Anderson localization on a simplex

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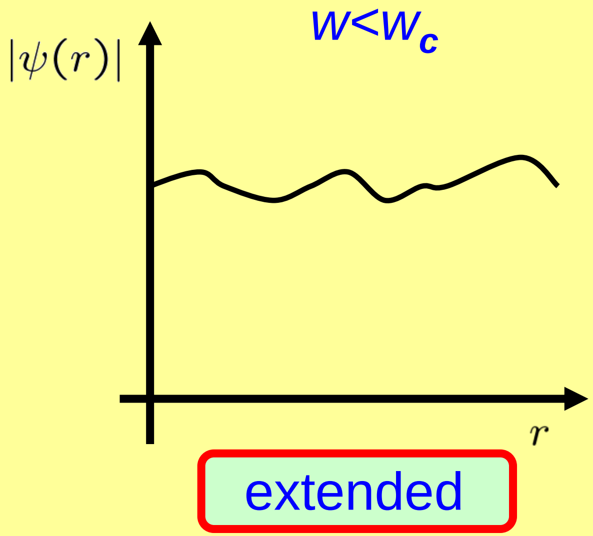
Anderson model

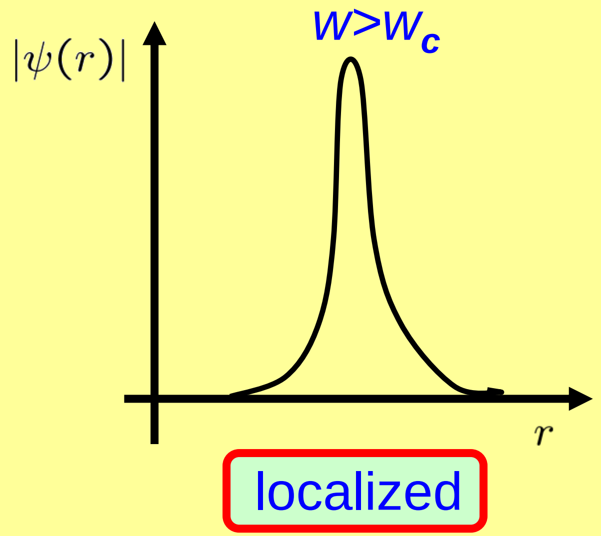
Hamiltonian on a *d*-dimensional lattice:

$$(\hat{H}\psi)_i = v_i\psi_i + \sum_{<ij>} \psi_j, \qquad \langle v_i \rangle = 0,\ \langle v_i v_k \rangle = w^2 \delta_{ik}$$

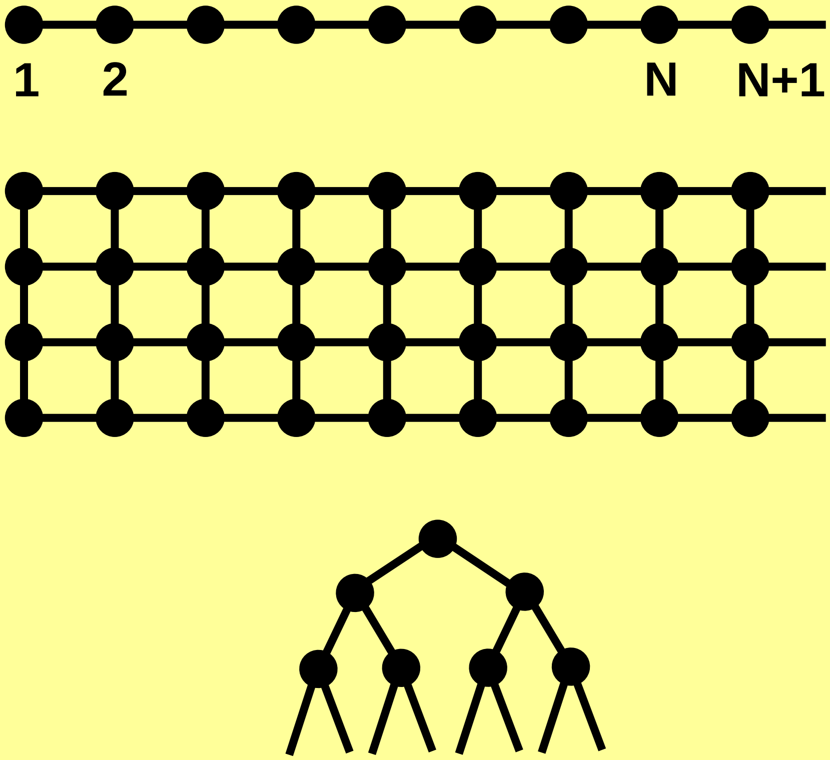
d≤2 eigenstates are localized

d>2 metal-insulator transition:





