortices are quantized.

Point of interest: the vortex fluid

- ∗ Fast motion: fluid precessing around vortices;
- ∗ Slow motion of vortices.

Digression:
Classical Incompressible rotating 2D fluid

Incompressibility	$\nabla \cdot \mathbf{u} = 0,$
Vorticity	$\varpi = \nabla \times \mathbf{u}$

Vorticity is transported along the velocity field:

the material derivative of the vorticity in that flow vanishes:

$$\frac{D\varpi}{Dt} \equiv \dot{\varpi} + \mathbf{u} \cdot \nabla \varpi = 0.$$

Kirchhoff Equations

$$\frac{D\varpi}{Dt} \equiv \dot{\varpi} + \mathbf{u} \cdot \nabla \varpi = 0.$$

Helmholtz (and later Kirchhoff)

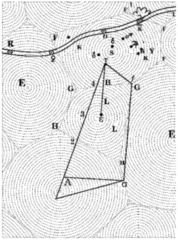
$$\mathrm{u}(z,t) = -i\Omega\bar{z} + \mathrm{i}\sum_{i=1}^{N} \frac{\Gamma_i}{z - z_i(t)}$$

Kirchhoff equations

$$\boxed{\mathrm{i}\,\dot{\bar{z}}_i = \Omega\bar{z}_i - \sum_{i\neq j}^{N} \frac{\Gamma_j}{z_i(t) - z_j(t)}}$$

Cartesian cosmogony

1644, Principia philosophiae



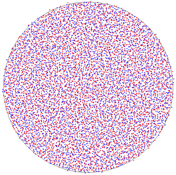
Chiral Flow

Large number of vortices largely compensating rotation

$$\Omega = \pi \bar{\rho} \sum_i \Gamma_i$$

Kirchhoff equations $\Gamma_i = \Gamma$

$$\mathrm{i}\dot{\bar{z}}_i = \Omega \bar{z}_i - \sum_{i \neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)}$$



$$\Psi_0(z_1, \ldots, z_N) = \prod_{i \neq j}^{N} (z_i - z_j)^{\beta} e^{-\sum_i |z_i|^2/4\ell^2},$$

$$\partial_{z_i} \log \Psi_0 = \Omega \bar{z}_i - \sum_{i \neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)}$$

Kirchoff Equations are Hamiltonian

$$i\dot{\bar{z}}_i = \Omega \bar{z}_i - \sum_{i \neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)}$$

$$\mathscr{H} = m_* \Omega \left(\sum_i [\Omega |z_i|^2 - \Gamma \sum_{j \neq i} \log |z_i - z_j|^2 \right)$$

$$(m_* \Omega)\{\bar{z}_i, z_j\}_{P.B.} = -i\delta_{ij}$$

Canonical quantization of Kirchhoff equations

Kirchhoff equations are Hamiltonian

- Poisson brackets → commutator:

$$\{\bar{z}_i, z_j\}_{P.B.} \to [\bar{z}_i, z_j] = 2\ell^2\delta_{ij};$$

- Representation: the operator $\bar{z}_i$ becomes a canonical momentum of the coordinate z_i

$$\boxed{\bar{z}_i = 2\ell^2\partial_{z_i}}$$

- Quantum states: Holomorphic polynomials $\Psi(z_1, \ldots, z_N)$.
- Hermitian conjugation (*chiral condition*)

$$\boxed{\bar{z}_i = z_i^\dagger}$$

Bargmann space - a Hilbert space of analytic polynomials with the inner product

$$\langle \Psi' | \Psi \rangle = \int \overline{\Psi'}\Psi \, d\mu, \qquad d\mu = \prod_i e^{-\frac{|z_i|^2}{2\ell^2}} d^2 z_i$$

Laughlin w.f.

