Non-iterative Reconstruction Schemes for Active Thermography

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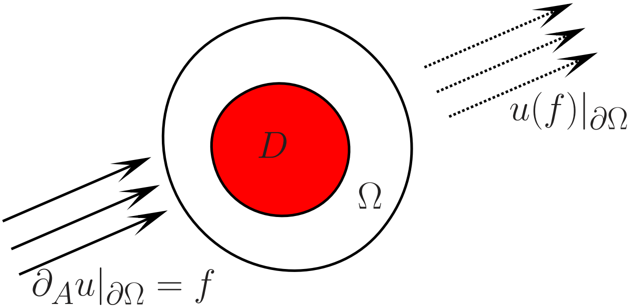
Outline of my talk

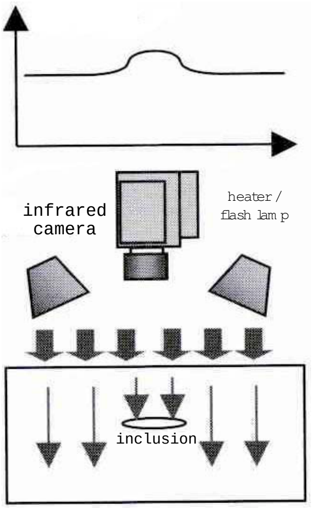
1. **Active Thermography and Its Mathematical Formulation**
   - Active thermography
   - Forward and inverse problems
2. **Dynamical probe method**
   - Runge's approximation, pre-indicator function and reflected solution
   - Identifying seperated inclusions and isotropic conductivity
3. **Linear sampling type method**
4. **Future work**

Active Thermography and Its Mathematical Formulation

Active thermography



Principle of active thermography



Mixed problem (set up)

$\Omega \subset \mathbb{R}^n$ $(1 \leq n \leq 3)$: bounded domain (heat conductor),

$$\partial\Omega \, : \, C^2 \, (n = 2, 3), \quad \partial\Omega = \overline{\Gamma^D} \cup \overline{\Gamma^N},$$

where Γ^D, Γ^N are open subsets of $\partial\Omega$ such that $\Gamma^D \cap \Gamma^N = \emptyset$ and $\partial\Gamma^D$, $\partial\Gamma^N$ are C^2 if they are nonempty.

$D \subset \Omega$: open set (**(separated) inclusion(s)**), $\overline{D} \subset \Omega$,
$\partial D : C^{1,\alpha}$ $(0 < \alpha \leq 1)$, $\Omega \setminus \overline{D}$: connected.

Heat conductivity:

$$\gamma(x) = A(x) + (\tilde{A}(x) - A(x))\chi_D \, : \text{positive definite for each } x \in \overline{\Omega},$$

where A, $\tilde{A} \in C^1(\overline{\Omega})$ are positive definite and $\tilde{A} - A$ is always positive definite or negative in a neigh. of ∂D, χ_D is the char func of D.

Sobolev spaces

Let $X \subset \mathbb{R}^n$ be a bounded domain and ∂X be its boundary.

anisotropic Sobolev spaces :

$H^{p,q}(\mathbb{R}^n \times \mathbb{R}) := \{u \,:\, (1+|\xi|^2)^{p/2}\hat{u},\ (1+|\tau|^2)^{q/2}\hat{u} \in L^2(\mathbb{R}^{n+1})\,\}$
if $p,\, q \geq 0$, where $\hat{u}$ is the Fourier transform of u.

$H^{p,q}(\mathbb{R}^n \times \mathbb{R}) := \Big(H^{-p,-q}(\mathbb{R}^n \times \mathbb{R})\Big)'$ (duality)
if $p,\, q \leq 0$.

$H^{p,q}(X_T) :=$ restriction of $H^{p,q}(\mathbb{R}^n \times \mathbb{R})$ to $X_T := X \times (0,T)$.

$\tilde{H}^{1,1/2}(X_T) := \{u \in H^{1,1/2}(X \times (-\infty, T)) \,:\, u(x,t) = 0 \ (t<0)\}$.

$H^{p,q}((\partial X)_T)$ is defined in a similar way.

$L^2((0,T);E) :=$ set of Hilbert space $E-$valued L^2 functions over $(0,T)$.