Solvable models



Recursion relation:

size **N** → size **N+1**

Necessary condition:

absence of the loops

Outline

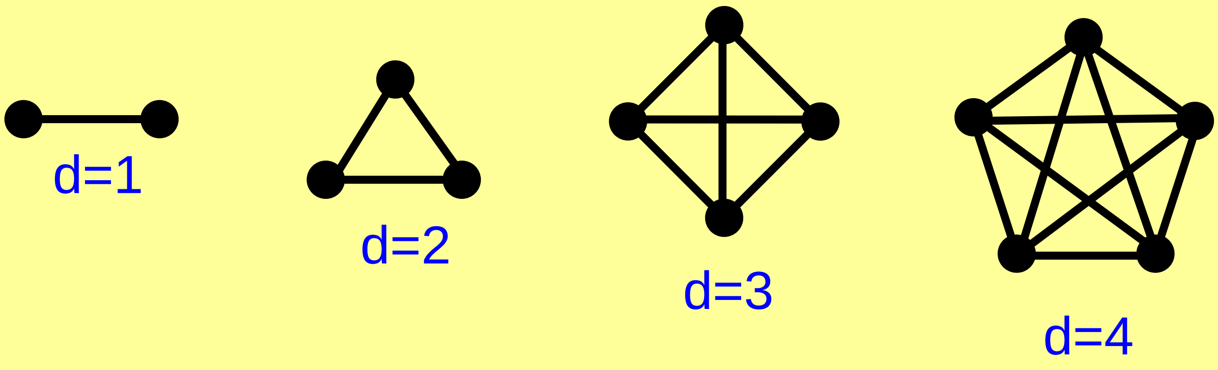
1. Definition of the simplex model and the moments of the eigenstates

2. Field-theoretical representation for the moments of the eigenstates

3. Moments of the eigenstates in the simplex model

Simplex model

d-simplex



Hamiltonian:

$$H = T + V, \quad T_{ij} = \frac{1}{N}, \quad V_{ij} = v_i \delta_{ij}, \quad i, j = 1, \ldots, N, \quad N = d + 1$$

v_i Gaussian random variable $\langle v_i \rangle = 0, \quad \langle v_i^2 \rangle = w^2$

Spectrum of the clean system $w = 0$:

$$T f_0 = f_0, \quad f_0 = (1, 1, \ldots, 1)^{\mathsf{T}} \quad \Rightarrow \quad \lambda = 1$$

$Tf = 0 \quad \forall f \perp f_0$ ⇒ $\lambda = 0$ $(N-1)$-fold degenerate

In the presence of disorder $w \neq 0$:

Expectation: $w \gg 1$ localization, $w \ll 1$ delocalization?

Moments of the eigenstates

$$\boxed{I_q(n) = \frac{1}{\rho(E)} \sum_{\alpha} \left\langle |f_\alpha(n)|^{2q} \, \delta(E - E_\alpha) \right\rangle}$$

$\rho(E)$ - the density of states $\qquad H f_\alpha = E_\alpha f_\alpha$

$$I_q \propto N^{-d_q(q-1)}$$

Extended states: $d_q = 1$ $\qquad$ Localized states: $d_q = 0$

Green's functions: $\qquad G^{R/A} = (E \pm i\epsilon - H)^{-1}$

$$K_{l,m}(n,\epsilon) = \left(G^R_{nn}\right)^l \left(G^A_{nn}\right)^m, \quad l, m = 1, 2, \ldots$$

$$I_q(n) = \frac{C_{l,m}}{\rho(E)} \lim_{\epsilon \to 0} (2\epsilon)^{l+m-1} \langle K_{l,m}(n,\epsilon) \rangle, \quad q = l + m$$

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Supersymmetric representation

$$\Phi_i = \begin{pmatrix} s_R(i) \\ \chi_R(i) \\ s_A(i) \\ \chi_A(i) \end{pmatrix}, \quad \Phi_i^\dagger = (s_R^*(i), \chi_R^*(i), s_A^*(i), \chi_A^*(i)), \quad i = 1, \ldots, N$$

