On the symmetry classification of topological insulators and superconductors

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@ Euler-Symposium

St. Petersburg (July 14, 2013)

“Periodic Table” of topological insulators/superconductors

	Symmetry			d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

TABLE I Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using the notation of Altland and Zirnbauer (1997)

Quantum Hall Effect

He-3 (B phase)

QSHI: HgTe

Majorana

$Bi_2\ Se_3$

from Hasan & Kane, Rev. Mod. Phys. (2011)

Schnyder, Ryu, Furusaki, Ludwig (2008); Kitaev (2009);
Teo & Kane (2010); Stone, Chiu, Roy (2011); Freed & Moore (2013)

[overview:]

I. Ten-Way Classification
II. Free Fermion Ground States

Quasi-particle vacua (I)

Fock operators: $c_\alpha^\dagger$ (creation), c_α (annihilation)

Fock vacuum: $c_\alpha \left|\mathrm{vac}\right\rangle = 0 \qquad (\alpha = 1, 2, \ldots)$

Quasi-particle vacuum: $\widetilde{c}_\alpha \left|\widetilde{\mathrm{vac}}\right\rangle = 0 \qquad (\alpha = 1, 2, \ldots)$

where $$\widetilde{c}_\alpha = \sum_{\alpha'} \left(c_{\alpha'} u_{\alpha'\alpha} + c_{\alpha'}^\dagger v_{\alpha'\alpha} \right)$$

Remark: quasi-particle vacua are also referred to as ground states in the Hartree-Fock-Bogoliubov mean-field approximation.

Special case (n-particle Slater determinant):

$$\widetilde{c}_\alpha = c_\alpha^\dagger \quad (1 \le \alpha \le n), \quad \widetilde{c}_\alpha = c_\alpha \quad (\alpha > n)$$

Quasi-particle vacua (II)

Vector space of annihilation operators: $\sum u_\alpha c_\alpha \in U$

Creation operators: $\sum v_\alpha c_\alpha^\dagger \in V$

Nambu space $W = U \oplus V$ comes with a Hermitian scalar product

$$\langle \psi \mid \psi' \rangle := \{\psi^\dagger, \psi'\}, \quad \psi = \sum \left(u_\alpha c_\alpha + v_\alpha c_\alpha^\dagger \right) \in W.$$

Fact: Quasi-particle vacua are in one-to-one correspondence with Hermitian subspaces $A \subset W$ (annihilation operators) subject to

$$\{A, A\} = 0, \quad \dim A = \tfrac{1}{2} \dim W.$$

In the presence of a group G of symmetries, we require

$$g \cdot A = A \qquad \text{(for all } g \in G\text{)}.$$

Classification of G-invariant ground states ?

Symmetries in quantum mechanics

Q: What's a symmetry in quantum mechanics?

A: An operator $T : \mathscr{R}\psi_1 \mapsto \mathscr{R}\psi_2$ on Hilbert rays that preserves all transition probabilities: $\left|\langle T\mathscr{R}\psi_2 , T\mathscr{R}\psi_1\rangle\right|^2 = \left|\langle \mathscr{R}\psi_2 , \mathscr{R}\psi_1\rangle\right|^2.$

Wigner's Theorem:

A symmetry T in quantum mechanics can always be represented on Hilbert space by an operator $\hat{T}$ which is either unitary or anti-unitary.

$$\langle \hat{T}\psi_2 | \hat{T}\psi_1 \rangle = \overline{\langle \psi_2 | \psi_1 \rangle}$$

Eugene P. Wigner

Remark 1: The symmetries form a group, G.

Remark 2: Symmetries commute with the Hamiltonian ($\hat{T}H = H\hat{T}$).
Thus "chiral symmetry" ($\gamma_5 D \gamma_5 = -D$) is not a symmetry.