Mixed problem (forward problem)

Given $f \in L^2((0,T); \overline{H}^{\frac{1}{2}}(\Gamma^D))$, $g \in L^2((0,T); \dot{H}^{-\frac{1}{2}}(\overline{\Gamma^N}))$ (input),

$(?)\,\exists!$ weak solution
$u = u(f,g) \in W(\Omega_T) := \{u \in H^{1,0}(\Omega_T),\ \partial_t u \in L^2((0,T); H^1(\Omega)^*)\}$:

$$\begin{cases} \mathcal{P}_D u(x,t) := \partial_t u(x,t) - \mathrm{div}_x(\gamma(x)\nabla_x u(x,t)) = 0 \text{ in } \ \Omega_T \\ \\ u(x,t) = f(x,t) \text{ on } \Gamma_T^D,\ \ \partial_A u(x,t) := \nu \cdot A\nabla u(x,t) = g(x,t) \text{ on } \Gamma_T^N \\ \\ u(x,0) = 0 \text{ for } x \in \Omega, \end{cases}$$

where ν is the outer unit normal of $\partial\Omega$,
$\overline{H}^{\frac{1}{2}}(\Gamma^D)$, $\dot{H}^{-\frac{1}{2}}(\overline{\Gamma^N})$ are Hörmander's notations of Sobolev sp,

$\Omega_T = \Omega_{(0,T)} := \Omega \times (0,T)$, $\partial\Omega_T = \partial\Omega_{(0,T)} := \partial\Omega \times (0,T)$.
(cylindrical sets)

This is a well-posed problem.

Measured data

Neumann-to-Dirichlet map Λ_D:

For fixed $f \in L^2((0,T);\overline{H}^{\frac{1}{2}}(\Gamma^D))$, define

$$\Lambda_D : L^2((0,T);\dot{H}^{-\frac{1}{2}}(\overline{\Gamma^N})) \to L^2((0,T);\overline{H}^{\frac{1}{2}}(\Gamma^N))$$
$$g \mapsto u(f,g)|_{\Gamma^N_T}.$$

Inverse boundary value problem

Reconstruct the unknown inclusion(s) D and $\tilde{A}\Big|_{\partial D}$ from Λ_D.

Known results I

$*$ H. Bellout (1992): Local uniqueness and stability.

$*$ A. Elayyan and V. Isakov (1997): Global uniqueness using the localized *Neumann-to-Dirichlet map* even for time dependent inclusions.

$*$ M. Di Cristo and S. Vessella (2010): Stability estimate (i.e. log type stability estimate) even for time dependent inclusions.

$*$ Y. Daido, H. Kang and G. Nakamura (2007) (Inverse Problems) : Introduced the dynamical probing method for 1-D case.

$*$ Y. Daido, Y. Lei, J. Liu and G. Nakamura (2009) (Applied Mathematics and Computation) Numerical implementations of 1-D dynamical probe method for non-stationary heat equation.

Known results II

∗ Y. Lei, K. Kim and G. Nakamura (2009) (Journal of Computational Mathematics) Theoretical and numerical studies for 2-D dynamical probe method.

∗ V. Isakov, K. Kim and G. Nakamura (2010) (Ann. Scoula Superior di Pisa) Gave the theoretical basis of dynamical probe method.

∗ K. Kim and G. Nakamura (2011) Inverse boundary value problem for anisotropic heat operators.

∗ M. Ikehata (2007) Extracting discontinuity in a heat conductive body: one-space-dimensional case.

∗ M. Ikehata (2007) Two analytical formulae of the temperature inside a body by using partial lateral and initial data.

Known results III

∗ M.Ikehata and M. Kawashita (2009) The enclosure method for the heat equation.

∗ M. Ikehata and M. Kawashita (2010) On the reconstruction of inclusions in a heat conductive body from dynamical boundary data over a finite time interval.

∗ H. Isozaki, P. Gaitan, O. Poisson and S. Siltanen (2011) Gave the enclosure method for many boundary measurements (isotropic case).

∗ G. Nakamura and S. Sasayama(2013) Reconstructed the conductivities of inclusions at their boundary (isotropic case).

∗ G. Nakamura and H. Wang (2013) Gave a linear sampling type method for many measurements (isotropic case).

