Spectral properties of Laplacians on non-compact manifolds with general ends

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Inverse Problems and Related Topics

Outline

1. Introduction
   - Manifolds with ends

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   - Manifolds with ends

Works I

- H. Isozaki, Y. Kurylev and M. Lassas,
  Forward and inverse scattering problem on manifolds with asymptotically hyperbolic ends,
  J. Funct. Anal. 258 (2010), 2010-2118.
- H. Isozaki, Y. Kurylev and M. Lassas,
  Conic singularities, genealized scattering matrix, and inverse scattering on asymptotically hyperbolic surfaces,
  to appear in *J. Reine angew. Math.*.

Works II

H. Isozaki, Y. Kurylev and M. Lassas,
Spectral theory and inverse problems on asymptotically hyperbolic orbifolds,
arXiv : 1312.0421

We consider a connected, non-compact manifold (or orbifold) of the form

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_{N+N'}$$

where, $\mathcal{K}$ = relatively compact,

$$\mathcal{M}_i \simeq (0, \infty) \times M_i, \quad \text{(diffeomprphic)},$$

M_i = compact manifold (orbifold) of dim $n - 1$ equipped with the metric $h_M(x, dx)$.

$\mathcal{M}_i$ is equipped with the metric

$$ds^2 = (dr)^2 + \rho_i(r)^2 h_i(r, x, dx)$$

$$h_i(r, x, dx) - h_{M_i}(x, dx) = O(r^{-\gamma}), \quad \gamma > 0.$$

Typical examples

- (1) $\rho(r) = e^{c_0 r}, \quad c_0 > 0,$ hyperbolic regular end
- (2) $\rho(r) = r$ Euclidean (or conical)
- (3) $\rho(r) = 1$ cylindrical end (waveguide)
- (4) $\rho(r) = e^{c_0 r}, \quad c_0 < 0,$ hyperbolic cusp

We know that

- For (1), (2) No embedded eigenvalues in the continuous spectrum.
- For (3), (4) $\exists$ embedded eigenvalues
- For (1), (2), (3) One can reconstruct $\mathcal{M}$ from the physical S-matrix for all energies associated with one end
- For (4), One can reconstruct $\mathcal{M}$ from the generalized S-matrix for all energies associated with one end

We deal with the metric of the form

$$\rho(r) \sim \begin{cases} \exp\left(c_0 r + \frac{\beta}{\alpha} r^\alpha\right), & 0 \leq \alpha < 1, \\ r^\beta, \end{cases}$$

where

- For the regular end, $c_0 > 0$ or $c_0 = 0, \beta > 0$,
- For the cusp, $c_0 < 0$, or $c_0 = 0$, $\beta < 0$.

Sometimes, it is more convenient to state the assumption in the form

$$\rho(r)^{-1} \in S^{-\beta}.$$

In this setting, the exponentially growing metric corresponds to the case $\beta = \infty$.

Outline

- Assumptions
- Instability of the short-range assumption

2 Rellich-Vekua theorem

3 Laplacian on $\mathcal{M}$

4 Spectral representation and S-matrix

5 Inverse scattering

Perturbed warped product metric

Consider the following metric on $(0, \infty) \times M$,

$$ds^2 = (dr)^2 + g_M(r, x, dx),$$

where $g_M(r, x, dx) = g_{M,ij}(r, x)dx^i dx^j$ is a metric on M depending smoothly on $r > 0$. Let

$$g = g(r, x) = \det \left(g_{M,ij}(r, x)\right).$$

Define

$$f(r, x) \in S^\kappa \Longleftrightarrow \partial_r^m \partial_x^\alpha f(r, x) = O(r^{\kappa - m}), \quad \forall m, \alpha.$$

The 1st assumption

$$\text{(A-1)} \qquad \frac{g'}{4g} - \frac{(n-1)c_0}{2} - \frac{(n-1)\beta}{2} r^{\alpha-1} \in S^{-1-\epsilon}, \quad \epsilon > 0.$$

$$0 \le \alpha < 1, \ \epsilon > 0,$$

$$\beta \neq 0, \ \text{if } c_0 = 0.$$

Integrating

$$\frac{g'}{4g} - \frac{(n-1)c_0}{2} - \frac{(n-1)\beta}{2} r^{\alpha-1} = O(r^{-1-\epsilon}),$$

we have

$$g = \rho(r)^{2(n-1)} O(1),$$

where

$$\rho(r) = \begin{cases} \exp\left(c_0 r + \frac{\beta}{\alpha} r^\alpha\right), & 0 < \alpha < 1, \\ \exp\left(c_0 r\right) r^\beta, & \alpha = 0. \end{cases}$$

We can then rewrite $g_M(r, x, dx)$ as

$$g_M(r, x, dx) = \rho(r)^2 h(r, x, dx),$$

where $h(r, x, dx)$ is bounded in r.

