Inverse problem for the wave equation with Dirichlet data and Neumann data on disjoint sets

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joint work with Y. Kurylev and M. Lassas

Hyperbolic inverse problem with restricted Cauchy data

We consider the wave equation

$$\begin{aligned}
&\partial_t^2 u - (\Delta_g + A)u = 0 \quad \text{in } (0,\infty)\times M,\\
&u|_{x\in\partial M} = f,\\
&u|_{t=0} = 0,\ \partial_t u|_{t=0} = 0,
\end{aligned}$$

where A is a first order differential operator on a compact Riemannian manifold with boundary (M,g). "Everything is smooth."

Let $\mathcal{S},\mathcal{R}\subset\partial M$ be open and non-empty.
Inverse problem. Does the restricted Cauchy data set

$$\mathcal{C}^T_{\mathcal{S},\mathcal{R}} = \{(f,\partial_\nu u|_{(0,T)\times\mathcal{R}});\ f\in C_0^\infty((0,T)\times\mathcal{S})\}$$

determine the geometry (M,g) and the lower order terms A (up to the natural obstructions)?

Local reconstruction of lower order terms

Theorem [Kurylev, Lassas and L.O.]. Suppose that

- $\mathcal{S} \subset \partial M$ and $T > 0$ satisfy the condition: "any billiard trajectory (reflecting from $\partial M \setminus \mathcal{S}$) exits M through $\mathcal{S}$ before time T",
- $\mathcal{R} \subset \partial M$ is strictly convex.

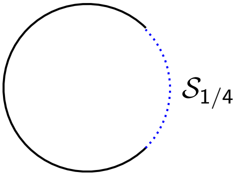
Then there is a neighborhood U of $\mathcal{R}$ such that $\mathcal{C}_{\mathcal{S},\mathcal{R}}^{2T}$ determines (U, g) up to the diffeomorphism invariance, and the operator $\Delta_g + A$ on U up to the gauge invariance

$$\Delta_g + A \mapsto \kappa(\Delta_g + A)\kappa^{-1}, \tag{1}$$

where $\kappa \in C^\infty(U)$ is nowhere vanishing and satisfies $\kappa|_{\mathcal{R}} = 1$.

- The Cauchy data $\mathcal{C}_{\mathcal{S},\mathcal{R}}^{2T}$ does not change under the gauge (1).
- If the convexity assumption could be removed, then the local reconstruction method could be iterated to give a global result.

The billiard condition on the disk



Let (M, g) be the Euclidean disk $\{z \in \mathbb{C};\ |z| \leq 1\}$ and define

$$\mathcal{S}_\alpha := \{e^{i2\pi\theta};\ \theta \in (0, \alpha)\}.$$

- If $\alpha > 1/2$ then the billiard condition holds (for large enough $T > 0$).
- If $\alpha < 1/2$ then the billiard condition does **not** hold.

Brief summary of earlier results with restricted data

Let us assume that $T > 0$ is "large enough".

- If $A = 0$ and $\mathcal{S}$ satisfies the billiard condition, then $\mathcal{C}^{2T}_{\mathcal{S},\mathcal{R}}$ determines (M, g) [LASSAS AND L.O.'14]. (Non-empty open $\mathcal{R}$ can be arbitrary.)
- If $\mathcal{S} = \mathcal{R}$ and it satisfies the billiard condition, then $\mathcal{C}^{2T}_{\mathcal{S},\mathcal{S}}$ determines (M, g) and A [KURYLEV-LASSAS'99].
- If $\mathcal{S} = \mathcal{R}$ and $A = 0$, then $\mathcal{C}^{2T}_{\mathcal{S},\mathcal{S}}$ determines (M, g) [KATCHALOV-KURYLEV'98].

The convexity assumption on $\mathcal{R}$ is **not** needed here and the results are global.

[ESKIN'07] allows analytic time dependence in A:
if $\mathcal{S} = \mathcal{R}$ and it satisfies the billiard condition, then $\mathcal{C}^{2T}_{\mathcal{S},\mathcal{S}}$ determines (M, g) and A.