$s_R,\ s_A$ commutative (bosonic) variables

$\chi_R,\ \chi_A$ anti-commutative (fermionic) variables

$$\boxed{\begin{aligned} K_{l,m}(n,\epsilon) \;=\; & \frac{i^{l-m}}{l!\,m!} \int \prod_{p=1}^{N} d\Phi_p d\Phi_p^\dagger\, (s_R^*(n) s_R(n))^l (s_A^*(n) s_A(n))^m \\ & \mathrm{exp}\left[i \sum_{p,q=1}^{N} (H_{pq} - E\delta_{pq})(\Phi_p, \Phi_q) - \epsilon \sum_{p=1}^{N} (\Phi_p, \Lambda \Phi_p) \right] \end{aligned}}$$

$$\Lambda = \mathrm{diag}(1, 1, -1, -1)$$

$$(\Phi_p, \Phi_q) = s_R^*(p) s_R(q) + \chi_R^*(p) \chi_R(q) - s_A^*(p) s_A(q) + \chi_A^*(p) \chi_A(q)$$

Reduced representation

7 out 8 variables can be integrated out in the limit $\epsilon \longrightarrow 0$

$$I_q(n) = c_q \prod_{p=1}^{N} \left(\int_{-\infty}^{\infty} \frac{du_p}{w u_p} \right) u_n^{2q-3} \det B \; e^{-\sum_p \left(\frac{\left(\sum_q T_{pq} \frac{u_q}{u_p} - E \right)^2}{2w^2} + u_p^2 \right)}$$

$$B_{pq} = -T_{pq} + \delta_{pq} \sum_r T_{pr} \frac{u_r}{u_p}, \quad p, q = 1, \ldots, N; \; p, q \neq n$$

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Anderson model on a simplex

$$T_{pq} = \frac{1}{N}$$

$$I_q(n) = c_q \prod_{p=1}^{N} \left(\int_{-\infty}^{\infty} \frac{du_p}{w u_p} \right) u_n^{2q-3} \det B \; e^{-\sum_p \left(\frac{\left(\sum_q u_q\right)^2}{2N^2 w^2 u_p^2} + u_p^2 \right)}$$

$$\det B = \frac{u_n^2}{N^{N-1}} \left(\sum_{r=1}^{N} u_r \right)^{N-2} \prod_{p=1}^{N} \frac{1}{u_p}$$

$s = \frac{1}{wN} \sum_q u_q$ "collective" variable

$$1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} d\theta e^{-i\theta\left(s - \frac{1}{wN}\sum_q u_q\right)}$$

Anderson model on a simplex

$$I_q(N) = r_q \int_{-\infty}^{\infty} d\theta e^{-i\theta w N} \int_{-\infty}^{\infty} ds\, |s|^{2q-3}\, f^{N-1}(s,\theta) g(s,\theta)$$

$$f(s,\theta) = \int_{-\infty}^{\infty} dx\, x^{-2} e^{-\frac{1}{2x^2} - s^2 x^2 + i\theta x}$$

$$g(s,\theta) = \int_{-\infty}^{\infty} dx\, x^{2q-2} e^{-\frac{1}{2x^2} - s^2 x^2 + i\theta x}$$

Moments of the eigenstates in the thermodynamic limit

$$\alpha = N\theta, \quad t = Ns$$

$$I_q(N) = \frac{r_q}{N^{2q-1}} \int_{-\infty}^{\infty} d\alpha e^{-i\alpha w} \int_{-\infty}^{\infty} dt\, |t|^{2q-3} f^{N-1}\left(\frac{t}{N}, \frac{\alpha}{N}\right) g\left(\frac{t}{N}, \frac{\alpha}{N}\right)$$

$$g\left(\frac{t}{N}, \frac{\alpha}{N}\right) = N^{2q-1}\left[|t|^{-2q+1} F_q\left(\frac{\alpha}{2t}\right) + O(N^{-2})\right],$$

$$F_q(z) = \sqrt{\pi} e^{-z^2} \sum_{p=0}^{q-1} 2^p (-z^2)^{q-1-p} \frac{(2q-2)!}{p!(2q-2-2p)!}$$

$$f\left(\frac{t}{N}, \frac{\alpha}{N}\right) = 1 - \sqrt{2}\frac{|t|}{N} e^{-\left(\frac{\alpha}{2t}\right)^2} - \sqrt{\frac{\pi}{2}} \frac{|\alpha|}{N} \mathrm{erf}\left(\left|\frac{\alpha}{2t}\right|\right) + O(N^{-2})$$

