On the Borg-Levinson type theorems

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The starting point is the following result of Ambartsumyan (1929) :
If $q(x) \in L^{\infty}(0,1)$ be real-valued and the spectrum of the boundary value problem (with appropriate normalization)

$$-y''(x) + q(x)y(x) = \lambda_k y(x), \quad y'(0) = 0, y'(1) = 0$$

is equal to

$$\lambda_k = (\pi k)^2, \quad k = 0, 1, ...$$

then $q(x) = 0$ a.e. on the interval (0,1). Hence, the Neumann spectrum uniquely determines the potential.
Borg in 1946 proved that a single spectrum in general does not suffice to determine potential uniquely, and therefore the result of Ambartsumyan is an exception.
A positive result can be provided by the celebrating Borg-Levinson theorem.

The classical one-dimensional Borg-Levinson theorem is formulated as follows : let q be real-valued and belongs to $L^\infty(0,1)$, and $y(x,\lambda)$ solves the initial value problem

$$-y''(x,\lambda) + q(x)y(x,\lambda) = \lambda y(x,\lambda), \quad y(0,\lambda) = 0, y'(0,\lambda) = 1.$$

Define the Dirichlet eigenvalues $\lambda_k(q)$ by the condition

$$y(1,\lambda_k) = 0$$

and define the norming constants $c_k(q)$ by

$$c_k(q) = \int_0^1 |y_k(x,\lambda_k)|^2 \, dx.$$

The result of Borg (1946) and Levinson (1949) is : if for two different potentials q_1 and q_2 their Dirichlet spectrums and norming constants are equal

$$\lambda_k(q_1) = \lambda_k(q_2), \quad c_k(q_1) = c_k(q_2), \quad k = 1, 2, ...,$$

then $q_1 = q_2$.
It can be reformulated as : if for all $k = 1, 2, ...$

$$\lambda_k(q_1) = \lambda_k(q_2), \quad y_k'(1, \lambda_k; q_1) = y_k'(1, \lambda_k; q_2),$$

then $q_1 = q_2$. Thus, the Dirichlet eigenvalues and normal derivatives of the eigenfunctions at the boundary uniquely determine a potential.
We generalize the classical Borg-Levinson theorem for the case of multidimensional elliptic differential operators.

We start with magnetic Schrödinger operators. Let $\Omega \subset R^n, n \geq 2,$ be a smooth bounded domain and let $\lambda_k(A, V), k = 1, 2, ...,$ be the Dirichlet eigenvalues of the magnetic Schrödinger operator

$$H_{A,V} = -(\nabla + iA)^2 + V\cdot$$

with real valued electric potential V and real-valued magnetic vector potential A. Associated to the eigenvalues $\lambda_k(A, V)$, we have the eigenfunctions $\phi_k(x; A, V), k = 1, 2, ...,$ forming an orthonormal basis in $L^2(\Omega)$.

Inverse spectral problem is : given the spectrum λ_k and normal derivatives $\partial_\nu \phi_k$ of the corresponding eigenfunctions at the boundary of Ω of the magnetic Schrödinger operator, can we determine V and A?

Borg-Levinson theorem for the Schrödinger operators for the first time was proved by Nachman, Sylvester and Uhlmann (1988) for the potentials $V \in C^\infty(\overline{\Omega})$. Their proof remains however valid if one assumes that $V \in L^\infty(\Omega)$.

The same result independently was obtained by Roman Novikov (1988).

For singular (meaning not bounded) potentials $V \in L^p(\Omega)$ for $p > \frac{n}{2}$ this theorem was proved by Päivärinta and Serov (2002).

For magnetic Schrödinger operator with singular potentials this theorem was proved by Serov (2010).

For Riemannian manifolds this result was proved by Katchalov, Kurylev and Lassas (they call this problem as the Gelfand inverse problem for quadratic pencil) in series publications (1999-2000).

For zero order perturbation of the bi-harmonic operator see Ikehata (1991) who proved the first result for the operator of order 4.

