Hamiltonian and small action variables for periodic defocussing non-linear Schrödinger equation

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Introduction. Consider the defocussing cubic non-linear Schrödinger equation (dNLS) (the real form)

$$J\frac{\partial \psi}{\partial t} = -\psi_{xx} + 2|\psi|^2\psi, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, i.e. $\psi(x+1, t) = \psi(x, t)$ for $x, t \in \mathbb{R}$, with the initial conditions:

$$\psi(\cdot, 0) = q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in L^2(\mathbb{T}) \oplus L^2(\mathbb{T}),$$

where q_1, q_2 are real functions. The dNLS equation has the Hamiltonian H given by

$$H(q) = \frac{1}{2}\int_0^1 (|q'(x)|^2 + |q(x)|^4)dx. \tag{0.1}$$

For functionals $E = E(q)$ and $G = G(q)$ the Poisson bracket $\{E, G\}_P$ has the form

$$\{E, G\}_P = \int_0^1 \left(\frac{\partial E}{\partial q_1(x)} \frac{\partial G}{\partial q_2(x)} - \frac{\partial G}{\partial q_1(x)} \frac{\partial E}{\partial q_2(x)} \right) dx.$$

Using the Poisson bracket and the Hamiltonian we rewrite the DNLS in the following form

$$\frac{\partial \psi}{\partial t} = \{H(\psi), \psi\}_P = -J \begin{pmatrix} \frac{\partial H}{\partial \psi_1} \\ \frac{\partial H}{\partial \psi_2} \end{pmatrix} = -J(-\psi_{xx} + 2|\psi|^2\psi).$$

The dNLS equation admits globally defined real analytic action-angle variables $I_n \geqslant 0, \phi_n \in [0, 2\pi), n \in \mathbb{Z}$ (see Veselov-Novikov, McKean-Vaninsky and Grebert-Kappeler- Pöschel and McKean- Vaninsky). The action-angle variables are canonical varables:

$$\{I_n, I_m\}_P = 0, \qquad \{\phi_n, \phi_m\}_P = 0, \qquad \{I_n, \phi_m\}_P = \delta_{n,m},$$

$\forall\ n, m \in \mathbb{Z}$.

The main goal of my talk is to demonstrate the new approach to study integrable systems. This approach is based on so-called Löwner type equations.

The dNLS equation has the Hamiltonian H and other two integrals H_0 and H_1 given by

$$H_0 = \|q\|^2 = \sum_{n\in\mathbb{Z}} I_n,$$

$$H_1 = \int_0^1 (q_2'(x)q_1(x) - q_1'(x)q_2(x))dx = \sum_{n\in\mathbb{Z}} (2\pi n) I_n,$$

$$H = \sum_{n\in\mathbb{Z}} (2\pi n)^2 I_n + 2H_0^2 - V,$$

E.K. [05]. Here V is some nonlinear functional. In fact we rewrite H_0, H_1 and the main part of H in terms of simple functions of the actions $I = (I_n)_{n\in\mathbb{Z}}$.

Introduce the real spaces

$$\ell^p = \left\{ f = (f_n)_{n\in\mathbb{Z}}, \quad \|f\|_p^p = \sum_n f_n^p < \infty \right\}, \qquad p \geqslant 1.$$

Recall that we have:
(i) $|q| \in L^2(\mathbb{T})$ iff $(I_n)_{n\in\mathbb{Z}} \in \ell^1$,
(ii) $|q'| \in L^2(\mathbb{T})$ iff $(nI_n)_{n\in\mathbb{Z}} \in \ell^1$.

Mein results. Recall that the Hamiltonian H depends only on the actions I. Introduce the frequencies Ω_n by

$$\Omega_n = \frac{\partial H}{\partial I_n}, \quad n \in \mathbb{Z}. \tag{0.2}$$

The parameters Ω_n are very important, since the angle variables $\phi_n(t)$ as functions of time $t \geqslant 0$ have the form

$$\phi_n(t) = \phi_n(0) + \Omega_n t, \quad t \geqslant 0, \quad n \in \mathbb{Z}.$$

Due to the identity $H = \sum_{n\in\mathbb{Z}}(2\pi n)^2 I_n + 2H_0^2 - V$ we deduce that the gradient is given by

$$\Omega_n = (2\pi n)^2 + 4H_0 - \frac{\partial V}{\partial I_n}. \tag{0.3}$$

Thus in order to study Ω_n we need to study $\frac{\partial V}{\partial I_n}$ only. Our goal is to give a new method to study Hamiltonian as a function of action variables. In this paper we reformulate the problems for the dNLS equation as the problems of the conformal mapping theory. The main technical tool is Theorem 3 about the Löwner type equation.

Why the action-angle variables are important?