Setting: Fock space & symmetries

Freeman J. Dyson

Single-particle Hilbert space V

Fock space $\mathscr{F}$ for (identical) fermions: $\mathscr{F}_n = \wedge^n(V)$

Particle creation $(c^\dagger)$ and annihilation operators (c) satisfy CAR,

$$c_\alpha^\dagger c_\beta + c_\beta c_\alpha^\dagger = \delta_{\alpha\beta} \ , \quad c_\alpha c_\beta + c_\beta c_\alpha = 0 = c_\alpha^\dagger c_\beta^\dagger + c_\beta^\dagger c_\alpha^\dagger \, .$$

Symmetry group acts G acts by (anti-)unitary ops on Fock space:

- **Unitary** symmetries: any group of unitary operators defined on V and extended to $\mathscr{F}$ in the natural way.
- **Anti-unitary** symmetries:
  1. Time reversal $T : V \to V$, extended to $T : \mathscr{F}_n \to \mathscr{F}_n$
  2. Particle-hole conjugation (twisted) C: n particles $\to$ n holes

Step of reduction

For any reductive group G_0 (here: unitary symmetries, $G_0 \subset G$)
Nambu space W decomposes into G_0-isotypic components:

$$W = \bigoplus\nolimits_{\lambda} W_{\lambda}, \quad W_{\lambda} = \mathscr{H}_{\lambda} \otimes R_{\lambda}.$$

multiplicity space

irreps of G_0

$$\mathscr{H}_{\lambda} = \mathrm{Hom}_{G_0}(R_{\lambda}, W) \simeq \mathbb{C}^{m_{\lambda}}$$

Example 1: Take $G_0 = \Gamma$ (space translations). Then
$\lambda = k$ (momentum), $R_{\lambda} = R_k \simeq \mathbb{C}$, and $\mathscr{H}_k = U_k \oplus V_{-k}$
(all Fock operators lowering the momentum by k).

Example 2: Take $G_0 = \mathrm{SU}_2$ (spin rotations). Then
$\lambda = j$ (single-particle spin), $R_{\lambda} = R_j \simeq \mathbb{C}^{2j+1}$.

See also: MRZ, *Symmetry Classes*, arXiv:1001.0722

Ten-Way Classification

Let $G = G_0 \cup G_1$ be any group of unitary & anti-unitary symmetries acting on $\mathscr{F}$ (as described above) and hence on Nambu space W.

Theorem. Every G-invariant quasi-particle vacuum A decomposes as an orthogonal sum

$$A = \bigoplus\nolimits_\lambda A(\lambda), \quad A(\lambda) = x(\lambda) \otimes R_\lambda ,$$

where each vector space $x(\lambda)$ lies in some classifying space X_λ , of which there exist 10 different types. The latter are in bijection with the 10 large families of symmetric spaces.

Remark. This is an immediate corollary of the classification result by Heinzner, Huckleberry & MRZ.

Communications in
Mathematical Physics

Symmetry Classes of Disordered Fermions

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Received: 10 June 2004 / Accepted: 9 December 2004
Published online: ■■■ – © Springer-Verlag 2005

Abstract: Building upon Dyson's fundamental 1962 article known in random-matrix theory as *the threefold way*, we classify disordered fermion systems with quadratic Hamiltonians by their unitary and antiunitary symmetries. Important physical examples are afforded by noninteracting quasiparticles in disordered metals and superconductors, and by relativistic fermions in random gauge field backgrounds.

The primary data of the classification are a Nambu space of fermionic field operators which carry a representation of some symmetry group. Our approach is to eliminate all

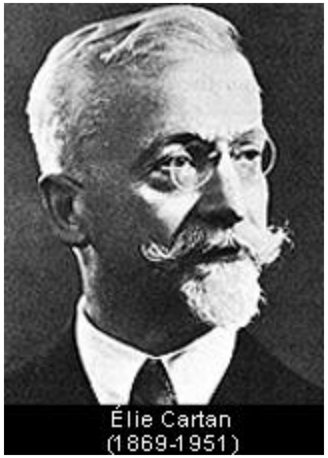
What's a symmetric space?

Riemann tensor: $R^{i}{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}$

Def.: A (locally) symmetric space is a Riemannian manifold $X = U/K$ with covariantly constant curvature: $\nabla R = 0$.

Ex. 1: the round two-sphere $X = \mathrm{S}^2$, $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

Ex. 2: the set $X = \mathrm{Gr}_n(\mathbb{C}^N) = \mathrm{U}(N)/\mathrm{U}(n)\times\mathrm{U}(N-n)$ of all subspaces $\mathbb{C}^n \simeq V \subset \mathbb{C}^N$



Some facts:

- Complete classification by E. Cartan (1926)
  Large families: *A*, *A*I, *A*II, *A*III, *BD*, *BD*I, *C*, *C*I, *C*II, *D*III
- Metric tensor g_{ij} is the only G-invariant (0,2) tensor on X

Quasi-particle vacua & symmetric spaces (I)

Example 1: $G = \{e\}$ (no symmetries at all).