Objectives of this talk

(i) Introduce two reconstruction schemes called dynamical probe method (DP method) and linear sampling type method (LS method).

(ii) Show some improvement on the DP method.

(iii) The DP method is good at probing D from its outside and LS method is good at probing D from its inside. By combining these two methods, I will propose a sampling type reconstruction scheme.

Dynamical Probe Method

Dynamical probe method (fundamental solutions)

For $(y, s), (y', s') \in \mathbb{R}^n \times \mathbb{R}$, $(x, t) \in \Omega_T$,

$\Gamma(x, t; y, s)$: fundamental solution of $\mathcal{P}_\emptyset := \partial_t - \nabla \cdot (A(x)\nabla)$

$\Gamma^*(x, t; y', s')$: fundamental solution of $\mathcal{P}^*_\emptyset := -\partial_t - \nabla \cdot (A(x)\nabla)$

$G(x, t; y, s)$, $G^*(x, t; y', s')$:

$$\begin{cases} \mathcal{P}_\emptyset G(x, t; y, s) = \delta(x - y)\delta(t - s) \text{ in } \Omega_T, \\ G(\cdot, \cdot; y, s) = 0 \text{ on } \Gamma^D_T, \\ G(x, t; y, s) = 0 \text{ for } x \in \Omega,\ t \le s \end{cases}$$

$$\begin{cases} \mathcal{P}^*_\emptyset G^*(x, t; y', s') = \delta(x - y)\delta(t - s') \text{ in } \Omega_T, \\ G^*(\cdot, \cdot; y', s') = 0 \text{ on } \Gamma^D_T, \\ G^*(x, t; y', s') = 0 \text{ for } x \in \Omega,\ t \ge s' \end{cases}$$

$G(x, t; y, s) - \Gamma(x, t; y, s),\ G^*(x, t; y', s') - \Gamma^*(x, t; y', s')$: C^1_t, C^2_x in Ω_T.

Dynamical probe method (Runge's approximation)

$\exists\ \{v^{0j}_{(y,s)}\},\ \{\psi^{0j}_{(y',s')}\} \in H^{2,1}(\Omega_{(-\varepsilon,T+\varepsilon)})$ for $\forall\varepsilon > 0$ s.t.

$$\begin{cases} \mathcal{P}_\emptyset v^{0j}_{(y,s)} = 0 & \text{in } \Omega_{(-\varepsilon,T+\varepsilon)}, \\ v^{0j}_{(y,s)} = 0 & \text{on } \Gamma^D \times (-\varepsilon, T+\varepsilon), \\ v^{0j}_{(y,s)}(x,t) = 0 & \text{if } -\varepsilon < t \le 0, \\ v^{0j}_{(y,s)} \to G(\cdot,\cdot;y,s) & \text{in } H^{2,1}(U \times (-\varepsilon', T+\varepsilon')) \text{ as } j \to \infty, \end{cases}$$

$$\begin{cases} \mathcal{P}^*_\emptyset \psi^{0j}_{(y',s')} = 0 & \text{in } \Omega_{(-\varepsilon,T+\varepsilon)}, \\ \psi^{0j}_{(y',s')} = 0 & \text{on } \Gamma^D \times (-\varepsilon, T+\varepsilon), \\ \psi^{0j}_{(y',s')}(x,t) = 0 & \text{if } T \le t < T+\varepsilon, \\ \psi^{0j}_{(y',s')} \to G^*(\cdot,\cdot;y',s') & \text{in } H^{2,1}(U \times (-\varepsilon', T+\varepsilon')) \text{ as } j \to \infty \end{cases}$$

for $0 < \forall\varepsilon' < \varepsilon$, $\forall U \subset \Omega$: open s.t.

$\overline{U} \subset \Omega$, $\Omega \setminus \overline{U}$: connected, ∂U : Lipschitz, $\overline{U} \not\ni y,\ y'$, and $0 < s,\ s' < T$.