For elliptic partial differential operator (with constant coefficients) with L^∞-potentials Borg-Levinson theorem was proved by Krupchyk and Päivärinta (2012).

For the first order perturbation of the poly-harmonic operator see Krupchyk, Lassas and Uhlmann (2012 and 2014).

One-dimensional so-called "magnetic" equation can be written as

$$-\left(\frac{d}{dx}+iA(x)\right)^2 u(x)+V(x)u(x)=\lambda u(x), \quad x\in(0,1)$$

with real-valued functions $A(x)$ and $V(x)$. Assuming that $A\in W_1^1(0,1)$ and $V\in L^1(0,1)$ we can rewrite it as

$$-\frac{d^2}{dx^2}v(x)+V(x)v(x)=\lambda v(x)$$

with the same λ and with $v(x)=e^{ih(x)}u(x)$, where $h(x)=\int\limits_0^x A(y)\,dy$.

This transformation saves the Dirichlet eigenvalues and the norming constants. Hence, the result of Borg and Levinson implies immediately that

$$V_1(x) = V_2(x)$$

from the "magnetic" equation. And even more is true : we may conclude that

$$\int_0^1 A_1(y)\,dy = \int_0^1 A_2(y)\,dy,$$

$$A_1(0) = A_2(0), \quad A_1(1) = A_2(1).$$

So, more detailed analysis is required even in one-dimensional case.

Consider in the smooth bounded domain $\Omega \subset R^n, n \geq 2,$ the magnetic Schrödinger operator

$$H_{A,V} = -(\nabla + iA)^2 + V\cdot$$

with electric potential V and magnetic vector potential A. We assume that A and V are real-valued and satisfy the following quite general conditions :

$$(i) \quad A = (a_1, a_2, ..., a_n), \quad a_j \in L^2(\Omega),$$

$$(ii) \quad V \in L^1(\Omega), \quad V \geq 0.$$

Under these conditions on A and V for any function $u \in C_0^\infty(\Omega)$

$$(H_{A,V}u, u)_{L^2} \geq \|(\nabla + iA)u\|_{L^2}^2 \geq \|\nabla|u|\|_{L^2}^2.$$

These inequalities allow us to conclude that there is self-adjoint extension denoted by $H_{A,V}$ which is positive and with domain

$$D(H_{A,V}) = \{f \in D(\nabla + iA) : H_{A,V}f \in L^2(\Omega)\}.$$

The conditions (i) and (ii) for the coefficients are very general, but nevertheless we can use so-called "diamagnetic inequality" and obtain the needed result.

The diamagnetic inequality (DMI) of Barry Simon says that for any $t \geq 0$ and any $f \in L^2(\Omega)$

$$|e^{-tH_{A,V}} f(x)| \leq e^{-tH_{0,0}} |f|(x), \quad a.e. \quad x \in \Omega.$$

The operators $e^{-tH_{A,V}}$ and $e^{-tH_{0,0}}$ can be understood via J. von Neumann spectral theorem as follows :

$$e^{-tH_{A,V}} = \int_0^\infty e^{-t\lambda}\, dE_\lambda^{A,V}, \quad e^{-tH_{0,0}} = \int_0^\infty e^{-t\lambda}\, dE_\lambda^{0,0},$$

where $E_\lambda^{A,V}$ and $E_\lambda^{0,0}$ are spectral families corresponding to $H_{A,V}$ and $H_{0,0}$, respectively.