We fix time $t = 0$ and we construct the action-angle variables $\psi(x,0) \to (I_n(0), \phi_n(0))_{n\in\mathbb{Z}}$. This construction is still very complicated and very very technical!!!!

We consider the NLS equation $J\frac{\partial\psi}{\partial t} = -\psi_{xx} + 2|\psi|^2\psi$ and we fix $t > 0$. The action variables $I_n(t) = I_n(0)$ do not depend on time t and the angle variables satisfy

$\phi_n(t) = \phi_n(0) + \Omega_n t, \quad t \geqslant 0, \; n \in \mathbb{Z}.$

After this we solve the **inverse problem**

$$(I_n(t), \phi_n(t))_{n\in\mathbb{Z}} \to \psi(x,t),$$

which gives the solution of the NLS equation for any time $t > 0$. Thus we have

$$\psi(x,0) \to (I_n(0), \phi_n(0))_{n\in\mathbb{Z}} \to (I_n(t), \phi_n(t))_{n\in\mathbb{Z}} \to \psi(x,t).$$

Theorem 1. *i) The following estimates hold true:*

$$\frac{\pi}{6}\|I\|_2^2 \leqslant V(I) \leqslant \frac{2\pi}{3} e^{\|I\|_2}\|I\|_2^2.$$

ii) Let $\mathscr{A} = \{I \in \ell^1 : \sum_{n\in\mathbb{Z}} I_n \leqslant \frac{1}{8^2}\}$*. Then the function* $V : \mathscr{A} \to [0,\infty)$ *has the derivative* $\frac{\partial V}{\partial I_n}$ *for each* $n \in \mathbb{Z}$*, which is continuous on* $\mathscr{A}$ *and satisfies*

$$|V(I) - \|I\|_2^2| \leqslant 7\pi\|I\|_2^3,$$

$$\|\frac{\partial V(I)}{\partial I} - 2I\|_2 \leqslant 11\pi^2\|I\|_\infty\|I\|_2.$$

Remark. i) In order to study the perturbation of the dNLS the estimates the Hamiltonian $H(q)$ in terms of $\|I\|_2$ are important. This is the motivation of these estimates.

ii) $V(I)$ is a well defined function of action $I = (I_n)_{n\in\mathbb{Z}} \in \ell^2$.

Preliminaries 3. The dNLS equation is integrable and admits a Lax-pair formalism. Consider the corresponding self-adjoint Zakharov-Shabat operator T_{zs} acting in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and given by

$$T_{zs} = J\frac{d}{dx} + \mathcal{Q}, \qquad \mathcal{Q} = \begin{pmatrix} q_1 & q_2 \\ q_2 & -q_1 \end{pmatrix},$$

where $|q| \in L^2(0,1)$. The spectrum of T_{zs} is purely absolutely continuous and is union of spectral bands $\sigma_n, n \in \mathbb{Z}$, where

$$\sigma_n = [z_{n-1}^+, z_n^-], \ \cdots < z_{2n-1}^- \leqslant z_{2n-1}^+ < z_{2n}^- \leqslant z_{2n}^+ < \dots,$$

$$z_n^\pm = n\pi + o(1) \ \text{as}\, |n| \to \infty.$$

The intervals σ_n and σ_{n+1} are separated by gap $\gamma_n = (z_n^-, z_n^+)$ with the length $|\gamma_n| \geqslant 0$. If a gap γ_n is degenerate, i.e., $|\gamma_n| = 0$, then the corresponding segments σ_n, σ_{n+1} merge. We need the Zakharov-Shabat equation

$$Jf' + \mathcal{Q}f = zf, \quad z \in \mathbb{C}, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad (\,')=\frac{\partial}{\partial x}.$$

Here $z_{2n}^{\pm}, n \in \mathbb{Z}$ are the eigenvalues of the equation $Jf' + \mathcal{Q}f = zf$ with the periodic boundary condition $f(x) = f(x+1)$. And $z_{2n+1}^{\pm}, n \in \mathbb{Z}$ are the eigenvalues of the equation $Jf' + \mathcal{Q}f = zf$ with the anti-periodic boundary condition $f(x) = -f(x+1)$. Define the 2×2-matrix valued fundamental solution $\Psi = \Psi(x,z)$ by

$$J\frac{d}{dx}\Psi + \mathcal{Q}\Psi = z\Psi, \quad \Psi(0,z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{C}. \tag{0.4}$$

Introduce the Lyapunov function $\Delta(z)$ by

$$\Delta(z) = \frac{1}{2}\mathrm{Tr}\Psi(1,z), \quad z \in \mathbb{C}.$$

The function Δ is entire and $\Delta(z_n^{\pm}) = (-1)^n$ for all $n \in \mathbb{Z}$.