Dynamical probe method (Runge approx funcs)

Let v, ψ satisfy

$$\begin{cases} \mathcal{P}_\emptyset v = 0 \text{ in } \Omega_T, \\ v = f \text{ on } \Gamma_T^D, \\ \partial_A v = 0 \text{ on } \Gamma_T^N, \\ v(x,0) = 0 \text{ for } x \in \Omega, \end{cases} \qquad \begin{cases} \mathcal{P}_\emptyset^* \psi = 0 \text{ in } \Omega_T, \\ \psi = 0 \text{ on } \Gamma_T^D, \\ \partial_A \psi = (\text{different fix})\ g \text{ on } \Gamma_T^N, \\ \psi(x,T) = 0 \text{ for } x \in \Omega. \end{cases}$$

For $j = 1, 2, \cdots$, we define

$$\begin{cases} v_{(y,s)}^j := v + v_{(y,s)}^{0j} \to V_{(y,s)} := v + G(\cdot,\cdot;y,s) \\ \psi_{(y',s')}^j := \psi + \psi_{(y',s')}^{0j} \to \Psi_{(y',s')} := \psi + G^*(\cdot,\cdot;y',s'). \end{cases}$$

in $H^{2,1}(U_T)$ as $j \to \infty$.

$\{v_{(y,s)}^j\}, \{\psi_{(y',s')}^j\}$: *Runge's approximation functions*

Pre-indicator function

Definition 1

$(y,s),(y',s') \in \Omega_T$

$\{v^j_{(y,s)}\},\{\psi^j_{(y',s')}\} \subset W(\Omega_T)$: *Runge's approximation functions*

Pre-indicator function :

$$I(y',s';y,s) = \lim_{j\to\infty}\int_{\Gamma^N_T}\left[\partial_A v^j_{(y,s)}|_{\Gamma^N_T}\ \psi^j_{(y',s')}|_{\Gamma^N_T} - \Lambda_D(\partial_A v^j_{(y,s)})|_{\Gamma^N_T}\ \partial_A\psi^j_{(y's')}|_{\Gamma^N_T}\right]$$

whenever the limit exists.

Reflected solution

Lemma 2

$y \notin \overline{D}$, $0 < s < T$, $\{v^j_{(y,s)}\} \subset W(\Omega_T)$: *Runge's approximation functions,*

$u^j_{(y,s)} := u(f, \partial_A v^j_{(y,s)}|_{\Gamma^N_T})$, $w^j_{(y,s)} := u^j_{(y,s)} - v^j_{(y,s)}$

Then, $w^j_{(y,s)}$ *has a limit* $w_{(y,s)} \in W(\Omega_T)$ *satisfying*

$$\begin{cases} \mathcal{P}_D w_{(y,s)} = \mathrm{div}_x((\tilde{A} - A)\chi_D \nabla_x V_{(y,s)}) \text{ in } \ \Omega_T, \\ w_{(y,s)} = 0 \text{ on } \Gamma^D_T, \ \partial_A w_{(y,s)} = 0 \text{ on } \ \Gamma^N_T \\ w_{(y,s)}(x,0) = 0 \text{ for } \ x \in \Omega. \end{cases}$$

$w_{(y,s)}$: reflected solution

Representation formula

Theorem 3

For $y, y' \notin \overline{D}$, $0 < s,\ s' < T$ such that $(y, s) \neq (y', s')$, the pre-indicator function $I(y', s'; y, s)$ has the representation formula in terms of the reflected solution $w_{(y,s)}$:

$$I(y', s'; y, s) = -w_{(y,s)}(y', s') - \int_{\partial\Omega_T} w_{(y,s)} \partial_A \Psi_{(y',s')} d\sigma dt$$

indicator function

Definition 4

$C := \{c(\lambda)\,;\, 0 \le \lambda \le 1\}$ *: non-selfintersecting* C^0 *curve in* $\overline{\Omega}$*,*

$c(0), c(1) \in \partial\Omega$ *(We call this* C *a needle.)*

Then, for each $c(\lambda) \in \Omega$ *and each fixed* $s \in (0, T)$*,*

indicator function (mathematical testing machine)

$$J(c(\lambda), s) := \lim_{\epsilon \downarrow 0} \limsup_{\delta \downarrow 0} |I(c(\lambda - \delta), s + \epsilon^2; c(\lambda - \delta), s)|$$

whenever the limit exists.

