Inverse scattering at high energies for classical particles in a long range electromagnetic field

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I. Forward problem

• Multidimensional relativistic Newton equation in a static external electromagnetic field [Einstein, 1907]

(1)
$$\dot{p} = -\nabla V(x) + \frac{1}{c}B(x)\dot{x},$$
$$p = \frac{\dot{x}}{\sqrt{1-\frac{|\dot{x}|^2}{c^2}}},\ x(t)\in\mathbb{R}^n,\ n\geq 2.$$

• Smoothness and long range assumptions for the external field

(2)
$$V\in C^2(\mathbb{R}^n,\mathbb{R}),\ B(x)=(B_{i,k})\in C^1(\mathbb{R}^n,A_n(\mathbb{R})),$$
$$\frac{\partial B_{i,k}}{\partial x_l}(x)+\frac{\partial B_{l,i}}{\partial x_l}(x)+\frac{\partial B_{k,l}}{\partial x_i}(x)=0,$$
$$V=V^s+V^l,\ B=B^s+B^l,$$

$$|\partial_x^{j_1}V^l(x)|\leq \beta^l_{|j_1|}(1+|x|)^{-\alpha-|j_1|},\ |\partial_x^{j_2}B^l_{i,k}(x)|\leq \beta^l_{|j_2|+1}(1+|x|)^{-\alpha-1-|j_2|},$$

$$|\partial_x^{j_1}V^s(x)|\leq \beta^s_{|j_1|+1}(1+|x|)^{-\alpha-1-|j_1|},|\partial_x^{j_2}B^s_{i,k}(x)|\leq \beta^s_{|j_2|+2}(1+|x|)^{-\alpha-2-|j_2|},$$

for $|j_1|\leq 2$, $|j_2|\leq 1$, $i,k,l=1\ldots n$ and for some $\alpha\in(0,1]$, where $j=(j^1,\ldots,j^n)\in(\mathbb{N}\cup\{0\})^n$, $|j|=\sum_{i=1}^n j^i$ and where $\beta^s_{|j|+1}$ and $\beta^l_{|j|}$ are positive constants).

- Integral of motion, the energy of the classical relativistic particle

$$E = c^2\sqrt{1+\frac{|p(t)|^2}{c^2}} + V(x(t)) \qquad (3)$$

- Parametrization of the scattering solutions

For $v \in B_c$, $v \neq 0$, let $z_\pm(v,.)$ be a solution of the equation (1) with $F^s \equiv 0$ so that

$$\dot z_\pm(v,t) \to v, \ \text{as } t \to \pm\infty.$$

Then for any $(v_-, x_-) \in B_c \times \mathbb{R}^n$, $v_- \neq 0$, there exists a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ of equation (1) so that

$$x(t) = z_-(v_-, t) + x_- + y_-(t), \ |y_-(t)| + |\dot y_-(t)| \to 0, \ \text{as } t \to -\infty;$$

and for a.e. $(v_-, x_-) \in B_c \times \mathbb{R}^n$, $v_- \neq 0$,

$$x(t) = z_+(v_+, t) + x_+ + y_+(t), \ |y_+(t)| + |\dot y_+(t)| \to 0, \ \text{as } t \to +\infty,$$

for some (v_+, x_+), $|v_+| = |v_-|$.

- Scattering map and scattering data for equation (1) :

$$S(v_-, x_-) := (v_+, x_+) =: (v_- + a_{sc}(v_-, x_-), x_- + b_{sc}(v_-, x_-))$$