So, our metric has the form

$$ds^2 = (dr)^2 + \rho(r)^2 h(r, x, dx).$$

The 2nd assumption

(A-2) There exists a smooth metric $h_M(x, dx)$ on M such that

$$h(r, x, dx) - h_M(x, dx) \in S^{-\gamma}, \quad \gamma > 0.$$

We say that

- $\mathcal{M}$ has a regular infinity if

$$\text{either } c_0 > 0, \quad \text{or} \quad c_0 = 0, \quad \beta > 0.$$

- $\mathcal{M}$ has a cusp if

$$\text{either} c_0 < 0, \quad \text{or} \quad c_0 = 0, \quad \beta < 0.$$

The 3rd assumption

(A-3) $\mathcal{M}_1, \cdots, \mathcal{M}_N$ have regular infinities, and $\mathcal{M}_{N+1}, \cdots, \mathcal{M}_{N+N'}$ have cusp.

The perturbation term

$$h(r, x, dx) - h_M(x, dx) \in S^{-\gamma}$$

is said to be short-range if $\gamma > 1$, and long-range if $\gamma \leq 1$.

Usually, the latter is more compilcated than the former.

It seems that there exist threshold growth orders of the metric :

Assuming that $\rho(r) = r^{\beta}$, they are (think of $x_{n+1} = (x_1^2 + \cdots + x_n^2)^{\beta/2}$)

- $\beta = 1$: conic surface
- $\beta = 1/2$: parabola
- $\beta = 1/3$: ?

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Instability of the short-range assumptioption

Consider the following metric with a cross term

$$ds^2 = a(t,z)(dt)^2 + 2w(t)b_i(t,z)dtdz^i + w(t)^2 c_{ij}(t,z)dz^i dz^j.$$

Assume

$$w(t)^{-1} \in S^{-\kappa}, \quad a(t,z) - 1 \in S^{-\lambda},$$

$$b_i(t,z) \in S^{-\mu}, \quad c_{ij}(t,z) - h_{ij}(t,z) \in S^{-\nu},$$

with the condition

$$\kappa > 1/2, \quad \lambda > 1, \quad \kappa + \mu > 1, \quad \nu > 0.$$

Note that κ corresponds to the volume growth of the manifold.

Theorem

One can transform the metric with a cross term

$$ds^2 = a(t,z)(dt)^2 + 2w(t)b_i(t,z)dtdz^i + w(t)^2 c_{ij}(t,z)dz^i dz^j.$$

into the perturbed warped product form

$$ds^2 = (dr)^2 + w(r)^2 \overline{h}(r,x,dx),$$

where $\overline{h}(r,x,dx)$ is an r-dependent metric on M satisfying

$$\overline{h}(r,x,dx) - h(z(x),dx) \in S^{-\min\{\nu,\epsilon_0\}},$$

$$\epsilon_0 = \min\{\lambda, \kappa+\mu, 2\kappa\} - 1.$$

Therefore, the metric with cross term can be transformed to the perturbed warped product if

$$w(t) \sim \exp\left(c_0 t + \frac{\beta}{\alpha} t^{\alpha}\right), \quad \text{or} \quad t^{\beta}, \quad \text{with} \quad \beta > 1/2.$$

However, even if ν is large, the resulting metric $(dr)^2 + w(r)^2 \overline{h}(r, x, dx)$ is

- a short-range perturbed metric of $(dr)^2 + \rho(r)^2 h_M(x, dx)$ only when $\beta > 1$,
- a long-range perturbed metric if $1/2 < \beta \leq 1$.

Note that the case $\beta = 1$ corresponds to the standard asymptotically Euclidean metric (see [Bouclet, 2012]).