Velocity $\mathrm{v} = \mathrm{v}_x - i\mathrm{v}_y$

$$\text{classical}: \quad i\,\mathrm{v}_i = \Omega \bar{z}_i - \sum_{i \neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)}$$

$$\text{quantum}: \quad i\mathrm{p}_i = \frac{\hbar}{\nu}\big(\nu\,\partial_{z_i} - \sum_{i \neq j} \frac{1}{z_i - z_j}\big)$$

Stationary state (no flow):

$$\mathrm{p}_i|\text{ground state}\rangle = 0$$

Solution is the Laughlin's w.f.

$$\Psi_0 = \prod_{i>j}(z_i - z_j)^{\beta}, \quad \beta = \nu^{-1}$$

Laughlin state is a ground state of a rotating incompressible quantum fluid

Stochastic Quantization

$$d\bar{z}_i = \Omega\bar{z}_i - \sum_{i\neq j}^{N} \frac{\Gamma dt}{z_i(t) - z_j(t)} + d\xi_i, \quad \mathbb{E}[d\xi_i d\bar{\xi}_i] = \frac{2\Gamma}{\nu}\delta_{ij}dt$$

Hydrodynamics of Vortex Flow

* Fast motion: fluid precessing around vortices - velocity u;
* Slow motion of vortices - velocity v:

Task: Hydrodynamics of vortex flow (secondary fluid)

Vorticity - density of vortices : $\rho(r) = \sum_i \delta(r - r_i),$

Momentum of vortices : $P = \rho \mathrm{v} = \sum_i \delta(r - r_i) \mathrm{v}_i.$

$$\mathrm{v}_i \to \mathrm{v}(r) \quad \Leftrightarrow \quad \mathrm{u}(r)$$

$$\mathrm{iu} = \Omega \bar{z} + \sum_i \frac{\Gamma}{z - z_i} = \Omega \bar{z} + \Gamma \int \frac{\rho(\zeta}{z - \zeta} d^2 \zeta$$

Vortex flow $\Leftrightarrow$ Fluid flow

Subtlety: Short distance anomaly

* Short-distance anomaly: Relation between $P = \rho \mathrm{v}$ and $\rho \mathrm{u}$

$$\rho \mathrm{v} = \rho \mathrm{u} + \frac{\hbar}{4\nu} \nabla \times \rho, \quad \mathrm{v} = \mathrm{u} + \frac{\Gamma}{4} \rho^{-1} \nabla \times \rho$$

* Effective change of velocity

$$\mathrm{u} \to \mathrm{v} = \mathrm{u} + \frac{\Gamma}{4} \nabla \times \log \rho$$

* Origin of the short distance anomaly:

$$\boxed{\text{vortex does not interact with itself}}$$

CALCULATIONS

$$\rho \mathrm{v} = \sum_i \delta(r - r_i)\mathrm{v}_i, \quad \mathrm{v}_i = -\mathrm{i}\Omega\bar{z}_i + \mathrm{i}\sum_{i \neq j}^{N} \frac{\Gamma}{z_i - z_j}$$

$$\rho \mathrm{v} = \mathrm{i}\Omega\bar{z}\rho(z) + i\frac{n}{2\pi}\bar{\partial}[(\sum_i \frac{1}{z - z_i})^2 - \sum_i \frac{1}{(z - z_i)^2}] = \rho \mathrm{u} + +\mathrm{i}\frac{\hbar}{2\nu}\partial\rho$$

$$\boxed{\rho \mathrm{v} = \rho \mathrm{u} + \tfrac{\hbar}{4\nu}\nabla \times \rho}$$

LORENTZ SHEAR FORCE:

AVRON, SEILER, ZOGRAF, 1995

* Effective change of velocity

$$\mathrm{v} \to \mathrm{v} - \frac{\Gamma}{4} \nabla \times \log \rho$$

* Anomalous viscous term emerges in the momentum flux tensor

fluid : $$\Pi_{ij} = \rho \mathrm{v}_i \mathrm{v}_j + p\delta_{ij},$$

vortex fluid : $$\Pi_{ij} \to \Pi_{ij} - \sigma'_{ij}$$

odd or Hall viscosity : $$\sigma'_{ij} = -\frac{\hbar}{4\nu}\rho(\epsilon_{ik}\nabla_j \mathrm{v}_k + \epsilon_{jk}\nabla_k \mathrm{v}_i) =$$