- Direct problem : $\text{Given } (V, B), \text{ find } S.$

  Inverse problem : $\text{Given } (V^l, B^l, S), \text{ find } (V^s, B^s).$

II. Inverse scattering at high energies

II.1 The "free" solutions

Let $v \in B_c$, $v \neq 0$. When $|v| > \rho(n, c, \beta_1^l, \beta_2^l)$ then there exists a unique solution $z_\pm(v, .)$ of the equation (1) with $F^s \equiv 0$ so that

$$\lim_{t \to \pm\infty} \dot{z}_\pm(v, t) = v, \ \ z_\pm(v, 0) = 0, \qquad \text{and}$$

$$\sup_{t \in \mathbb{R}} |\dot{z}_\pm(v, t) - v| \leq \frac{C n^{\frac{3}{2}} \beta_1^l \sqrt{1 - \frac{|v|^2}{c^2}}}{\alpha |v|} \quad .$$

II.2 Asymptotic of the scattering data

• X-ray transform : $$Pf(\theta, x) = \int_{-\infty}^{+\infty} f(t\theta + x)dt, \ (\theta, x) \in T\mathbb{S}^{n-1}.$$

for $f \in C(\mathbb{R}^n, \mathbb{R}^m)$, $f(x) = O(|x|^{-1-\varepsilon})$ when $|x| \to +\infty$, $\varepsilon > 0$,

and where $T\mathbb{S}^{n-1} := \{(\theta', x') \in \mathbb{S}^{n-1} \times \mathbb{R}^n \mid \theta' \cdot x' = 0\}$.

First study and inversion of P in $\mathbb{R}^2$: Radon (1917).

Application to X-ray Tomography : Cormack (1963).

Theorem 1 [J1]. *Let $(\theta, x) \in T\mathbb{S}^{n-1}$ and $0 < r \le 1$, $r < \frac{c}{\sqrt{2}}$. Under conditions (2) we have*

$$\lim_{\substack{\rho \to c \\ \rho < c}} \frac{\rho}{\sqrt{1-\frac{\rho^2}{c^2}}} a_{sc}(\rho\theta, x) = -P(\nabla V)(\theta, x) + \int_{-\infty}^{+\infty} B(x+\tau\theta)\theta d\tau, \qquad \textit{and}$$

$$\left| \frac{\rho}{\sqrt{1-\frac{\rho^2}{c^2}}} a_{sc}(\rho\theta, x) + P(\nabla V)(\theta, x) - \frac{\rho}{c}\int_{-\infty}^{+\infty} B(x+\tau\theta)\theta d\tau \right| \le \frac{Cn^4\beta^2(1+|x|)\rho\left(\frac{1}{c}+\frac{1}{\frac{\rho}{2\sqrt{2}}-r}\right)}{\alpha^2(\frac{\rho}{2\sqrt{2}}-r)^2(1-r)^{2\alpha+2}}\sqrt{1-\frac{\rho^2}{c^2}}$$

for $\rho_1(c, n, \beta, \alpha, r) < \rho < c$, $(\beta = \max(\beta_1, \beta_2, \beta_3))$; *In addition*

$$\left| \frac{\rho^2}{\sqrt{1-\frac{\rho^2}{c^2}}}\big(b_{sc}(\rho\theta, x) - W(\rho\theta, x)\big) - \frac{\rho^2}{c^2} PV^s(\theta, x)\theta + \int_{-\infty}^{0}\int_{-\infty}^{\tau} \nabla V^s(\sigma\theta + x) d\sigma d\tau \right.$$

$$\left. - \int_0^{+\infty}\int_{\tau}^{+\infty} \nabla V^s(\sigma\theta+x)d\sigma d\tau - \frac{\rho}{c}\int_{-\infty}^{0}\int_{-\infty}^{\tau} B^s(\sigma\theta+x)\theta d\sigma d\tau + \frac{\rho}{c}\int_0^{+\infty}\int_{\tau}^{+\infty} B^s(\sigma\theta+x)\theta d\sigma d\tau \right|$$

$$\le \frac{Cn^4\beta^2\big(1+|x|\big)\big(\frac{1}{c}+1\big)\rho^2\big(1+\frac{1}{\frac{\rho}{2\sqrt{2}}-r}\big)}{\alpha^2(\alpha+1)(\frac{\rho}{2\sqrt{2}}-r)^3(1-r)^{2\alpha+1}}\sqrt{1-\frac{\rho^2}{c^2}},$$

for $\rho_2(c, n, \beta, \alpha, r) < \rho < c$.