Rellich-Vekua theorem

Consider $\mathcal{M} = (0,\infty) \times M$ with metric

$$ds^2 = (dr)^2 + \rho(r)^2 h(r,x,dx),$$

$$h(r,x,dx) - h_M(x,dx) \in S^{-\epsilon}, \quad \epsilon > 0.$$

Assume that

(B-1) *There exist constants* $c_0, \epsilon_0, \delta_0, \epsilon$ *such that*

$$\frac{\rho'}{\rho} - c_0 \in S^{-\epsilon_0}, \quad (r\partial_r + \delta_0)\rho^{-1} \leq 0$$

and satisfying either (i) or (ii) :

(i) $c_0 \geq 0$, $\delta_0 > 1/3$, $\epsilon_0 > 0$, $\epsilon > 0$.

(ii) $c_0 = 0$, $\delta_0 > 0$, $\epsilon_0 = 1$, $\epsilon > 0$.

We fix a point p_0 in $\mathcal{M}$, and let

$$S(r) = \{p \in \mathcal{M}\,;\, \mathrm{dist}(p, p_0) = r\}.$$

Theorem

Suppose there exist constants $R > 0$, $\lambda > \left((n-1)c_0/2\right)^2$ and $u \in H^2_{loc}(\mathcal{M})$ such that

$$(-\Delta_{\mathcal{M}} - \lambda)u = 0, \quad \text{for} \quad r > R,$$

$$\liminf_{r\to\infty} r^{\gamma} \int_{S(r)} \left(\left|\frac{\partial u}{\partial r}\right|^2 + |u|^2 \right) dS(r) = 0, \quad \exists \gamma > 0.$$

Then $u = 0$ for $r > R$.

This theorem covers all of the cases :

$$\rho(r) = \begin{cases} \exp\left(c_0 r + \frac{\beta}{\alpha} r^{\alpha}\right), & 0 < \alpha < 1, \\ \exp\left(c_0 r\right) r^{\beta}, & \alpha = 0. \end{cases}$$

To prove this theorem, we consider an abstract differential equation

$$-u''(t) + B(t)u(t) + V(t)u(t) - \lambda u(t) = 0, \quad t > 0$$

for an Hilbert space - valued functions, and apply the classical method of T. Kato ([1959], CPAM), or Eidus ([1969], Russ. Math. Survey).

Here, $B(t)$ (corresponding to $-\rho(r)^{-2}\Delta_M$) is a non-negative self-adjoint operator having the property

$$t\frac{dB(t)}{dt} + \delta B(t) \leq Ct^{-\epsilon}, \quad \delta > 0.$$

Basic spectral properties

Consider the case of one end : $\mathcal{M} = (0, \infty) \times M$, on which the metric is

$$ds^2 = (dr)^2 + \rho(r)^2 h(r, x, dx).$$

Then, the Laplacian is

$$-\Delta_{\mathcal{M}} = -\partial_r^2 - \frac{g'}{2g}\partial_r + \rho^{-2}B(r),$$

$$g = \rho^{2(n-1)}h, \quad h = h(r, x) = \det\left(h_{ij}\right),$$

$$B(r) = -\frac{1}{\sqrt{h}}\partial_{x_i}\left(\sqrt{h}h^{ij}\partial_{x_j}\right).$$

We put

$$\mathbf{h}(r) = L^2\left(M;\ \sqrt{h(r,x)}dx\right),$$

$$L^{2,s}(\mathcal{M}) \ni f \Longleftrightarrow \|f\|_s^2 = \int_0^\infty (1+r)^{2s}\|f(r)\|^2_{\mathbf{h}(r)}\rho(r)^{n-1}dr < \infty,$$

$$\mathcal{B} \ni f \Longleftrightarrow \sum_{j=1}^{\infty} 2^{j/2}\|f\|_{L^2(I_j)} < \infty,$$

$$I_0 = (0,1), \quad I_j = (2^{j-1}, 2^j), \quad (j \geq 1),$$

$$\mathcal{B}^* \ni v \Longleftrightarrow \sup_{R>1}\frac{1}{R}\int_0^R \|v(r)\|^2_{\mathbf{h}(r)}\rho(r)^{n-1}dr < \infty,$$

$$\mathcal{B}_0^* \ni v \Longleftrightarrow \lim_{R\to\infty}\frac{1}{R}\int_0^R \|v\|^2_{\mathbf{h}(r)}\rho(r)^{n-1}dr = 0.$$