Fact. If $\dim V = N$, the space X of quasi-particle vacua is a symmetric space $X = \mathrm{O}(2N)/\mathrm{U}(N)$. (Cartan type: *D*III).

Proof. Apply the most general Bogoliubov transformation in $\mathrm{O}(2N)$ to the Fock vacuum $A_0 = \mathrm{span}_{\mathbb{C}}\{c_1, \ldots, c_N\}$. The latter is invariant under unitary transformations $\mathrm{U}(N) \subset \mathrm{O}(2N)$.

Remarks:

1. X is also the space of complex structures of $\mathbb{R}^{2N}$. (Interpretation: decomposition of Majorana fermions into creation and annihilation parts)
2. X has two connected components $\longleftrightarrow$ even/odd fermion parity
3. X_0 is the space of general BCS-states $\exp\left(\sum Z_{\alpha\beta} c_\alpha^\dagger c_\beta^\dagger\right) |\mathrm{vac}\rangle$

Quasi-particle vacua & symmetric spaces (II)

Example 2. $G = \mathrm{U}(1)$ (conservation of charge): quasi-particle vacua are Hartree-Fock ground states (a.k.a. Slater determinants).

Decomposition into isotypic components: $W = U \oplus V$.

Q.p. vacua are sums $A = A(-) \oplus A(+)$ with $A(-) \subset U$, $A(+) \subset V$.

If $\dim V = N$ and $\dim A(+) = n$ (particle number), then $A(+)$ is a point of the Grassmannian $X = \mathrm{U}(N)/\mathrm{U}(n) \times \mathrm{U}(N-n)$.

Q: How is the structure of Riemannian manifold determined?

A: Our variable quasi-particle vacua $x \equiv \mathscr{R}\,|\widetilde{\mathrm{vac}}\rangle$ constitute a homogeneous space, which is Riemann by the (geodesic) distance function

$$\mathrm{dist}(x_1, x_2) = t \iff \mathscr{R}\,|\widetilde{\mathrm{vac}}_2\rangle = \mathscr{R}\,\mathrm{e}^S |\widetilde{\mathrm{vac}}_1\rangle, \ \| S \| = t.$$

Proof of classification theorem (HHZ)

1. Recall the decomposition of W into G_0-isotypic components:

$$W = \bigoplus_\lambda W_\lambda \,, \quad W_\lambda = \mathscr{H}_\lambda \otimes R_\lambda \,.$$

2. Transfer all remaining structure (CAR, T, C) to the blocks $\mathscr{H}_\lambda$.

3. Show that, in the process, the only change that may occur is a change of involution type, i.e. $T = T_{\text{eff}} \otimes \beta$ with $\beta^2 = \pm 1$, etc.

4. Enumerate the possible cases.

Example 1. $G_0 = \mathrm{SU}(2)_{\text{spin}} \,, \, R_\lambda = (\mathbb{C}^2)_{\text{spin}} \; : \; \mathrm{CAR} \to \mathrm{CCR}.$

Example 2. Same as above, but T present: $T^2 = -1 \; \to \; T_{\text{eff}}^2 = +1.$

Free fermion ground states

Quasi-particle vacua with translation symmetry

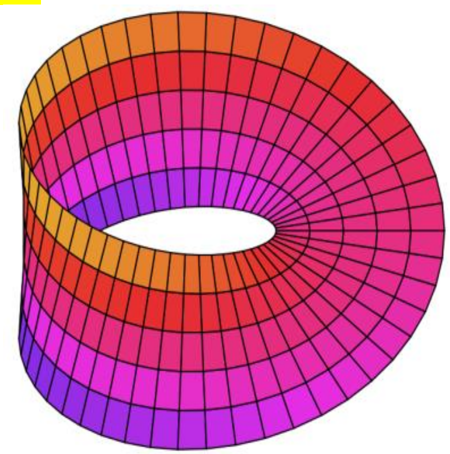
Notation. Group of translations Γ (symmetries).
Fourier dual $\widehat{\Gamma} \equiv M$ (Brillouin zone, momentum space).
U_k (V_k) annihilation (resp. creation) operators at $k \in M$.
Hermitian vector spaces $W_k = U_k \oplus V_{-k}$ with $\dim W_k = 2N$
and scalar product $\langle \psi \mid \psi' \rangle = \{ \psi^\dagger, \psi' \}$.