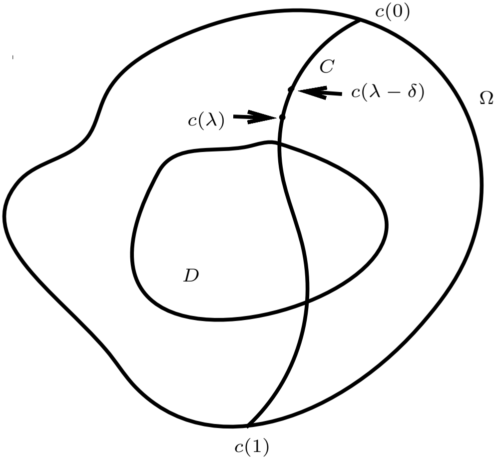


Figure 1 : Domains Ω, D, and a curve C

Seperated inclusions case result (theorem)

Theorem 5

Let D consist of separated inclusions, and C, $c(\lambda)$ be as in the definition above. Fix $s \in (0, T)$.

(i) $C \subset \Omega \setminus \overline{D}$ *except* $c(0)$ *and* $c(1)$

$\Longrightarrow J(c(\lambda), s) < \infty$ for all λ, $0 \le \lambda \le 1$

(ii) $C \cap \overline{D} \neq \emptyset$

λ_s $(0 < \lambda_s < 1)$ *s.t.* $c(\lambda_s) \in \partial D$, $c(\lambda) \in \Omega \setminus \overline{D}$ $(0 < \lambda < \lambda_s)$

$\Longrightarrow$

$$\lambda_s = \sup\{\, 0 < \lambda < 1\,;\, J(c(\lambda'), s) < \infty \quad \text{for any} \quad 0 < \lambda' < \lambda \,\}.$$

Remarks :

(i) A numerical realization of this reconstruction method has been done for isotropic conductivities.

(ii) If $\Gamma^D \neq \emptyset$, $u(f,g)$ decays exponentially after a time from which there is not any input. Hence, in this case, we can repeat many measurements. in a short time.

(iii) What is the advantage of the freedom to choose $s\,(0 < s < T)$?

Can parameterize $c(\lambda)$ by s.

(iv) All other arguments are OK for non-separated inclusions except the behavior of reflected solution.

(v) What about the case if there are inner boundaries or buried inclusions in D ?

(vi)Overshooting $c(\lambda_0)$ may happen in numerical implementation.

Identifying isotropic conductivity

Let the conductivity γ be isotropic and piecewise homogeneous:

$$\gamma = 1 + (k-1)\chi_D \tag{1}$$

with $0 < k \neq 1\,(\mathrm{constant})$.

Theorem 1

By explicitly computing the asymptotic behavior of the limit of pre-indicator function $I(c(\lambda_0), s+\varepsilon^2; c(\lambda_0), s)$, we can recover k. Here $c(\lambda_0)$ is the first touching point of needle C to ∂D.

Remark We note that the result is from the short time asymptotic of the reflected solution.

Improvement of the identification results

Let $k > 1$ for example. By directly considering the pre-indicator function $I(\varepsilon) = I(y, s + \varepsilon^2, y, s)$ with $\varepsilon^{1-\gamma} = \mathrm{dist}\,(y, \partial D)$ for any fixed $\gamma\,(0 < \gamma < 1/10)$,

$$I(\varepsilon) = -\frac{1}{8k(k-1)\pi}\varepsilon^{-3+3\gamma} + O(\varepsilon^{-3+5\gamma})\ (\varepsilon \to 0).$$

From this, by looking at the asymptotic behavior as y tends to ∂D, we can know the distance to ∂D and k.

Linear Sampling Type Method

Linear sampling type method

Let $\Gamma^D = \emptyset$, and for the conductivity $\gamma = A_0 + (\tilde{A} - A_0)\chi_D$, $A_0 = I$ and for example $\tilde{A} > I$ on $\overline{\Omega}$.