We then have the following inclusion relations

$$L^{2,s} \subset \mathcal{B} \subset L^{2,1/2} \subset L^2 \subset L^{2,-1/2} \subset \mathcal{B}^* \subset L^{2,-s}, \quad s > 1/2.$$

We return to our manifold

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cdots \mathcal{M}_{N+N'}.$$

Take a partition of unity $\{\chi_j\}_{j=0}^{N+N'}$, and define the norms on $\mathcal{M}$, e.g.

$$\|f\|_s = \|\chi_0 f\|_{L^2(\mathcal{M})} + \sum_{j=1}^{N+N'} \|\chi_j f\|_{L^{2,s}(\mathcal{M}_j)}.$$

Recall that on each end $\mathcal{M}_i = (0,\infty) \times M_i$,

$$ds^2 = (dr)^2 + \rho_i(r)^2 h_i(r, x, dx),$$

$$\rho_i(r) = \begin{cases} \exp\left(c_{0,i} r + \frac{\beta_i}{\alpha_i} r^{\alpha_i}\right), & 0 < \alpha_i < 1, \\ \exp\left(c_{0,i} r\right) r^{\beta_i}, & \alpha_i = 0, \end{cases}$$

$$h_i(r, x, dx) - h_{M_i}(x, dx) \in S^{-\gamma_i},$$

where

for regular ends $1 \leq i \leq N$,
either $c_{0,i} > 0$, or $c_{0,i} = 0, \beta_i > 0$,

for cusp ends $N + 1 \leq i \leq N + N'$,
either $c_{0,i} < 0$, or $c_{0,i} = 0$, $\beta_i < 0$.

Let

$$H = -\Delta_{\mathcal{M}} \quad \text{on} \quad \mathcal{M},$$

$$R(z) = (H - z)^{-1}.$$

Let

$$E_{0,i} = \big((n-1)c_{0,i}/2\big)^2, \quad \mathcal{T} = \{E_{0,1}, \cdots, E_{0,N+N'}\}.$$

$$E_0 = \min_{1\le i\le N+N'} E_{0,i}, \quad E_{0,reg} = \min_{1\le i\le N} E_{0,i}.$$

Lemma

(1) $\sigma_e(H) = [E_0, \infty)$.
(2) $\sigma_p(H) \cap \big((E_{0,reg}, \infty) \setminus \mathcal{T}\big) = \emptyset$.
(3) Suppose $E_0 < E_{0,reg}$. Then, the eigenvalues in $(E_0, E_{0,reg}) \setminus \mathcal{T}$ are of finite multiplicities with possible accumulation points at $\mathcal{T}$.
(4) If all ends are cusp, the eigenvalues in $(E_0, \infty) \setminus \mathcal{T}$ are of finite multiplicities with possible accumulation points at $\mathcal{T}$ and infinity.

Take any compact interval

$$I \subset (E_0, \infty) \setminus \left(\sigma_p(H) \cup \{E_{0,1}, \cdots, E_{0,N+N'}\} \right).$$

Assume long-range, i.e. for $h_i(r, x, dx) - h_{M_i}(x, dx) \in S^{-\gamma_i}$,

$$\gamma_i > 0, \quad \forall i = 1, \cdots, N + N'.$$

Theorem

For any $f \in \mathcal{B}$ and $\lambda \in I$, there exists a wak $*$-limit $\lim_{\epsilon \to 0} R(\lambda \pm i\epsilon) f$, i.e. for $f, g \in \mathcal{B}$

$$\lim_{\epsilon \to 0} (R(\lambda \pm i\epsilon) f, g) = (R(\lambda \pm i0) f, g).$$

Moreover

$$\|R(\lambda \pm i0) f\|_{\mathcal{B}^*} \leq C \|f\|_{\mathcal{B}}, \quad \lambda \in I.$$