The Hassell-Tao spectral condition

In [Hassell-Tao'10] it is shown that all *non-trapping* Riemannian manifolds (M, g) satisfy the spectral condition,

$$\lambda_j \leq C \, \|\partial_\nu \phi_j\|^2_{L^2(\partial M)}, \quad j = 1, 2, \dots,$$

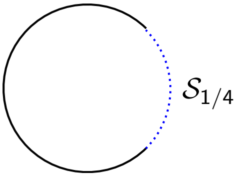
where λ_j are the Dirichlet eigenvalues of $-\Delta_g$ and ϕ_j are the corresponding $L^2(M)$ normalized eigenfunctions.

- If $\mathcal{S} \subset \partial M$ satisfies the billiard condition (for some $T > 0$) then

$$\lambda_j \leq C \, \|\partial_\nu \phi_j\|^2_{L^2(\mathcal{S})}, \quad j = 1, 2, \dots. \tag{2}$$

- If $A = 0$ and $\mathcal{S}$ satisfies (2), then $\mathcal{C}^{2T}_{\mathcal{S},\mathcal{R}}$ determines (M, g) [Lassas and L.O.'14].

Example: the disk again



Let (M, g) be the Euclidean disk $\{z \in \mathbb{C};\ |z| \leq 1\}$ and define

$$\mathcal{S}_\alpha := \{e^{i2\pi\theta};\ \theta \in (0, \alpha)\}.$$

- If $\alpha > 1/2$ then the billiard condition holds (for large enough $T > 0$).
- If $\alpha < 1/2$ then the billiard condition does **not** hold.
- The spectral condition holds for all $\alpha > 0$.

Open questions

Theorem [Kurylev, Lassas and L.O.]. Suppose that

- $\mathcal{S} \subset \partial M$ and $T > 0$ satisfy the billiard condition,
- $\mathcal{R} \subset \partial M$ is strictly convex.

Then there is a neighborhood U of $\mathcal{R}$ such that $\mathcal{C}_{\mathcal{S},\mathcal{R}}^{2T}$ determines (U, g) as a Riemannian manifold, and the operator $\Delta_g + A$ on U up to the gauge invariance.

- Is the convexity assumption necessary?
- Can the billiard assumption be replaced by a spectral assumption? or removed altogether? The most natural context for this question is when $\mathcal{R} = \mathcal{S}$ and convexity is not assumed.

Idea of the proof that the lower order terms are determined

We suppose that (M, g) is known and focus on reconstruction of A. We use a version of the Boundary Control method [Belishev'87].

- A Blagoveščenskiĭ type identity implies that the inner products

$$(u^f(T), v^\psi(T))_{L^2(M)}, \quad f \in C_0^\infty((0,T) \times \mathcal{S}),\ \psi \in C_0^\infty((0,T) \times \mathcal{R}),$$

  are determined by the Cauchy data set. Here u^f is the solution with the source f and v^ψ is the solution of the adjoint wave equation with the source ψ.
- We will use the above inner products to enforce $u^f(T)$ to have small support near a point in a neighborhood of $\mathcal{R}$.
- The billiard condition implies that the set

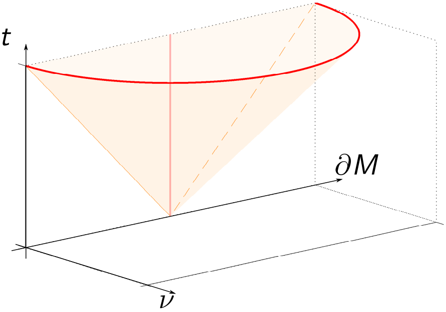
$$\{u^f(T);\ f \in L^2((0,T) \times \mathcal{S})\}$$

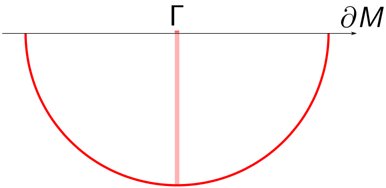
  is the whole $L^2(M)$ [Bardos-Lebeau-Rauch'92]. In particular, there exist f's such that $u^f(T)$ has a small support.