The vector W is known from F^l and the scattering data:

$$W(v,x) := \int_{-\infty}^{0} \Big(g\big(g^{-1}(v)+\int_{-\infty}^{\sigma} F^l(z_-(v,\tau)+x, \dot z_-(v,\tau))d\tau\big) - g\big(g^{-1}(v)+\int_{-\infty}^{\sigma} F^l(z_-(v,\tau), \dot z_-(v,\tau))d\tau\big)\Big)d\sigma$$

$$+\int_{0}^{+\infty} \Big(g\big(g^{-1}(a(v,x)) - \int_{\sigma}^{+\infty} F^l(z_+(a(v,x),\tau)+x, \dot z_+(a(v,x),\tau))d\tau\big)$$

$$-g\big(g^{-1}(a(v,x)) - \int_{\sigma}^{+\infty} F^l(z_+(a(v,x),\tau), \dot z_+(a(v,x),\tau))d\tau\big)\Big)d\sigma, \quad \text{for } (v,x)\in \mathcal{D}(S).$$

Proposition 1 [J3]. *Under conditions* (2) *we have*

$$P(\nabla V)(\theta,x) = -\frac{1}{2}\left(\omega_1(V,B,\theta,x)+\omega_1(V,B,-\theta,x)\right).$$

for $(\theta,x)\in T\mathbb{S}^{n-1}$ *; in addition*

$$P(B_{i,k})(\theta,x) = \frac{\theta_k}{2}\left(\omega_1(V,B,\theta,x)_i - \omega_1(V,B,-\theta,x)_i\right)$$

$$-\frac{\theta_i}{2}\left(\omega_1(V,B,\theta,x)_k - \omega_1(V,B,-\theta,x)_k\right)$$

for $i,k=1\ldots n$ *and for every* $(\theta,x)\in T\mathbb{S}^{n-1}$, $\theta=(\theta_1,\ldots,\theta_n)$ *such that* $\theta_j=0$ *for* $j\neq i$ *and* $j\neq k$.

II.3 Idea of the proof

Theorem 1 was obtained by developing the method of R. Novikov (1999).
Equation (1) is rewritten in an integral equation and we have

$$y_- = A_{v_-,x_-}(y_-), \qquad \text{where}$$

$$\begin{cases} A_{v_-,x_-}(f)(t) = \int_{-\infty}^{t} \dot{A}_{v_-,x_-}(f)(\sigma)d\sigma, \\ \dot{A}_{v_-,x_-}(f)(t) = g\left(g^{-1}(v_-) + \int_{-\infty}^{t} F(z_-(v_-,\sigma) + x_- + f(\sigma), \dot{z}_-(v_-,\sigma) + \dot{f}(\sigma))d\sigma\right) \\ -g\left(g^{-1}(v_-) + \int_{-\infty}^{t} F^l(z_-(v_-,\sigma), \dot{z}_-(v_-,\sigma))d\sigma\right), \end{cases}$$

$$\textit{and where } F(x,v) = -\nabla V(x) + \tfrac{1}{c}B(x)v \textit{ for } (x,v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

We consider the operator A_{v_-,x_-} on the complete metric space

$$M_{r,v_-} := \{f \in C^1(\mathbb{R},\mathbb{R}^n) \mid \sup_{\mathbb{R}} |\dot{z}_-(v_-,.) + \dot{f}| \le c,$$
$$\|f\| := \max\Big(\sup_{(-\infty,0)} |f|, \sup_{t\in(-\infty,0)} \max(1,(1-r+(\frac{|v_-|}{2\sqrt{2}}-r)|t|))|\dot{f}(t)|, \sup_{(0,+\infty)} |\dot{f}|\Big) \le r\},$$
$$0 < r < \min(1, \frac{|v_-|}{2\sqrt{2}}).$$

Hence we study a small angle scattering regime compared to the dynamics generated by F^l .