The representation space

Let

$$\mathbf{h}_j = L^2(M_j).$$

$$\mathbf{h}_{j,\infty} = \begin{cases} \mathbf{h}_j, & 1 \leq j \leq N, \\ \mathbf{C}, & N+1 \leq j \leq N+N', \end{cases}$$

$$\mathbf{h}_\infty = \bigoplus_{j=1}^{N+N'} \mathbf{h}_{j,\infty},$$

$$\widehat{\mathcal{H}} = \bigoplus_{j=1}^{N+N'} L^2((E_{0,j},\infty)\,;\,\mathbf{h}_{\infty,j}\,;\,d\lambda).$$

Let $\mathcal{H}_{ac}(H)$ be the absolutely continupus subspace for H, i.e.

$$\mathcal{H}_{ac}(H) \ni f \Longleftrightarrow d(E_H(\lambda)f, f) \text{ is absolutely continuous.}$$

where $E_H(\lambda)$ is the spectral decomposition of H.

We assume either

$$\beta_i \geq 1, \ \gamma_i > 0,$$

or

$$\beta_i > 0, \ \gamma_i > 1.$$

As in the case of $\mathbf{R}^n$, there exists a spectral reresentation (or generalized Fourier transformation) associated with H. It is first defined on $\mathcal{H}$,

$$\mathcal{F}^{(\pm)}(\lambda) = (\mathcal{F}_1^{(\pm)}(\lambda), \cdots, \mathcal{F}_{N+N'}^{(\pm)}(\lambda)) \in \mathbf{B}(\mathcal{B}\,;\,\mathbf{h}_\infty),$$

(to be be explained later),
and then extended to $L^2(\mathcal{M})$ by the formula

$$(\mathcal{F}^{(\pm)} f)(\lambda) = \mathcal{F}^{(\pm)}(\lambda) f.$$

Theorem

There exists a unitary operator $\mathcal{F}^{(\pm)} : \mathcal{H}_{ac}(H) \to \widehat{H}$ having the following properties.

(1) $(\mathcal{F}^{(\pm)}f)(\lambda) = \mathcal{F}^{(\pm)}(\lambda)f$, $\forall f \in \mathcal{B}$.
(2) $(\mathcal{F}^{(\pm)}Hf)(\lambda) = \lambda(\mathcal{F}^{(\pm)}f)(\lambda)$, $\forall f \in D(H)$.
(3) $\mathcal{F}^{(\pm)}(\lambda)^* \in \mathbf{B}(\mathbf{h}_\infty; \mathcal{B}^*)$, and

$$(H - \lambda)\mathcal{F}^{(\pm)}(\lambda)^* = 0.$$

(4) For any $f \in \mathcal{H}_{ac}(H)$, the inversion formula holds

$$f = \sum_{j=1}^{N+N'} \int_{E_{0,j}}^{\infty} \mathcal{F}_j^{(\pm)}(\lambda)^* (\mathcal{F}_j^{(\pm)}f)(\lambda) d\lambda.$$

Asymptotic expansion of the resolvent at infity

Each operator $\mathcal{F}_j^{(\pm)}(\lambda)$ is constructed as follows.
Note that $-\Delta_{h_j}$ has

- eigenvalues $B_\ell^{(j)}$, $\ell = 0, 1, 2, \cdots$,
- eigenprojections $P_\ell^{(j)}$, $\ell = 0, 1, 2, \cdots$.

Then for $1 \leq j \leq N$,

$$\mathcal{F}_j^{(\pm)}(\lambda) = \sum_{\ell=0}^{\infty} \mathcal{F}_{j,\ell}^{(\pm)}(\lambda) \otimes P_\ell^{(j)},$$

for $N + 1 \leq j \leq N + N'$,

$$\mathcal{F}_j^{(\pm)}(\lambda) = \mathcal{F}_{j,0}^{(\pm)}(\lambda).$$

Here $\mathcal{F}_{j,\ell}^{(\pm)}(\lambda)$ are related to the asymptotic behavior of the resolvent at infinity:

For $f \in \mathcal{B}$, $R(\lambda \pm i0)f$ behaves like as $r \to \infty$

- on $\mathcal{M}_j$, $1 \le j \le N$,

$$\sum_{\ell \ge 0} c_j(\lambda)\rho_j(\lambda)^{-(n-1)/2} e^{i\varphi_{j,\ell}^{(\pm)}(\lambda,r)} \mathcal{F}_{j,\ell}^{(\pm)}(\lambda) \otimes P_\ell^{(j)} f,$$

