Zeta invariants of the Steklov spectrum for a planar domain

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Outline

1. Three forms of the inverse problem
2. Zeta invariants $Z_k(a)$
3. Zeta invariants and the conformal group
4. Explicit formulas for Z_1 and Z_2
5. Some open questions

1. Three forms of the inverse problem

1.1. Form I

$D = \{(x, y) \mid x^2 + y^2 \le 1\} \subset \mathbb{R}^2 = \mathbb{C}, \quad \gamma = \partial D = \{e^{i\theta}\}$

$$\Lambda_e = \sqrt{-d^2/d\theta^2} : C^\infty(\gamma) \to C^\infty(\gamma)$$

is the Dirichlet-to-Neumann operator of the Euclidean metric.
Equivalently, $\Lambda_e e^{in\theta} = |n|\, e^{in\theta}$.
For a positive function $a \in C^\infty(\gamma)$, the operator $a\Lambda_e$ has a discrete eigenvalue spectrum

$$\mathrm{Sp}(a\Lambda_e) = \{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots\}$$

that will be called the Steklov spectrum of the operator $a\Lambda_e$.
Inverse problem: To what extent is a positive function $a \in C^\infty(\gamma)$ determined by the Steklov spectrum $\mathrm{Sp}(a\Lambda_e)$?

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Definition

Two functions $a, b \in C^\infty(\gamma)$ are said to be conformally equivalent if there exists a conformal or anticonformal transformation $\Phi : D \to D$ such that

$$b = a \circ \varphi \left| \frac{d\varphi}{d\theta} \right|^{-1}, \quad \textit{where} \quad \varphi = \Phi|_\gamma.$$

Here $\frac{d\varphi}{d\theta} \in C^\infty(\gamma)$ is defined by $\varphi^*(d\theta) = \frac{d\varphi}{d\theta}\, d\theta$.

If two positive functions $a, b \in C^\infty(\gamma)$ are conformally equivalent, then $\mathrm{Sp}(a\Lambda_e) = \mathrm{Sp}(b\Lambda_e)$. The converse statement is still open

Conjecture (1)

For two positive functions $a, b \in C^\infty(\gamma)$, the equality

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1.2. Form II

Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain bounded by a smooth closed curve $\partial\Omega$.

$\lambda \in \mathrm{Sp}(\Omega)$

$$\Delta u = 0 \quad \text{in} \quad \Omega, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega} = -\lambda u|_{\partial\Omega}$$

Inverse problem: To what extent is a simply connected smooth bounded domain $\Omega \subset \mathbb{R}^2$ determined by its Steklov spectrum?

Conjecture (2)

A simply connected smooth bounded domain $\Omega \subset \mathbb{R}^2$ is determined by its Steklov spectrum uniquely up to an isometry of $\mathbb{R}^2$ endowed with the standard Euclidean metric e.

1.3. Form III

Let g be a Riemannian metric on the unit disc D. The Dirichlet-to-Neumann operator of the metric is defined by

$$\Lambda_g : C^\infty(\gamma) \to C^\infty(\gamma), \quad \Lambda_g(f) = -\left.\frac{\partial u}{\partial \nu}\right|_\gamma,$$

where u is the solution to the Dirichlet problem

$$\Delta_g u = 0 \quad \text{in} \quad D, \quad u|_\gamma = f.$$

Inverse problem: to what extent is a Riemannian metric g on D determined by the spectrum $\mathrm{Sp}(\Lambda_g)$?

Conjecture (3)

A Riemannian metric on the unit disk is determined by its Steklov spectrum uniquely up to a conformal equivalence. More precisely, for two Riemannian metrics g and g' on D, the equality $\mathrm{Sp}(\Lambda_g) = \mathrm{Sp}(\Lambda_{g'})$ holds if and only if there exist a diffeomorphism $\Psi : D \to D$ and function $0 < \rho \in C^\infty(D)$ such that $\rho|_\gamma = 1$ and $g' = \rho\Psi^* g$.

These three conjectures are equivalent. The first version of the inverse problem seems easier from the analytic viewpoint since it is a problem of recovering one function of one real argument. On the other hand, two last versions seem, probably, more interesting from the geometric viewpoint. In what follows, we discuss Conjecture 1.