In the bounded domain Ω the Laplacian $H_{0,0}$ has pure discrete spectrum and corresponding eigenfunctions form an orthonormal basis in $L^2(\Omega)$. That's why $e^{-tH_{0,0}}$ is an integral operator with kernel denoted by $P_0(t,x,y)$. Since P_0 is a Heat kernel it can be proved the following estimate

$$0 \le P_0(t,x,y) \le \frac{1}{(\sqrt{4\pi t})^n} e^{-\frac{|x-y|^2}{4t}}.$$

This fact allows us to rewrite DMI as

$$|e^{-tH_{A,V}} f(x)| \le \frac{1}{(\sqrt{4\pi t})^n} \int_\Omega e^{-\frac{|x-y|^2}{4t}} |f(y)|\, dy, \quad a.e. \quad x \in \Omega.$$

Let $\lambda > 0$ is an eigenvalue and $\phi(x)$ is corresponding normalized eigenfunction of $H_{A,V}$. Using this general type of DMI we can obtain the inequality for an eigenfunction

$$\|\phi\|_{L^\infty(\Omega)} \leq \left(\frac{e}{2\pi n}\right)^{\frac{n}{4}} \lambda^{\frac{n}{4}}.$$

This is very nice estimate (if we take into account so general conditions for A and V). But in the applications it is usually needed more, namely the estimates for the "bundle" of eigenfunctions

$$\sum_{\lambda \leq \lambda_k < 2\lambda} |\phi_k(x)|^2 \leq C\lambda^{\frac{n}{2}}$$

uniformly with respect to $x \in \Omega$, and the estimates of the normalized eigenfunctions in some Sobolev spaces. For these purposes we need more restrictive conditions for the coefficients of the magnetic Schrödinger operator.

Let us assume that $A \in L^n(\Omega)$ for $n \geq 3$, $A \in L^s(\Omega), s > 2,$ for $n = 2$ and $V \in L^1(\Omega)$ and $V \geq 0$, as it was before. Then instead of the first inequality for the quadratic form we can obtain for any $u \in C_0^\infty(\Omega)$ the Gårding's inequality

$$(H_{A,V}u, u)_{L^2} \geq c_1 \|u\|^2_{W_2^1} - c_2 \|u\|^2_{L^2},$$

where $0 < c_1 < 1$ and $c_2 > 0$. This inequality implies that $H_{A,V}$ has the Friedrichs self-adjoint extension with pure discrete spectrum

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k ... \to +\infty$$

of finite multiplicity with only one accumulation point at $+\infty$. The corresponding orthonormal eigenfunctions $\{\phi_k(x)\}_{k=1}^\infty$ form orthonormal basis in $L^2(\Omega)$.

Even more is true in that case. Namely, $e^{-tH_{A,V}}$ is an integral operator with kernel denoted by $P(t,x,y)$. And DMI can be rewritten as

$$|\int_{\Omega} P(t,x,y)f(y)\,dy| \leq \frac{1}{(\sqrt{4\pi t})^n}\int_{\Omega} e^{-\frac{|x-y|^2}{4t}}|f(y)|\,dy,$$

where $f \in L^2(\Omega)$. It holds a.e. $x \in \Omega$.
Using then the Hardy-Littlewood maximal functions from this inequality we can obtain the inequality

$$|P(t,x,y)| \leq \frac{1}{(\sqrt{4\pi t})^n}e^{-\frac{|x-y|^2}{4t}}$$

that holds a.e. in $x \in \Omega$.

Let $\lambda > 0$, then $(H_{A,V} + \lambda I)^{-1}$ exists and it is an integral operator. The kernel of this operator is called Green's function denoted by $G(x, y, \lambda)$. It can be calculated as a Laplace transform of $P(t, x, y)$

$$G(x, y, \lambda) = \int_0^\infty e^{-t\lambda} P(t, x, y)\, dt.$$

Using this we can easily obtain that

$$|G(x, y, \lambda)| \leq (2\pi)^{-\frac{n}{2}} \left(\frac{|x-y|}{\sqrt{\lambda}} \right)^{-\frac{n-2}{2}} K_{\frac{n-2}{2}}(\sqrt{\lambda}|x-y|),$$

where K_ν is the Mcdonald function of order ν.