- on $\mathcal{M}_j$, $N+1 \le j \le N+N'$,

$$c_j(\lambda)\rho_j(\lambda)^{-(n-1)/2} e^{i\varphi_{j,0}^{(\pm)}(\lambda,r)} \mathcal{F}_{j,0}^{(\pm)}(\lambda) \otimes P_0^{(j)} f,$$

$$\varphi_{j,\ell}^{(\pm)}(\lambda, r) \sim \pm r\sqrt{\lambda - \frac{(n-1)^2 c_{0,j}^2}{4}}.$$

Helmholtz equation

Let

$$\mathbf{h}_\infty(\lambda) = \bigoplus_{j=1}^{N+N'} \chi_{(E_{0,j},\infty)}(\lambda)\mathbf{h}_{\infty,j}.$$

Theorem

$$\{u \in \mathcal{B}^* \,;\, (H-\lambda)u = 0\} = \mathcal{F}^{(\pm)}(\lambda)^*\mathbf{h}_\infty(\lambda).$$

S-matrix

Take $\forall\psi^{(in)} = (\psi_1^{(in)}, \cdots, \psi_{N+N'}^{(in)}) \in \mathbf{h}_\infty(\lambda)$, and let

$$\psi_{j,\ell}^{(in)} = P_\ell^{(j)}\psi_j^{(in)}.$$

Then $\exists 1 u \in \mathcal{B}^*$ and $\exists 1 \psi^{(out)} \in \mathbf{h}_\infty(\lambda)$ such that $(H-\lambda)u = 0$, which behaves as follows :

(1) on the regular ends $\mathcal{M}_j$, $1 \leq j \leq N$,

$$u \simeq \sum_{\ell=0}^{\infty} \omega_-^{(j,\ell)}(\lambda)\rho_j(\lambda)^{-(n-1)/2} e^{-i\varphi_j(\lambda,r)}\psi_{j,\ell}^{(in)},$$

$$-\sum_{\ell=0}^{\infty} \omega_+^{(j,\ell)}(\lambda)\rho_j(r)^{-(n-1)/2} e^{i\varphi_j(\lambda,r)}\psi_{j,\ell}^{(out)}$$

(2) on the cusp ends $\mathcal{M}_j$, $N+1 \leq j \leq N+N'$

$$u \simeq \omega_-^{(j)}(\lambda)\rho_j(\lambda)^{-(n-1)/2}e^{-i\varphi_j(\lambda,r)}\psi_j^{(in)}$$

$$-\omega_+^{(j)}(\lambda)\rho_j(r)^{-(n-1)/2}e^{i\varphi_j(\lambda,r)}\psi_j^{(out)}$$

The S-matrix is then defined by

$$\widehat{S}(\lambda) : \mathbf{h}_\infty(\lambda) \ni \psi^{(in)} \to \psi^{(out)} \in \mathbf{h}_\infty(\lambda)$$

which is unitary.

Inverse Scattering

Theorem (Inverse scattering from regular ends)

Given two manifolds $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$, let $\mathcal{M}_1^{(1)}$, $\mathcal{M}_1^{(2)}$ be regular ends, and assume that

$$\widehat{S}_{11}^{(1)}(\lambda) = \widehat{S}_{11}^{(2)}(\lambda), \quad \forall\lambda,$$

moreover $\mathcal{M}_1^{(11)}$ and $\mathcal{M}_1^{(2)}$ are isometric. Then $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are isometric.

The physical S-matrix for the cusp end does not have sufficient information to recover the whole manifold.

Generalized S-matrix

Assume that for the cusp ends $\mathcal{M}_j$, $j = N+1, \cdots, N+N'$, the metric has the form

$$ds^2\Big|_{\mathcal{M}_j} = (dr)^2 + \rho_j(r)^2 h_{M_j}(x, dx),$$

$$\rho_j(r) = \begin{cases} \exp\left(c_{0,j} r + \dfrac{\beta_j}{\alpha_j} r^{\alpha_j}\right), & 0 < \alpha_j < 1, \\ \exp\left(c_{0,j} r\right) r^{\beta_j}, & \alpha_j = 0, \end{cases}$$

where $c_{0,j} < 0$, or $c_{0,j} = 0$, $\beta_j < 0$.