- Quantum analogs : Born, Faddeev (1956), Henkin-Novikov (1988), Enss-Weder (1995), H. Ito (1995), Isozaki (1997), Jung (1997), Hachem (1999).

III. A modified scattering map

III.1 Other "free" solutions

Let $(v, x) \in \mathcal{B}_c \times \mathbb{R}^n$, $v \neq 0$, $v \cdot x = 0$. When $|v| > \tilde{\rho}(n, c, \beta_1^l, \beta_2^l, |x|)$ then there exists a unique solution $z_\pm(w, x+q, .)$ of the equation (1) with $F^s \equiv 0$ so that

$$\lim_{t \to \pm\infty} \dot{z}_\pm(w, x+q, t) = w, \;\; z_\pm(w, x+q, 0) = x+q, \qquad \text{and}$$

$$\sup_{t \in \mathbb{R}} |\dot{z}_\pm(w, x+q, t) - w| \leq \frac{C n^{\frac{3}{2}} \beta_1^l \sqrt{1 - \frac{|v|^2}{c^2}}}{\alpha |v| (1 + \frac{|x|}{\sqrt{2}})^\alpha}$$

for $t \in \mathbb{R}$ and for $(w, q) \in \mathcal{B}_c \times \overline{\mathcal{B}_{\frac{1}{2}}}$, $|v| = |w|$ and $|v - w| \leq \frac{|v|}{2^{\frac{5}{2}}}$.

Consider $x(t)$ the unique solution of the equation (1) that satisfies

$$x(t) = z_-(v_-, x_-, t) + y_-(t), \;\; |y_-(t)| + |\dot{y}_-(t)| \to 0, \;\; \text{as } t \to -\infty.$$

When $|v_-| > \tilde{\rho}_0(n, c, \beta_1^l, \beta_2, |x_-|)$ then

$$x(t) = z_+(\tilde{a}(v_-, x_-), \tilde{b}(v_-, x_-), t) + y_+(t), \;\; |y_+(t)| + |\dot{y}_+(t)| \to 0, \;\; \text{as } t \to +\infty,$$

for some $(\tilde{a}(v_-, x_-), \tilde{b}(v_-, x_-))$.

- Modified scattering map : $\tilde{S}(v_-, x_-) = (\tilde{a}(v_-, x_-), \tilde{b}(v_-, x_-))$

- Direct problem : Given (V, B), find $\tilde{S}$.

- Inverse problem : Given $(V^l, B^l, \tilde{S})$ find (V^s, B^s).

III.2 High energies asymptotics of the modified scattering data

Theorem 2 [J1]. *Let* $(\theta, x) \in T\mathbb{S}^{n-1}$ *and* $0 < r < \min\left(2^{-\frac{3}{2}}c, \frac{1}{2}\right)$. *Under conditions* (2) *we have*

$$\lim_{\substack{\rho\to c\\ \rho<c}} \frac{\rho}{\sqrt{1-\frac{\rho^2}{c^2}}}\tilde{a}_{sc}(\rho\theta, x) = -P(\nabla V)(\theta, x) + \int_{-\infty}^{+\infty} B(x+\tau\theta)\theta d\tau, \qquad \textit{and}$$

$$\left|\frac{\rho}{\sqrt{1-\frac{\rho^2}{c^2}}}(\tilde{a}_{sc}(\rho\theta, x) - \tilde{W}(\rho\theta, x)) + P(\nabla V^s)(\theta, x) - \frac{\rho}{c}\int_{-\infty}^{+\infty} B^s(x+\tau\theta)\theta d\tau\right|$$