Using the estimates for the Mcdonald functions we can obtain more useful estimates for the Green's function

$$|G(x,y,\lambda)| \leq C|x-y|^{2-n}e^{-\sqrt{\lambda}|x-y|}, \quad n \geq 3,$$

and

$$|G(x,y,\lambda)| \leq C\left(1 + |\log(\sqrt{\lambda}|x-y|)|\right)e^{-\sqrt{\lambda}|x-y|}, \quad n = 2,$$

where $x, y \in \Omega$ and C does not depend on λ and $x, y \in \Omega$.
Remark.
It can be mentioned here that these estimates of the Green's function are obtained for very weak conditions of the coefficients of $H_{A,V}$.

Assume that $A \in W_p^1(\Omega)$ and $V \in L^p(\Omega)$ for some $p > \frac{n}{2}$, and consider the Dirichlet boundary value problem

$$(H_{A,V} + \lambda I)u(x) = 0, \quad x \in \Omega, \quad u(x) = f(x), \quad x \in \partial\Omega,$$

where $f(x)$ belongs to Besov space $B^t_{pp}(\partial\Omega)$ with

$$t > \frac{n-1}{p}, \quad \frac{n}{2} < p \leq n, \quad t = \frac{p-1}{p}, \quad p > n.$$

We can conclude (see Gilbarg and Trudinger) that there exists a unique solution u of the corresponding boundary value problem which belongs to

$$u \in W^2_{p,loc}(\Omega) \cap W_p^{t+\frac{1}{p}}(\Omega).$$

Thus, we may define the Dirichlet-to-Neumann map $\Lambda_{A,V+\lambda}$ by

$$\Lambda_{A,V+\lambda}f(x) := \frac{\partial u(x)}{\partial \nu} + iA\cdot\nu f(x), \quad x \in \partial\Omega,$$

where ν is outward normal vector.

Green's function estimates allow us also to obtain the estimates

$$\|\phi_k\|_{W_p^2(\Omega)} \leq C\lambda_k^{1+\frac{n}{4}},$$

where p is as above, and the convergence of the series

$$\sum_{k=1}^{\infty} \frac{1}{(\lambda_k+\lambda)^{\sigma}} < \infty, \quad \sigma > \frac{n}{2}.$$

These two facts imply

Theorem

Under the above conditions for A_j, V_j, $j = 1,2$ with $A_1(x) = A_2(x)$ at the boundary $\partial\Omega$ and f is as above, for any $0 < \delta < 1 - \frac{1}{p}$

$$\lim_{\lambda\to+\infty} \|\Lambda_{A_1,V_1+\lambda}f - \Lambda_{A_2,V_2+\lambda}f\|_{B^{\delta}_{pp}(\partial\Omega)} = 0.$$

Applying this Theorem, we can obtain

Theorem

Assume that $A_j \in W_p^1(\Omega)$ *and* $V_j \in L^p(\Omega)$, $j = 1, 2,$ *for some* $p > \frac{n}{2}$. *Assume in addition that* $A_1(x) = A_2(x)$ *at the boundary* $\partial\Omega$. *Assume also that for each* $k = 1, 2, \ldots$

$$\lambda_k(A_1, V_1) = \lambda_k(A_2, V_2),$$

$$\frac{\partial \phi_k}{\partial \nu}(x; A_1, V_1) = \frac{\partial \phi_k}{\partial \nu}(x; A_2, V_2).$$

Then for all $\lambda \geq \lambda_0$

$$\Lambda_{A_1, V_1 + \lambda} = \Lambda_{A_2, V_2 + \lambda}.$$

Now we can present the main result for the magnetic Schrödinger operator, i.e. Borg-Levinson theorem ($n \geq 3$)

Theorem

If $A_j \in W^1_\infty(\Omega)$ and $V_j \in L^\infty(\Omega)$, $j = 1, 2$, and all conditions of the previous theorem are satisfied, then

$$dA_1 = dA_2, \quad V_1 = V_2,$$

where the 2-form dA is defined by

$$dA = \sum_{j,k}^{n} \left(\frac{\partial a_k}{\partial x_j} - \frac{\partial a_j}{\partial x_k} \right) dx^k \wedge dx^j.$$