Fact. The ground state of a gapped system of free fermions is a rank-N Hermitian vector bundle

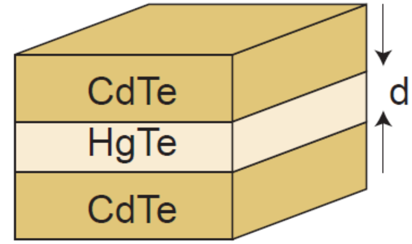
$$\mathscr{A} \xrightarrow{\pi} M, \quad \pi^{-1}(k) \equiv A(k) \subset W_k$$

with fibers subject to

$$\{A(k), A(-k)\} = 0 \qquad \text{(for all } k \in M\text{).}$$



Quantum Spin Hall Insulator



Let G be generated by Γ (translations),
U(1) (charge conservation) and time reversal ($T^2 = -1$).

Note: in this setting, q.p. vacua are Hartree-Fock ground states with an even particle number, n. If the system is gapped (band insulator), then the ground state is a vector bundle $M \ni k \mapsto V(k) \in \mathbb{C}^N$ with $V(k) \simeq \mathbb{C}^n$:= vector space of valence states.

Time-reversal symmetry implies $T\,V(k) = V(-k)$.
At T-invariant momenta $k_0 = -k_0$ one has $T\,V(k_0) = V(k_0)$.

For $M = \mathrm{S}^2$ such ground states fall into 2 top. classes (trivial band insulator; QSHI phase) distinguished by the Kane-Mele invariant.

Quantum Spin Hall Insulator (cont'd)

Kane & Mele, 2005:

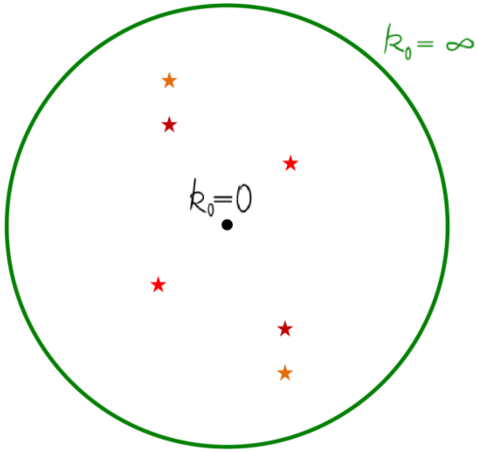
The Pfaffian $p(k) = \mathrm{Pf}\,\tau_k$ of the skew-symmetric $\mathbb{C}$-linear map

$\tau_k : V(k) \to V(k)^* = (\mathbb{C}^N)^*/V(k)^\perp,\ v \mapsto \langle Tv, \cdot\rangle$

is T-even: $p(k) = p(-k)$, and $p(k_0) \neq 0$.

The Kane-Mele $\mathbb{Z}_2$-topological invariant

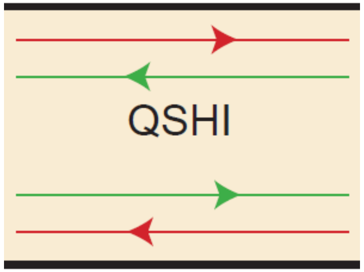
counts its pairs of zeroes (mod 2).

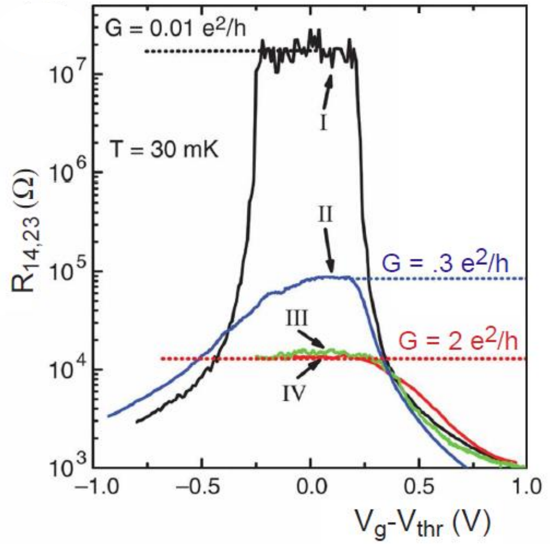


Bulk-boundary correspondence:

non-trivial bulk topological invariant

→ perfectly conducting surface mode





From vector bundles to classifying maps

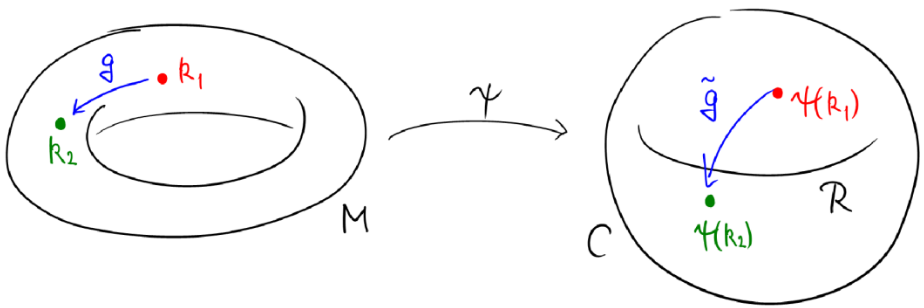
There exists an equivalent description by a so-called classifying map.

Example. $V(k) \simeq \mathbb{C}^n \subset \mathbb{C}^N$ determines $x \in C := \mathrm{U}(N)/\mathrm{U}(n) \times \mathrm{U}(N-n)$.

Thus $\{k \mapsto V(k)\}$ determines a mapping $\{k \mapsto \psi(k) \in C\}$.

Constraint $T\,V(k) = V(-k) \quad \Longrightarrow \quad \widetilde{T}\,\psi(k) = \psi(-k)$,

and $T\,V(k_0) = V(k_0) \quad \Longrightarrow \quad \psi(k_0) \in R = \mathrm{Sp}(N)/\mathrm{Sp}(n) \times \mathrm{Sp}(N-n)$.



General picture:

G-invariant free fermion ground states ($\Gamma \subset G$) are described by classifying maps into a symmetric space, $\psi:\ M \to C$, subject to an equivariance condition $\widetilde{g} \cdot \psi(k) = \psi(g \cdot k)$ for all $g \in G_{\mathrm{red}}$.

Majorana fermions in superconductors

No symmetries, "spinless fermions", single band, $D = 1$.

Vector bundle:

$$A(k) = \mathrm{span}_{\mathbb{C}}\left(u(k)\,c_k + v(k)\,c_k^\dagger\right), \quad v(k)/u(k) = z(k) = -z(-k).$$

Classifying map:

$$C = \mathrm{U}(2)/\mathrm{U}(1)\times\mathrm{U}(1) = \mathrm{S}^2, \quad R = \mathrm{O}(2)/\mathrm{U}(1) = \{\mathrm{N.P.}, \mathrm{S.P.}\}.$$

Weak pairing: bulk-boundary correspondence → gapless edge state

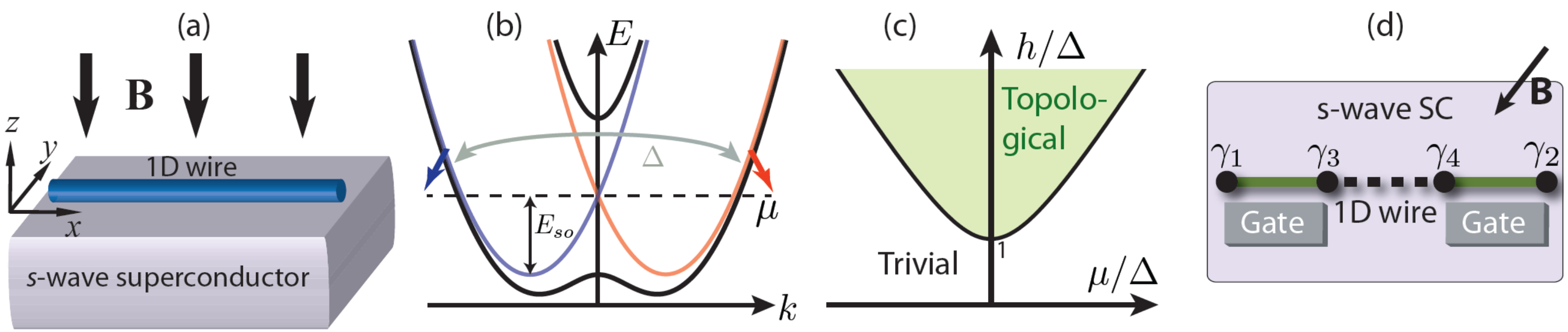


FIG. 6. (a) Basic architecture required to stabilize a topological superconducting state in a 1D spin-orbit-coupled wire. (b) Band structure for the wire when time-reversal symmetry is present (red and blue curves) and broken by a magnetic field (black curves). When the chemical potential lies within the field-induced gap at $k = 0$, the wire appears 'spinless'. Incorporating the pairing induced by the proximate superconductor leads to the phase diagram in (c). The endpoints of topological (green) segments of the wire host localized, zero-energy Majorana

J. Alicea, arXiv:1202.1293

How to classify?