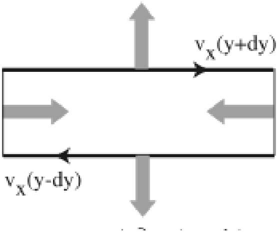
$$= -\frac{\hbar}{4\nu}\rho(\nabla_i \nabla_j - \frac{1}{2}\delta_{ij}\Delta)\Psi, \quad \mathrm{v}_i = -\epsilon_{ij}\Psi$$

Lorentz Shear Force

Pressure acts orthogonal to shear, proportional to a shear (with a universal coefficient $\hbar/4\nu$

$$\sigma'_{xx} = \sigma'_{yy} = -\frac{\hbar}{4\nu}\rho(\nabla_x v_y + \nabla_y v_x),$$

$$\sigma'_{xy} = \sigma'_{yx} = \frac{\hbar}{4\nu}\rho(\nabla_x v_x - \nabla_y v_y)$$



Chiral condition

Holomorphic states (all physics is constraint by the first Landau level):

- Incompressibility: $\nabla \cdot \mathrm{v} = 0,$
- Chiral condition: density determines velocity (or vorticity of the secondary flow)

$$\boxed{\frac{\nu}{2\pi}(\nabla \times \mathrm{v}) = \rho - \bar{\rho} + \frac{1}{4\pi}(\frac{1}{2} - \nu)\Delta \log \rho}$$

Lorentz shear force and trace anomaly

Stream function ψ

$$\mathrm{v} = \nabla \times \psi$$

$$\sigma'_{\mu\nu} = -\frac{\hbar}{2\nu}\rho(\nabla_\mu \nabla_\nu - \frac{1}{2}\delta_{\mu\nu}\Delta)\Psi$$

Metric space g^{ij}:

$$\text{Geometric Action} = \int\int \sigma'_{ij} g^{ij} \sqrt{g} d^2\xi = \underbrace{\frac{\hbar}{4\nu}\bar{\rho}\left(\int\int R\psi\sqrt{g}d^2\xi + \frac{1}{2}\int K\psi ds\right)}_{\text{trace anomaly}}$$

R is Riemann curvature, K is the boundary curvature.

Curved space (trace anomaly)

$$R - \text{Curvature}$$

$$\text{Force}: \quad -\sigma'_{\mu\mu} = \frac{\Delta_\nu}{16\pi^2 \nu} R,$$

$$\text{Charge}: \quad \frac{1}{8\pi\nu} R.$$

Algebra

$$\hbar^{-1}[\mathrm{P}(r),\mathrm{P}^{\dagger}(r')] = -\frac{1}{2}(P\times\nabla)\delta(r-r') + \underbrace{\frac{\hbar}{2\nu}\left(2\pi\rho^2\delta(r-r') + \frac{1}{4}\nabla\left[\rho\cdot\nabla\delta(r-r')\right]\right)}_{\text{anomalous term}}$$

$$[\mathrm{P}(r),\rho(r')] = -i\hbar\rho\,\partial\,\delta(r-r').$$

Obeys Jacobi condition

Potential Flow: Virasoro algebra

Potential flow:

$$\nabla \times \mathrm{v} = 0, \quad \rho = \mathrm{const}$$

Flux is harmonic

$$\mathrm{P} = \sum_{n \neq 0} \frac{1}{n} L_n z^{-n}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(m^3 - m)\delta_{m+n}.$$

$$c = 1 - 6(\sqrt{\nu} - 1/\sqrt{\nu})^2$$

SUMMARY

* Quantization of vortex fluid;
* Conformal symmetry of the vortex fluid;
* Known physical properties of FQHE can be obtained from the quantum vortex flow in incompressible 2D fluid;
* Possible applications to turbulence