Consider now in the smooth bounded domain $\Omega \subset R^n, n \geq 2,$ the following operator of order 4 :

$$H_4 = \Delta^2 + i\nabla(\nabla(A\nabla)) + i\nabla(A\Delta) - \nabla(F\nabla) - 2iG\nabla - i\nabla G + V$$

with vector-valued functions A and G, and with scalar functions F and V. We assume that all these coefficients are real-valued and satisfy the following quite general conditions :

$$A(x) \in W_p^3(\Omega), \quad F(x) \in W_p^2(\Omega), \quad G(x) \in W_p^1(\Omega),$$

$$V(x) \in L_p(\Omega), \quad p > \frac{n}{2}, \quad n \geq 2$$

with the same value of p. It can be mentioned here that for the operators of order 4 or higher DMI does not hold and the technique for getting Green's function estimates is completely different in this case.

Under the mentioned above conditions for the coefficients of H_4 we can obtain for any $u \in C_0^\infty(\Omega)$ the inequality

$$(H_4 u, u)_{L^2} \geq c_1 \|u\|_{W_2^2}^2 - c_2 \|u\|_{L^2}^2,$$

where $0 < c_1 < 1$ and $c_2 > 0$. This inequality implies that H_4 has the Friedrichs self-adjoint extension with pure discrete spectrum

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k ... \to +\infty$$

of finite multiplicity with only one accumulation point at $+\infty$. The corresponding orthonormal eigenfunctions $\{\phi_k(x)\}_{k=1}^\infty$ form orthonormal basis in $L^2(\Omega)$.

Let $\lambda > 0$, then $(H_4 + \lambda I)^{-1}$ exists and it is an integral operator. The kernel of this operator is called Green's function denoted by $G(x, y, \lambda)$. It can be proved the following estimates :

$$|G(x, y, \lambda)| \leq C|x - y|^{4-n} e^{-\delta|x-y|\lambda^{\frac{1}{4}}}, \quad n \geq 5,$$

$$|G(x, y, \lambda)| \leq C \left(1 + |\log(|x - y|\lambda^{\frac{1}{4}})|\right) e^{-\delta|x-y|\lambda^{\frac{1}{4}}}, \quad n = 4,$$

$$|G(x, y, \lambda)| \leq \frac{C}{\lambda^{\frac{4-n}{4}}} e^{-\delta|x-y|\lambda^{\frac{1}{4}}}, \quad n \leq 3,$$

where $x, y \in \Omega$ and C does not depend on λ and $x, y \in \Omega$.

Assume that $A \in W_p^3(\Omega)$, $F \in W_p^2(\Omega)$, $G \in W_p^1(\Omega)$ and $V \in L^p(\Omega)$ for some $p > \frac{n}{2}$, and consider the Dirichlet boundary value problem

$$(H_4 + \lambda I)u(x) = 0, \quad x \in \Omega,$$

$$u(x) = f_0(x), \quad \partial_\nu u(x) = f_1(x), \quad x \in \partial\Omega,$$

where $f_0(x)$ belongs to Besov space $B_{pp}^{3-\frac{1}{p}}(\partial\Omega)$ and $f_1(x)$ belongs to Besov space $B_{pp}^{2-\frac{1}{p}}(\partial\Omega)$. We may conclude (see Mazja) that there exists a unique solution u which belongs to

$$u \in W^4_{p,loc}(\Omega) \cap W_p^3(\Omega).$$

Thus, we may define the Dirichlet-to-Neumann map Λ_λ by

$$\Lambda_\lambda\{f_0, f_1\}(x) :=$$

$$\{\partial_\nu(\Delta u)(x) + i\partial_\nu(A\nabla u) + i\nu A\Delta u(x) - Ff_1(x) - i\nu Gf_0(x),$$

$$-\Delta u(x) - iA\nabla u(x)\}, \quad x \in \partial\Omega.$$

In the particular case of zero perturbation of bi-harmonic operator we have that the Dirichlet-to-Neumann map has the form

$$\Lambda_\lambda\{f_0, f_1\}(x) := \{\partial_\nu(\Delta u)(x), -\Delta u(x)\}.$$

This corresponds to earlier result of Ikehata for zero perturbation of bi-harmonic operator.