There exist several notions of topological equivalence:

1. Homotopy classes of classifying maps
2. Isomorphism classes of vector bundles

   (okay for "many conduction bands")

3. Stable equivalence of vector bundles (K-theory)

   (okay for "many conduction & many valence bands")

Example 1. "Hopf magnetic" insulator (Moore, Ran, Wen; 2008)
Two bands (one valence, one conduction).
Let $M = \mathrm{S}^3$. Then $\mathbf{0} = \mathrm{Vect}_{\mathbb{C}}(M) \neq \pi_3(\mathrm{S}^2) = \mathbb{Z}$.

Example 2. Let $M = \mathrm{S}^1$. Then $\mathbb{Z}_2 = \mathrm{Vect}_{\mathbb{R}}(M) \neq \pi_1(\mathrm{S}^1) = \mathbb{Z}$.

Notation for classifying spaces

s	$R_s(n)$	$C_s(n)$		$d=1$	2	3	4	5
0	$\mathrm{O}_{16n}/\mathrm{O}_{8n}\times\mathrm{O}_{8n}$	$\mathrm{U}_{16n}/\mathrm{U}_{8n}\times\mathrm{U}_{8n}$	AI	0	0	0	$\mathbb{Z}$	0
1	O_{8n}	U_{8n}	BDI	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
2	$\mathrm{O}_{8n}/\mathrm{U}_{4n}$	$\mathrm{U}_{8n}/\mathrm{U}_{4n}\times\mathrm{U}_{4n}$	D	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
3	$\mathrm{U}_{4n}/\mathrm{Sp}_{4n}$	U_{4n}	DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
4	$\mathrm{Sp}_{4n}/\mathrm{Sp}_{2n}\times\mathrm{Sp}_{2n}$	$\mathrm{U}_{4n}/\mathrm{U}_{2n}\times\mathrm{U}_{2n}$	AII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
5	Sp_{2n}	U_{2n}	CII	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
6	$\mathrm{Sp}_{2n}/\mathrm{U}_n$	$\mathrm{U}_{2n}/\mathrm{U}_n\times\mathrm{U}_n$	C	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
7	$\mathrm{U}_n/\mathrm{O}_n$	U_n	CI	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$

Statement of periodicity for $G_{\mathrm{red}}=\mathbb{Z}_2$:

$$[S^d, C_s(n)]_{\mathbb{Z}_2} \cong [S^{d+1}, C_{s+1}(2n)]_{\mathbb{Z}_2}$$

Homotopy-theoretic proof of "Periodic Table"

(R. Kennedy & MRZ, in preparation)

Ingredients:

1. Bott-Milnor isomorphism $[S^D, R_s] \cong [S^{D+1}, R_{s-1}]$
2. Whitehead theorem in G-equivariant homotopy for $G = \mathbb{Z}_2$

   (→ transfer to the required setting)

3. Long exact sequence in relative homotopy (fiber bundle)

Consider the generalization by Teo & Kane (2010)
(D space-like and d momentum-like components for $k \in M$)

$$[S^{D,d}, C_s(n)]_{\mathbb{Z}_2} \overset{1,2}{\cong} [S^{D+s,d}, C_0(n)]_{\mathbb{Z}_2}$$

$$\overset{3}{\cong} [S^{D+s,d+1}, C_1(2n)]_{\mathbb{Z}_2} \overset{1,2}{\cong} [S^{D,d+1}, C_{s+1}(2n)]_{\mathbb{Z}_2}$$

Summary & Outlook

- The ten-way symmetry classification of disordered fermions carries over to quasi-particle vacua / free fermion ground states.
- No new classifying spaces appear beyond the ten large families of symmetric spaces.
- The notions of vector bundle and classifying map are associated with inequivalent notions of topological equivalence.

- Homotopy-theoretic proof of “Periodic Table” is forthcoming.
- Our method gives bounds on the range of stable equivalence.
- It applies to other symmetry groups (including, e.g., reflections).

The End