$$\leq \frac{Cn^4\beta^2\rho(\frac{1}{c} + \frac{1}{\frac{\rho}{2^{\frac{3}{2}}}-r})}{\alpha(\alpha+1)(\frac{\rho}{2\sqrt{2}} - r)^2(1+|x|)^{2\alpha+1}}\sqrt{1-\frac{\rho^2}{c^2}}$$

for $\tilde{\rho}_1(c, n, \beta, |x|, \alpha, r) < \rho < c$,

$$\lim_{\substack{\rho\to c\\ \rho<c}} \frac{\rho^2}{\sqrt{1-\frac{\rho^2}{c^2}}}\tilde{b}_{sc}(\rho\theta, x) = \omega_2(V^s, B^s, \theta, x)$$

$$\Big|\frac{\rho^2}{\sqrt{1-\frac{\rho^2}{c^2}}}\tilde{b}_{sc}(\rho\theta, x) - \frac{\rho^2}{c^2}PV^s(\theta, x)\theta + \int_{-\infty}^{0}\int_{-\infty}^{\tau}\nabla V^s(\sigma\theta + x)d\sigma d\tau$$

$$-\int_{0}^{+\infty}\int_{\tau}^{+\infty}\nabla V^s(\sigma\theta + x)d\sigma d\tau - \frac{\rho}{c}\int_{-\infty}^{0}\int_{-\infty}^{\tau}B^s(\sigma\theta + x)\theta d\sigma d\tau + \frac{\rho}{c}\int_{0}^{+\infty}\int_{\tau}^{+\infty}B^s(\sigma\theta + x)\theta d\sigma d\tau\Big|$$

$$\leq \frac{Cn^4\beta^2\big(1+\frac{1}{c}\big)\rho^2\big(1+\frac{1}{\frac{\rho}{2\sqrt{2}}-r}\big)}{\alpha^2(\alpha+1)(\frac{\rho}{2\sqrt{2}}-r)^3(1+|x|)^{2\alpha}}\sqrt{1-\frac{\rho^2}{c^2}},$$

$for\ \tilde{\rho}_2(c, n, \beta, |x|, \alpha, r) < \rho < c.$

The vector $\tilde{W}$ is known from F^l:

$$\tilde{W}(v, x) = g\Big(g^{-1}(v) + \int_{-\infty}^{+\infty}F^l(z_-(v, x, \tau), \dot{z}_-(v, x, \tau))d\tau\Big) - v.$$

IV. Similar results for the multidimensional (nonrelativistic) Newton equation

$$\ddot{x} = -\nabla V(x) + B(x)\dot{x}.$$

- Energy of the particle : $E = \frac{1}{2}|\dot{x}|^2 + V(x)$.

- High energies asymptotics of the scattering data

Theorem 3. *Let* $(\theta, x) \in T\mathbb{S}^{n-1}$, $\theta = (\theta_1, \ldots, \theta_n)$. *Then*

$$a_{sc}^{nr}(\rho\theta, x) = \int_{-\infty}^{+\infty} B(\tau\theta + x)\theta d\tau + \rho^{-1}\Big(- P(\nabla V)(\theta, x) + \int_{-\infty}^{+\infty} B(\sigma\theta + x) \int_{-\infty}^{\sigma} B(\tau\theta + x)\theta d\tau d\sigma\Big)$$

$$+\rho^{-1}\sum_{j=1}^{n} \theta_j \Big(\int_{-\infty}^{+\infty} < \nabla B_{i,j}(\sigma\theta + x), \int_{-\infty}^{\sigma}\int_{-\infty}^{\tau} (B(\eta\theta + x) - B^l(\eta\theta))\theta d\eta d\tau + \int_0^{\sigma}\int_{-\infty}^{\tau} B^l(\eta\theta)\theta d\eta d\tau > d\sigma\Big)_{i=1\ldots n}$$