Moments of the eigenstates in the thermodynamic limit

$$I_q = -\frac{1}{\pi}\int_{-\infty}^{\infty} dz \frac{F_q(z)}{(q-2)!} \ln\left[4z^2w^2 + 2\left(e^{-z^2} + \sqrt{\pi}\,|z|\,\mathrm{erf}(|z|)\right)^2\right]$$

$$q = 2, 3, \ldots$$

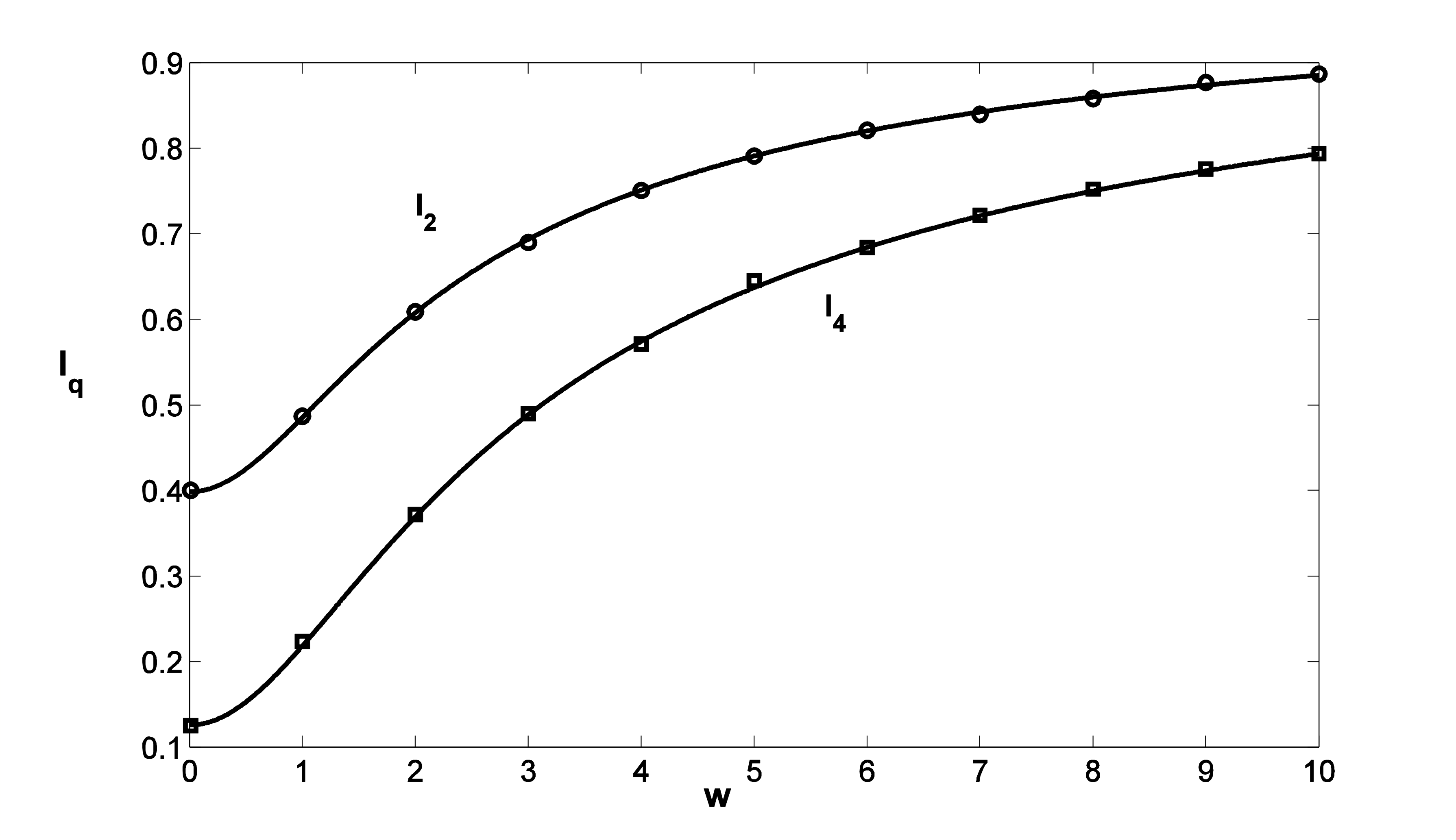
Eigenstates are **localized** at any strength of disorder

$$I_q = -\frac{1}{\pi}\int_{-\infty}^{\infty} dz\, \frac{\tilde{F}_q(z)}{\Gamma(q-1)} \ln\left[4z^2w^2 + 2\left(e^{-z^2} + \sqrt{\pi}\,|z|\,\mathrm{erf}(|z|)\right)^2\right],$$