Let $0 = B_0^{(j)} \leq B_1^{(j)} \leq B_2^{(j)} \leq \cdots$ be the eignevalues of $-\Delta_{M_j}$ with normarized eigenvectors $\phi_\ell^{(j)}, \ell = 0, 1, 2, \cdots$. We put

$$\Phi_j(r) = \int_1^r \Big(\frac{B_\ell^{(j)}}{\rho_j^2} - \lambda + \frac{n^2 - 2n}{4}\big(\frac{\rho_j'}{\rho_j}\big)^2\Big)^{1/2} dt.$$

Note

$$\Phi_j(r) \to \infty, \quad \text{as} \quad r \to \infty.$$

Then, there exist solutins of the equation

$$-u'' - \frac{(n-1)\rho_j'}{\rho_j} u' + \Big(\frac{B_\ell^{(j)}}{\rho_j^2} - \lambda\Big) u = 0,$$

which behave like

$$u_{\ell,\pm}^{(j)} \sim \rho_j(r)^{-(n-1)/2} e^{\pm\Phi_j(r)}, \quad r \to \infty.$$

Take any solution u of the equation

$$(-\Delta_{\mathcal{M}} - \lambda)u = 0, \quad \text{on} \quad \mathcal{M}_j, \quad j = N+1, \cdots, N+N'.$$

Expanding it by $\phi_\ell^{(j)}$, we have

$$(u, \phi_\ell^{(j)})_{L^2(M_j)} = a_\ell^{(j)} u_{\ell,+}^{(j)}(r) + b_\ell^{(j)} u_{\ell,-}^{(j)}(r).$$

Here, we introduce two sequences $\mathbf{A}_\pm^{(j)}$:

$$\mathbf{A}_\pm^{(j)} \ni \{c_{\ell,\pm}\}_{\ell=0}^\infty \Longleftrightarrow \sum_{\ell=0}^\infty |c_{\ell,\pm}|^2 |u_{\ell,\pm}^{(j)}(r)|^2 < \infty, \quad \forall r > 1.$$

Then, $c_{\ell,\pm}^{(j)}$ behaves like, roughly,

$$\mathbf{A}_\pm^{(j)} \ni \{c_{\ell,\pm}\}_{\ell=0}^\infty \Longleftrightarrow \sum_{\ell=0}^\infty |c_{\ell,\pm}|^2 e^{\pm 2B_\ell^{(j)} \Phi_j(r)} < \infty, \quad \forall r > 1.$$

So, $\mathbf{A}_+^{(j)}$ is the space of super-exponentially decaying sequences, and $\mathbf{A}_-^{(j)}$ is the space of super-exponentially growing sequences.

Lemma

If $\{a_\ell^{(j)}\} \in \mathbf{A}_+^{(j)}$, then $\{b_\ell^{(j)}\} \in \mathbf{A}_-^{(j)}$.

For the cusp end, we define the generalized incoming data and outgoing data by

$$\Psi_j^{(in)} = \sum_{\ell=0}^{\infty} a_\ell^{(j)} u_{\ell,+}^{(j)}(r)\phi_\ell^{(j)}(x), \quad \{a_\ell^{(j)}\}_{\ell=0}^{\infty} \in \mathbf{A}_+^{(j)},$$

(which is exponentially growing as $r \to \infty$),

$$\Psi_j^{(out)} = \sum_{\ell=0}^{\infty} b_\ell^{(j)} u_{\ell,-}^{(j)}(r)\phi_\ell^{(j)}(x), \quad \{b_\ell^{(j)}\}_{\ell=0}^{\infty} \in \mathbf{A}_-^{(j)},$$

(which is exponentially decaying as $r \to \infty$).