$$+o(\rho^{-1}),$$

$$
\begin{aligned}
\rho(b_{sc}^{nr}-W^{nr})(\rho\theta,x) \;=\;& \int_{-\infty}^{0}\int_{-\infty}^{\sigma} B^{s}(\tau\theta+x)\theta d\tau d\sigma - \int_{0}^{+\infty}\int_{\sigma}^{+\infty} B^{s}(\tau\theta+x)\theta d\tau d\sigma \\
&+\rho^{-1}\Big(\int_{-\infty}^{0}\int_{-\infty}^{\sigma}(-\nabla V^{s})(\tau\theta+x)d\tau d\sigma - \int_{0}^{+\infty}\int_{\sigma}^{+\infty}(-\nabla V^{s})(\tau\theta+x)d\tau d\sigma\Big) \\
&+\rho^{-1}\Big(\int_{-\infty}^{0}\int_{-\infty}^{\sigma} B^{s}(\tau\theta+x)\int_{-\infty}^{\tau} B(\eta\theta+x)\theta d\eta d\tau d\sigma - \int_{0}^{+\infty}\int_{\sigma}^{+\infty} B^{s}(\tau\theta+x)\int_{-\infty}^{\tau} B(\eta\theta+x)\theta d\eta d\tau d\sigma\Big) \\
&+\rho^{-1}\Big(\int_{-\infty}^{0}\int_{-\infty}^{\sigma} B^{l}(\tau\theta+x)\int_{-\infty}^{\tau} B^{s}(\eta\theta+x)\theta d\eta d\tau d\sigma - \int_{0}^{+\infty}\int_{\sigma}^{+\infty} B^{l}(\tau\theta+x)\int_{\tau}^{+\infty} B^{s}(\eta\theta+x)\theta d\eta d\tau d\sigma\Big) \\
&+\rho^{-1}\sum_{j=1}^{n}\theta_j\int_{-\infty}^{0}\int_{-\infty}^{\sigma}\Big(<\nabla B_{i,j}^{s}(\tau\theta+x),\int_{-\infty}^{\tau}\int_{-\infty}^{\eta_1}(B(\eta_2\theta+x)-B^{l}(\eta_2\theta))\theta d\eta_2 d\eta_1 \\
&\qquad\qquad + \int_{0}^{\tau}\int_{-\infty}^{\eta_1} B^{l}(\eta_2\theta)\theta d\eta_2 d\eta_1 >\Big)_{i=1\dots n} d\tau d\sigma \\
&+\rho^{-1}\sum_{j=1}^{n}\theta_j\int_{-\infty}^{0}\int_{-\infty}^{\sigma}\Big(<\nabla B_{i,j}^{l}(\tau\theta+x),\int_{-\infty}^{\tau}\int_{-\infty}^{\eta_1} B^{s}(\eta_2\theta+x)\theta d\eta_2 d\eta_1 >\Big)_{i=1\dots n} d\tau d\sigma \\
&-\rho^{-1}\sum_{j=1}^{n}\theta_j\int_{0}^{+\infty}\int_{\sigma}^{+\infty}\Big(<\nabla B_{i,j}^{s}(\tau\theta+x),\int_{-\infty}^{\tau}\int_{-\infty}^{\eta_1}(B(\eta_2\theta+x)-B^{l}(\eta_2\theta))\theta d\eta_2 d\eta_1 \\
&\qquad\qquad + \int_{0}^{\tau}\int_{-\infty}^{\eta_1} B^{l}(\eta_2\theta)\theta d\eta_2 d\eta_1 >\Big)_{i=1\dots n} d\tau d\sigma \\
&-\rho^{-1}\sum_{j=1}^{n}\theta_j\int_{0}^{+\infty}\int_{\sigma}^{+\infty}\Big(<\nabla B_{i,j}^{l}(\tau\theta+x),\int_{\tau}^{+\infty}\int_{\eta_1}^{+\infty} B^{s}(\eta_2\theta+x)\theta d\eta_2 d\eta_1 >\Big)_{i=1\dots n} d\tau d\sigma \\
&+o(\rho^{-1}), \qquad\qquad as\ \rho\to+\infty.
\end{aligned}
$$