Green's function estimates allow us to obtain two very important things. The first one is the estimate for the normalized eigenfunctions in Sobolev norms

$$\|\phi_k\|_{W_p^4(\Omega)} \leq C\lambda_k^{1+\frac{n}{8}}.$$

And the second is the convergence of the following number series :

$$\sum_{k=1}^{\infty} \frac{1}{(\lambda_k + \lambda)^{\sigma}} < \infty, \quad \sigma > \frac{n}{4}.$$

These two estimates play the crucial role in the inverse boundary spectral problems. At the same time these results certainly have some independent interest.

More precisely, these two facts imply one of the main results for this operator H_4. Namely, the following theorem holds.

Theorem

Assume that A_j, F_j, G_j and V_j, $j = 1, 2$, and f_0, f_1 are as above. Assume in addition that $A_1(x) = A_2(x)$, $F_1(x) = F_2(x)$, $G_1(x) = G_2(x)$ at the boundary. Then, for any $0 < \delta < 1 - \frac{1}{p}$

$$\lim_{\lambda \to +\infty} \|\Lambda_\lambda^{(1)}\{f_0, f_1\} - \Lambda_\lambda^{(2)}\{f_0, f_1\}\|_{B^{\delta}_{pp}(\partial\Omega)} = 0,$$

where $\Lambda_\lambda^{(j)}$ denotes the corresponding Dirichlet-to-Neumann map for $A_j, F_j, G_j, V_j + \lambda$, $j = 1, 2$.

Theorem

Assume that all conditions of the previous theorem are satisfied. Assume also that for each $k = 1, 2, \ldots$ *and for* $x \in \partial\Omega$

$$\lambda_k(A_1, F_1, G_1, V_1) = \lambda_k(A_2, F_2, G_2, V_2),$$

$$\nabla\phi_k(x; A_1, F_1, G_1, V_1) = \nabla\phi_k(x; A_2, F_2, G_2, V_2),$$

$$\partial_\nu(A_1\nabla\phi_k(x; A_1, F_1, G_1, V_1)) = \partial_\nu(A_2\nabla\phi_k(x; A_2, F_2, G_2, V_2)),$$

$$\Delta\phi_k(x; A_1, F_1, G_1, V_1) = \Delta\phi_k(x; A_2, F_2, G_2, V_2).$$

$$\partial_\nu\Delta\phi_k(x; A_1, F_1, G_1, V_1) = \partial_\nu\Delta\phi_k(x; A_2, F_2, G_2, V_2).$$

Then for all λ *big enough*

$$\Lambda_\lambda^{(1)}\{f_0, f_1\} = \Lambda_\lambda^{(2)}\{f_0, f_1\}.$$

Now we can present the main result for the operator H_4, i.e. Borg-Levinson theorem ($n \geq 3$).

Theorem

If $A = 0$, $F_j \in W^2_\infty(\Omega)$, $G_j \in W^1_\infty(\Omega)$ *and* $V_j \in L^\infty(\Omega)$, $\|F_j\|_{W^2_\infty(\Omega)}$ ($j = 1, 2$) *is small enough and for each* $k = 1, 2, \ldots$ *and* $x \in \partial\Omega$

$$\lambda_k(F_1, G_1, V_1) = \lambda_k(F_2, G_2, V_2),$$

$$\Delta\phi_k(x; F_1, G_1, V_1) = \Delta\phi_k(x; F_2, G_2, V_2),$$

$$\partial_\nu \Delta\phi_k(x; F_1, G_1, V_1) = \partial_\nu \Delta\phi_k(x; F_2, G_2, V_2),$$

then

$$F_1(x) = F_2(x), \quad G_1(x) = G_2(x), \quad V_1(x) = V_2(x)$$

a.e. in Ω.

Thanks very much for your attention !