For the regular end, we put

$$\Psi_j^{(in)} = \sum_{\ell=0}^{\infty} \omega_-^{(j)}(\lambda, B_\ell^{(j)})\Psi_j^{(-)}(\lambda, r; B_\ell^{(j)})\psi_{j,\ell}^{(in)},$$

$$\Psi_j^{(out)} = \sum_{\ell=0}^{\infty} \omega_-^{(j)}(\lambda, B_\ell^{(j)})\Psi_j^{(-)}(\lambda, r; B_\ell^{(j)})\psi_{j,\ell}^{(out)}.$$

Theorem

For any incoming data $\psi_j^{(in)} \in L^2(M_j)$ from regular ends and genaralized incoming data from cusp ends, there exist a solution u of the equation $(-\Delta_{\mathcal{M}} - \lambda)u = 0$ and the outgoing data such that

$$u - \sum_{j=N+1}^{N+N'} \Psi_j^{(in)} \in \mathcal{B}^*,$$

moreover

$$u \simeq \Psi_j^{(in)} - \Psi_j^{(out)}, \quad \text{on} \quad \mathcal{M}_j, \quad j = 1, \cdots, N,$$

$$u = \Psi_j^{(in)} - \Psi_j^{(out)}, \quad \text{on} \quad \mathcal{M}_j, \quad j = N+1, \cdots, N+N'.$$

We call the mapping

$$\mathcal{S}(\lambda) : (\psi_1^{(in)}, \cdots, \psi_N^{(in)}, \{a_\ell^{(N+1)}\}, \cdots, \{a_\ell^{(N+N')}\}) \to (\psi_1^{(out)}, \cdots, \psi_N^{(out)}, \{b_\ell^{(N+1)}\}, \cdots, \{b_\ell^{(N+N')}\})$$

the generalized scatteing matrix.

Theorem (Inverse scattering from cusp)

Let $\mathcal{M}^{(1)}_{N_1+N_1'}$, $\mathcal{M}^{(2)}_{N_2+N_2'}$ be cusp ends of $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$. Assume that
(i) $\mathcal{M}^{(1)}_{N_1+N_1'}$ and $\mathcal{M}^{(2)}_{N_2+N_2'}$ are isometric,
(ii) The components of the generalized S-matrix associated with the cusp ends coincide for all energies. Then $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are isometric.

Idea of the proof

Introduce an artificial boundary

$$S = \{(r, x)\,;\; r = 2, x \in M_1\} \subset \mathcal{M}_1.$$

Let

$$\mathcal{M}_{ext} = \mathcal{M}_1 \cap \{r \geq 2\},$$

$$\mathcal{M}_{int} = \mathcal{M} \setminus \mathcal{M}_{ext}.$$

Then $\widehat{S}_{11}(k)$ determines the N-D map

$$H^1(S) \ni f \to u\Big|_S,$$

where u is the solution to the equation

$$(H - \lambda)u = 0, \quad \text{in} \quad \mathcal{M}_{int},$$

$$\frac{\partial u}{\partial r} = f, \quad \text{on} \quad S,$$

$$u \text{ satisfies the radiation condition.}$$

Using this N-D map, one can apply the boundary control method due to Belishev-Kurylev to reconstruct the manifold $\mathcal{M}$.
For the case of cusp, one needs to introduce the generaized S-matrix.
(On the cusp end $\mathcal{M}_{N+N'}$, one uses eigenprojections associated with $-\Delta_h$ to reduce to the 1-dimensional problem, and observe the behavior of growing and decaying solutions.)

Remarks on the assumption

The decay assumption is roughly as follows. For the sake of simplicity, let us consider only the case of regular ends, and assume that

$$\rho(r)^{-1} \in S^{-\beta},$$

$$h(r,x,dx) - h_M(x,dx) \in S^{-\gamma}.$$

Then we assume :
(1) For the limiting absorption principle,

$$\beta > 0, \quad \gamma > 0.$$

(2) For the spectral repesentation, and the inverse scattering,

$$\text{either} \quad \beta \geq 1, \quad \gamma > 0, \quad \text{or} \quad \beta > 0, \quad \gamma > 1.$$

For Further Reading I

H. Isozaki, Y. Kurylev and M. Lassas,
Conic singularities, genealized scattering matrix, and inverse scattering on asymptotically hyperbolic surfaces,
to appear in *J. Reine angew. Math.*.

H. Isozaki, Y. Kurylev and M. Lassas,
Spectral theory and inverse problems on asymptotically hyperbolic orbifolds,
arXiv : 1312.0421

H. Isozaki, Y. Kurylev and M. Lassas,
Recent progrss of inverse scattering theory on non-compact manifolds,
to appear in *Contemp. Math.*

For Further Reading II