For $g \in H^{1/2,1/4}(\partial\Omega_T)$, let $v = v^g \in \tilde{H}^{1,1/2}(\Omega_T)$ be the solution to

$$\begin{cases} \partial_t v - \Delta v = 0 \quad \text{in } \Omega_T, \\ v = g \quad \text{on } \partial\Omega_T, \\ v = 0 \quad \text{at } t = 0. \end{cases} \tag{2}$$

Define

$S_D \, : H^{1/2,1/4}(\partial\Omega_T) \to \tilde{H}^{1,1/2}(D_T) \; g \mapsto v^g\Big|_{D_T}$,

$L_\Omega \, : H^{1/2,1/4}(\partial\Omega_T) \to H^{-1/2,-1/4}(\partial\Omega_T) \; g \mapsto \partial_\nu v^g$.

$G_{(y,s)}$: Green function of the heat equation satisfying Neumann boundary condition on $\partial\Omega_T$.

By using layer potentials, we have the following.

Theorem 2 (linear sampling type theorem) *Let* $s \in (0, T)$.
(i) *Assume that* $y \in D$. *Then for any* $\varepsilon > 0$, *there exists* g_ε^y *such that*

$$\|(\Lambda_D - \Lambda_\emptyset)L_\Omega g_\varepsilon^y - G_{(y,s)}\|_{H^{1/2,1/4}(\partial\Omega_T)} < \varepsilon \tag{3}$$

and is locally bounded in $\varepsilon > 0$. *Furthermore, we have*

$$\|S_D g_\varepsilon^y\|_{\tilde{H}^{1,1/2}(D_T)},\ \|g_\varepsilon^y\|_{H^{1/2,1/4}(\partial\Omega_T)} \to \infty \quad \text{as } y \to \partial D. \tag{4}$$

(ii) *Assume that* $y \in \Omega \setminus D$. *Then for any* $\varepsilon > 0$ *and* $\delta > 0$, *there exists* $g_{\varepsilon,\delta}^y \in H^{1/2,1/4}(\partial\Omega_T)$ *such that*

$$\|(\Lambda_D - \Lambda_\emptyset)L_\Omega g_{\varepsilon,\delta}^y - G_{(y,s)}\|_{H^{1/2,1/4}(\partial\Omega_T)} < \varepsilon + \delta, \tag{5}$$

$$\|S_D g_{\varepsilon,\delta}^y\|_{\hat{H}^{1,1/2}(D_T)},\ \|g_{\varepsilon,\delta}^y\|_{H^{1/2,1/4}(\partial\Omega_T)} \to \infty \quad \text{as } \delta \to 0. \tag{6}$$

Future Work

Sampling type method

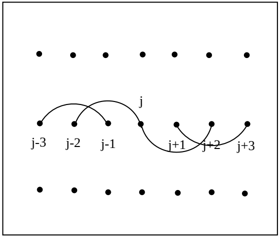


Figure 2 : Sampling type method

Let $y_j \in \Omega$, $s_j \in (0,T)$, $s_j \uparrow (j \uparrow)$. For each (y_j, s_j), find

$$g^{y_j,s_j} \in H^{1/2,1/4}(\partial\Omega_T) \,:\, (\Lambda_D - \Lambda_\emptyset)L_\Omega g^{y_j,s_j} \approx G_{(y_j,s_j)}$$

and compute

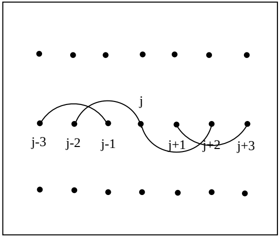


Figure 3 : Sampling type method

$$I_{\mathsf{LSM}}(y_j) := \|g^{y_j, s_j}\|_{H^{1/2,1/4}(\partial\Omega_T)},$$

$$I_{\mathsf{DP}}(y_j) := |I(y_j, s_j + \epsilon^2, y_j, s_j)|.$$

Note that $G_{(y_j,s_j)}$ can be used instead of $V_{(y_j,s_j)}$ for $I(y_j, s_j+\epsilon^2, y_j, s_j)$ and the whole $G_{(y_j,s_j)}$ can be put used as one set of input data over $(0,T)$.

Then, for each $y_j \in \Omega$, define

$$I(y_j) := \min\{\min_{1\le k\le 3} I_{\textsf{LSM}}\,(y_{j+(k-1)}),\, \min_{1\le k\le 3} I_{\textsf{DP}}\,(y_{j-(k-1)})\}.$$

This $I(y_j)$ can be used to sample $y_j \sim \partial D$ or not.

Thank you for your attention.