We recall results which is crucial for the present paper. For each $q \in L^2(\mathbb{T})$ there exists a unique conformal mapping (the quasimomentum) $k : \mathcal{Z} \to \mathcal{K}(h)$ (see Fig. 1 and 2) and such that

$$\cos k(z) = \Delta(z), \quad z \in \mathcal{Z} = \mathbb{C} \setminus \cup \overline{\gamma}_n, \quad \textit{and}$$

$$\mathcal{K}(h) = \mathbb{C} \setminus \cup \Gamma_n, \quad \Gamma_n = (\pi n - i h_n, \pi n + i h_n),$$

Here $h_n \geqslant 0$ are the Marchenko-Ostrovski height, defined by equations $\cosh h_n = |\Delta(z_n)| \geqslant 1$,

$$z_n \in [z_n^-, z_n^+], \quad \Delta'(z_n) = 0,$$

$$k(\sigma_n) = [\pi(n-1), \pi n], \qquad k([z_n^-, z_n^+]) = \Gamma_n,$$

$$k(z) = z + o(1) \quad as \qquad |z| \to \infty.$$

Here Γ_n is the vertical cut and $\gamma_n = [z_n^-, z_n^+]$ is the horizontal cut. We have

$$(h_n)_{n \in \mathbb{Z}} \in \ell^2 \Leftrightarrow q \in L^2(0,1)$$

If $|q'| \in L^2(0,1)$, then $k(\cdot)$ has asymptotics

$$k(z) = z - \frac{H_0}{2z} - \frac{H_1}{(2z)^2} - \frac{H_2 + o(1)}{(2z)^3} \quad as \quad z \to i\infty.$$

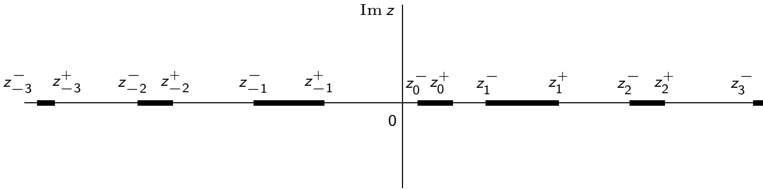

Figure: The domain $\mathcal{Z} = \mathbb{C} \setminus \cup \bar{\gamma}_n$, where $\gamma_n = (z_n^-, z_n^+)$

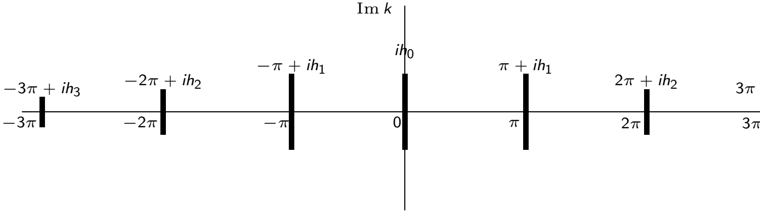

Figure: The domain $K(h) = \mathbb{C} \setminus \cup \Gamma_n$, where $\Gamma_n = (\pi n - ih_n, \pi n + ih_n)$

The Löwner type equation 4. Define the ball

$$\mathcal{B}(r) = \{\eta : \|\eta\|_2 \leqslant r\} \subset \ell^2, r > 0.$$

Let ℓ^p_C be the complexification of the space ℓ^p. In the complex space ℓ^2_C the corresponding ball is denoted by $\mathcal{B}_C(r) \subset \ell^2_C$.

Let $z(\cdot) = k^{-1} : \mathcal{K}(h) \to \mathcal{Z}$ be the inverse mapping for $k : \mathcal{Z} \to \mathcal{K}(h)$. Below we will sometimes write $z(k, h)$, instead of $z(k)$, when several h are being dealt with.

Recall the simple case about th Loewner equation.

Loewner equation, is an ordinary differential equation discovered by Charles Loewner in 1923 in complex analysis and geometric function theory. (Loewner, C. (1923), "Untersuchungen ber schlichte konforme Abbildungen des Einheitskreises, I", Math. Ann. 89: 103-121). Loewner's method was later developed in 1943 by the Russian mathematician Pavel Parfenevich Kufarev (1909-1968).

Loewner equation was used in the following cases:

1) De Brange proved the Bieberbach conjecture.

2) Smirnov, Stanislav (2001). "Critical percolation in the plane". Comptes Rendus de l'Acadmie des Sciences

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The classical case. Let $B \subset \mathbb{C}$ be "good" bounded domain and $B_t = B \setminus [z_0, z_0 + it]$ for $t \in [0, t_0]$, where $z_0 \in \partial B$ and the segment $(z_0, z_0 + it_0] \subset B$. Let $g(z,t)$ be a conformal mapping from the unit disc $\mathbb{D}$ onto the domain B_t such that

$$g(0,t) = 0, \quad g'_z(0,t) > 0.$$

Define the new function

$$f(z,t) = g^{-1}(g(z,0),t), \qquad z \in \mathbb{D}.$$

The function $f(z,t)$ is a conformal mapping from the unit disc $\mathbb{D}$ onto the disc $\mathbb{D}$ with the cut $\gamma(t)$, where $\gamma(t)$ is some curve. For each $|z| < 1$ the function $f(z,t)$ satisfies the **Löwner** equation

$$\frac{\partial f}{\partial t} = -f \frac{1 + kf}{1 - kf}$$

where $k = k(t), |k(t)| = 1$ is some continuous function and $f(0,t) = 0, \quad f'_z(0,t) > 0.$