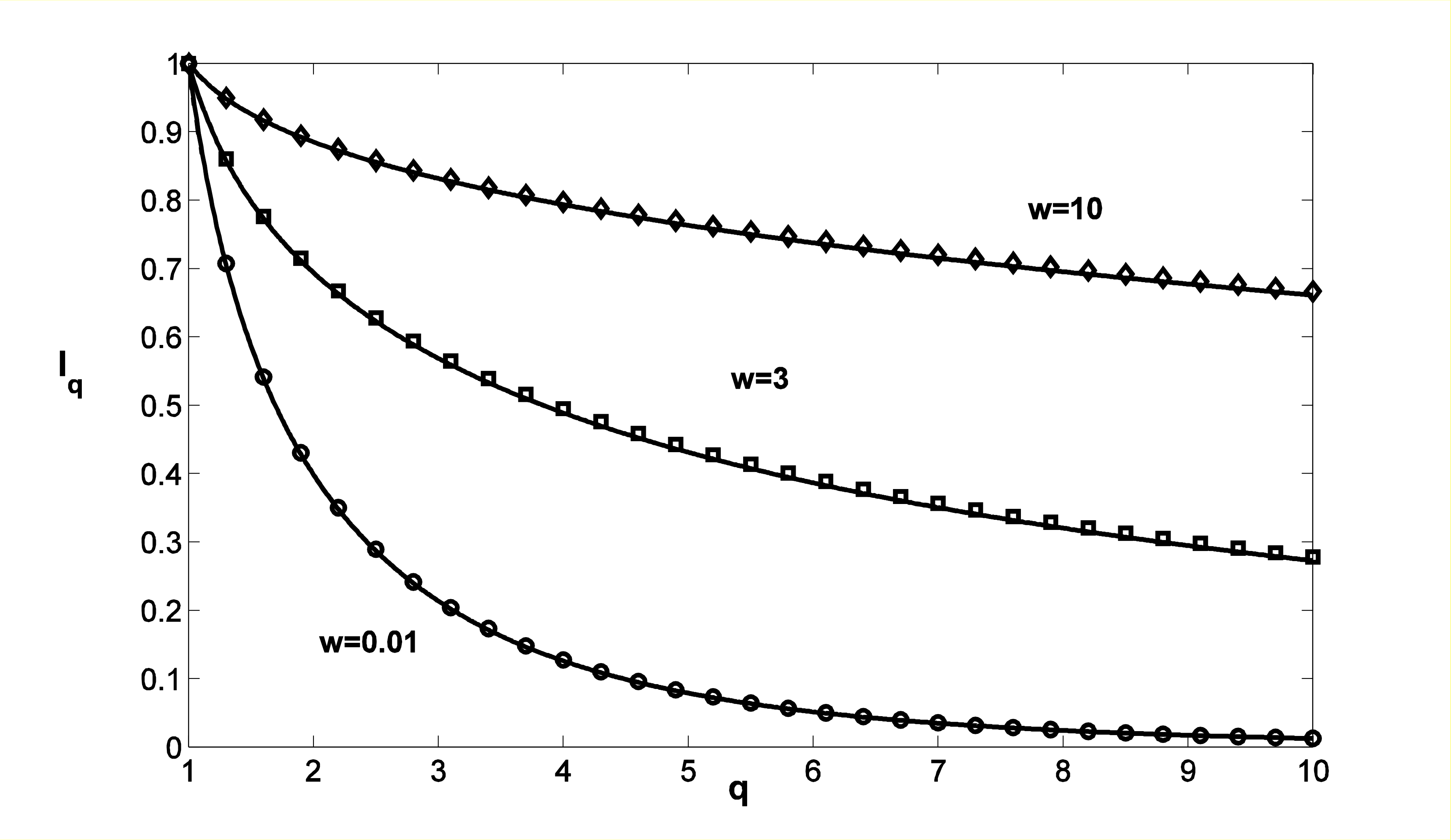
$$\tilde{F}_q = \Gamma\left(q - \frac{1}{2}\right)\, {}_1F_1\left(q - \frac{1}{2}, \frac{1}{2}, -z^2\right), \quad q > 1$$

$$\lim_{w\to\infty} I_q = 1$$

Comparison with numerical simulations



Comparison with numerical simulations



Why eigenstates are localized?

Mathematical explanation:

$$H = V + T = V + \frac{1}{N} |f_0\rangle \langle f_0| \,, \quad |f_0\rangle = (1, 1, \ldots, 1)^T, \; \langle f_0| = (1, 1, \ldots, 1)$$

$V \sim w$ - diagonal matrix

T - rank one matrix is a small perturbation at any w

Physical explanation:

Eigenstates are degenerate at $w = 0$

At $w > 0$ energy band of the width $\sim w$ $\Rightarrow$

Disorder is always strong

Summary

- Field-theoretical representation for the moments of the eigenstates in the generalized Anderson model

- Simplex model: localization at any disorder strength

- Analytical and numerical results for the moments of the eigenstates

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