Domains of influence

Let $\Gamma \subset \mathcal{R}$ be open and $r > 0$. We define the domain of influece

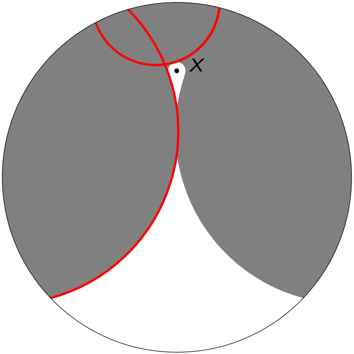
$$M(\Gamma, r) = \{x \in M;\ d(x, \Gamma) \le r\}.$$





Hyperbolic unique continuation [Tataru'95] implies that the inclusion $\{v^\psi(T);\ \psi \in C_0^\infty((T - r, T) \times \Gamma)\} \subset L^2(M(\Gamma, r))$ is dense.

Controlling the support of $u^f(T)$



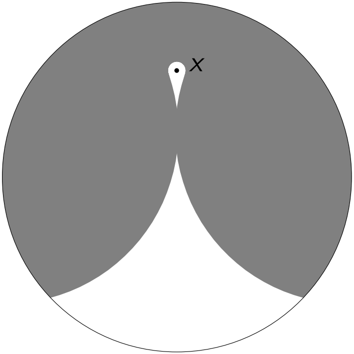
Let $f \in L^2((0,T) \times \mathcal{S})$. If

$$(u^f(T), v^\psi(T))_{L^2(M)} = 0, \quad \psi \in C_0^\infty((T-r, T) \times \Gamma),$$

then $u^f(T) = 0$ in $M(\Gamma, r)$. Let $x \in M$ be close to $\mathcal{R}$. We can enforce

$$u^f(T) = 0, \quad \text{in } \bigcup_{y \in \mathcal{R}} M(y, d(x,y) - \epsilon).$$

The gauge invariance



Taking $\epsilon \to 0$ in a careful way, we can enforce $u^f(T) \to \kappa \delta_x$ near x. We can also enforce $\kappa \in C^\infty(U)$, $\kappa|_{\mathcal{R}} = 1$ and $\kappa \neq 0$ in U. Here $U \subset M$ is a neighborhood of $\mathcal{R}$. Now $(u^f(T), v^\psi(T))_{L^2(M)}$ converges to $w = \kappa v^\psi$ at $t = T$ and

$$\partial_t^2 w + \kappa(\Delta_g + A^*)\kappa^{-1} w = 0.$$

About the billiard condition

- If the billiard condition does not hold, the set

$$\mathbb{D} := \{u^f(T);\ f \in L^2((0,T) \times \mathcal{S})\}$$

  is **only** dense in $L^2(M)$.
  - Density for large enough $T > 0$ follows from unique continuation [Tataru'95] by a duality argument.
  - There is $\phi \in L^2(M)$ such that the convergene $u^{f_j}(T) \to \phi$ implies that $(f_j)_{j=1}^\infty \subset L^2((0,T) \times \mathcal{S})$ is unbounded.
- Boundedness of the inner products $(u^f(T), h_j)_{L^2(M)}$ does **not** imply that the sequence $(h_j)_{j=1}^\infty \subset L^2(M)$ is bounded.
  - Adjoint of $f \mapsto u^f(T)$ is given by $Wh|_{(0,T)\times\mathcal{S}}$ where $Wh = w$ satisfies

$$\partial_t^2 w - (\Delta_g + A^*) w = 0 \quad \text{in } (0,T) \times M,$$
$$w|_{x\in\partial M} = 0, \quad w|_{t=T} = 0,\ \partial_t w|_{t=T} = h,$$

  - There is an unbounded sequence $(h_j)_{j=1}^\infty$ such that Wh_j concentrates on a billiard trajectory. In particular, if the trajectory does not intersect $\mathcal{S}$ before T, then $Wh_j|_{(0,T)\times\mathcal{S}} \to 0$.