$$
\begin{aligned}
W^{nr}(v,x) \;=\; & -\int_{-\infty}^{0}\int_{-\infty}^{\sigma}\big(\nabla V^l(z_-(v,\tau)+x)-\nabla V^l(z_-(v,\tau))\big)d\tau d\sigma \\
& +\int_{-\infty}^{0}\int_{-\infty}^{\sigma}\Big(B^l(z_-(v,\tau)+x+\int_{-\infty}^{\tau}\int_{-\infty}^{s_1}(B^l(z_-(v,s_2)+x)-B^l(z_-(v,s_2)))\dot z_-(s_2)ds_2ds_1\big) \\
& \qquad -B^l(z_-(v,\tau))\Big)\dot z_-(v,\tau)d\tau d\sigma \\
& +\int_{-\infty}^{0}\int_{-\infty}^{\sigma}B^l(z_-(v,\tau)+x)\Big(\int_{-\infty}^{\tau}\big(B^l(z_-(v,\eta)+x)-B^l(z_-(v,\eta))\big)\dot z_-(v,\eta)d\eta\Big)d\tau d\sigma \\
& +\int_{0}^{+\infty}\int_{\sigma}^{+\infty}\big(\nabla V^l(z_+(a,\tau)+b)-\nabla V^l(z_+(a,\tau))\big)d\tau d\sigma \\
& +\int_{0}^{+\infty}\int_{\sigma}^{+\infty}B^l(z_+(a,\tau)+b)\Big(\int_{\tau}^{+\infty}\big(B^l(z_+(a,\eta)+b)-B^l(z_+(a,\eta))\big)\dot z_+(a,\eta)d\eta\Big)d\tau d\sigma \\
& -\int_{0}^{+\infty}\int_{\sigma}^{+\infty}\Big(B^l(z_+(a,\tau)+b+\int_{\tau}^{+\infty}\int_{s_1}^{+\infty}(B^l(z_+(a,s_2)+b)-B^l(z_+(a,s_2)))\dot z_+(a,s_2)ds_2ds_1\big) \\
& \qquad -B^l(z_+(a,\tau))\Big)\dot z_+(a,\tau)d\tau d\sigma .
\end{aligned}
$$

- Quantum analogs : Nicoleau (1997), Arians (1997).

- Asymptotics of the modified scattering data when $|x| \to +\infty$.

$$\tilde{W}^{nr}(v,x) = \int_{-\infty}^{+\infty} F^l(z_-(v,x,\tau), \dot{z}_-(v,x,\tau))d\tau.$$

Theorem 4. *Let* $(\theta, x) \in T\mathbb{S}^{n-1}$, $\theta = (\theta_1, \ldots, \theta_n)$. *Then*

$$\rho\big(\tilde{a}_{sc}^{nr}(\rho\theta, x) - \tilde{W}^{nr}(\rho\theta, x)\big) = \int_{-\infty}^{+\infty} F^s(\tau\theta + x, \rho\theta)d\tau + O(\frac{1}{|x|^{2\alpha+1}}),$$

$$\rho^2\tilde{b}_{sc}^{nr}(\rho\theta, x) = \int_{-\infty}^{0}\int_{-\infty}^{\sigma} F^s(\tau\theta+x, \rho\theta)d\tau d\sigma - \int_{0}^{+\infty}\int_{\sigma}^{+\infty} F^s(\tau\theta+x, \rho\theta)d\tau d\sigma + O(\frac{1}{|x|^{2\alpha}}),$$

as $|x| \to +\infty$.

Other directions

- Inverse scattering at high energies for the N-body problem.

- Inverse scattering at fixed energy in a long range electromagnetic field: similar conjectures to those formulated in [Novikov, 1999] for the short range case in classical non-relativistic mechanics.

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