Theorem 2. *There exist $\varepsilon > r > 0$ such that for any fixed real $h \in \ell^2$ the function $z(k, h+\eta)$ has the analytic extension from $(k, \eta) \in \mathcal{K}(h, \varepsilon) \times \mathcal{B}(r)$ into the domain $K(h, \varepsilon) \times \mathcal{B}_C(r)$, where*

$$\mathcal{K}(h, \varepsilon) = \{k \in \mathbb{C} : \quad \mathrm{dist}(k, \cup \Gamma_n) > \varepsilon\} \subset \mathcal{K}(h).$$

Moreover, for each $n \in \mathbb{Z}$ the derivatives are given by

$$\frac{\partial z(k,h)}{\partial h_n} = \frac{\nu_n}{z(k,h) - z_n(h)}, \quad h_n \neq 0, \qquad k \in \mathcal{K}(h),$$

$k \neq \pi n \pm i h_n$ *(it is* **the Löwner type equation***), where*

$$\nu_n(h) = (-1)^{n-1} \frac{\sinh h_n}{\Delta''(z_n)}, \qquad \text{all} \quad n \in \mathbb{Z}.$$

Remark. 1) In the classical case we have only one cut. In our case we have infinitely many cuts.

2) In our case the point of the normalisation is the infinity, since $z(iv, h) = iv - (Q_0 + o(1))/iv$ as $v \to \infty$ i.e. this point belongs to boundary of our domain. Remark that in the classical case this point lies inside the domain.

We show applications. Due to Flashka, McLaughlin we define the actions I_n by

$$I_n = \frac{2}{\pi} \int_{\gamma_n} v(z + i0) dz \geqslant 0$$

since $v(z + i0) > 0$ on the open gap $\gamma_n \neq \emptyset$.

Theorem 3. *i) The mapping given by*

$$I^0 = (I_n^0)_{n\in\mathbb{Z}} \to I = (I_n)_{n\in\mathbb{Z}}, \qquad I_n^0 = h_n^2,$$

is a real analytic isomorphism of
$\ell^1_+ = \{f = (f_n)_{n\in\mathbb{Z}} \in \ell^1 : f_n \geqslant 0, \forall\ n\}$ *onto itself and*

$$\left\| \left(\frac{\partial I}{\partial I^0}\right)^{-1} \right\| \leqslant e^{\|h\|_\infty}.$$

ii) The derivatives $\frac{\partial I_n}{\partial I_j^0}$ *have the following forms:*

$$\frac{\partial I_n}{\partial I_j^0} = -\frac{2\widetilde{\nu}_j}{\pi}\int_{\gamma_n} \frac{v(z)dz}{(z-z_j)^2}, \quad j \neq n,$$

$$\frac{\partial I_n}{\partial I_n^0} = 2\nu_n + \frac{2\nu_n}{\pi}\int_{\gamma\setminus\gamma_n} \frac{v(z)dz}{(z-z_n)^2}, \qquad \gamma = \cup\gamma_s, \quad k = u + iv,$$

$$\widetilde{\nu}_j = (-1)^{n-1}\frac{\sinh h_n}{2h_n\Delta''(z_n)}.$$

Remarks: Define the operator $I' : \ell^1 \to \ell^1$ by

$$I' = \{I'_{n,j}\}, \qquad (I'f)_n = \sum_{j\in\mathbb{Z}} I'_{n,j} f_j, \quad f = (f_j)_{j\in\mathbb{Z}}, \qquad I'_{n,j} = \frac{\partial I_n}{\partial I_j^0}.$$

We have the identity

$$\Omega_j^0 = \frac{\partial H}{\partial I_j^0} = \sum_{n\in\mathbb{Z}} \frac{\partial H}{\partial I_n^0} \frac{\partial I_n}{\partial I_j^0},$$

or in the vector form

$$\Omega^0 = (\Omega_n^0)_{n\in\mathbb{Z}} = \frac{\partial H}{\partial I^0} = I'^{\top}\Omega, \qquad \Omega = \frac{\partial H}{\partial I}$$

Theorem 3. *The following identities hold true:*

$$(I'^{\top})^{-1}\Omega^0 = \Omega, \qquad \Omega_n^0 = \frac{\partial H}{\partial I_n^0} = 2\widetilde{\nu}_n(H_0 + 2z_n^2).$$