We are not very optimistic about the validity of the conjecture in the general case. Nevertheless, there are many versions of the problem which are worth of studying even if the answer is "no" in the general case.

For example, we can ask: given $0 < b \in C^\infty(\gamma)$, how many positive functions $a \in C^\infty(\gamma)$ satisfy

$$\mathrm{Sp}(a\Lambda_e) = \mathrm{Sp}(b\Lambda_e)?$$

We believe that, for a generic b, such a function a is unique up to the conformal equivalence.

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2. Zeta invariants

$\gamma = \{e^{i\theta}\}$. For a function $a \in C^\infty(\gamma)$, let $\hat{a}_n$ be its Fourier coefficients, i.e.,

$$a(\theta) = \sum_{n=-\infty}^{\infty} \hat{a}_n e^{in\theta}.$$

For every integer $k \geq 1$, we define

$$Z_k(a) = \sum_{j_1+\cdots+j_{2k}=0} N_{j_1\ldots j_{2k}} \hat{a}_{j_1} \hat{a}_{j_2} \ldots \hat{a}_{j_{2k}},$$

where

$$N_{j_1\ldots j_{2k}} = \sum_{n=-\infty}^{\infty} \Big[|n(n+j_1)(n+j_1+j_2)\ldots(n+j_1+\cdots+j_{2k-1})| - n(n+j_1)(n+j_1+j_2)\ldots(n+j_1+\cdots+j_{2k-1}) \Big].$$

There is only a finite number of nonzero summands on the right-hand side.

Theorem

For a function $0 < a \in C^\infty(\gamma)$ *normalized by the condition* $L = \frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{a(\theta)} = 1$ *and for every* $k \geq 1$*, the invariant* $Z_k(a)$ *is uniquely determined by the Steklov spectrum* $\mathrm{Sp}\,(a\Lambda_e)$*.*

Idea of the proof:

Let $\{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots\}$ be the spectrum of the operator $a\Lambda_e$. *The zeta-function* of the operator is defined by

$$\zeta_a(s) = \mathrm{Tr}\,[(a\Lambda_e)^{-s}] = \sum_{n=1}^{\infty} \lambda_n^{-s}.$$

The series converges for $\mathrm{Re}\, s > 1$ and $\zeta_a(s)$ extends to a meromorphic function on $\mathbb{C}$ with the unique simple pole at $s = 1$. Moreover, $\zeta_a(s) - 2\zeta_R(s)$ is an entire function, where $\zeta_R(s)$ is the classical Riemann zeta-function.

We prove that for every $k = 1, 2, \dots$

$$\zeta_a(-2k) = \zeta_a(-2k) - \zeta_R(-2k) = \mathrm{Tr}[(a\Lambda_e)^{2k} - (aD_\theta)^{2k}] = Z_k(a).$$

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If two positive functions $a, b \in C^\infty(\gamma)$ are isospectral

$$\mathrm{Sp}(a\Lambda_e) = \mathrm{Sp}(b\Lambda_e),$$

then, by the theorem, we have the infinite system of algebraic equations in Fourier coefficients

$$Z_k(a) = Z_k(b) \quad (k = 1, 2, \dots).$$

Question: given $0 < b \in C^\infty(\gamma)$, how many positive functions $a \in C^\infty(\gamma)$ satisfy the system?
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3. Zeta invariants and the conformal group

Rewrite the definition of $Z_k(a)$ in the form

$$Z_k(a) = \sum_{j_1,\dots,j_{2k}} Z_{j_1\dots j_{2k}}\, \hat{a}_{j_1} \dots \hat{a}_{j_{2k}},$$

where

$$Z_{j_1\dots j_{2k}} = \begin{cases} 0, & \text{if} \quad j_1 + \dots + j_{2k} \neq 0, \\ N_{(j_1\dots j_{2k})}, & \text{if} \quad j_1 + \dots + j_{2k} = 0. \end{cases},$$

$$N_{j_1\dots j_{2k}} = \sum_{n=-\infty}^{\infty} \Big[|n(n+j_1)(n+j_1+j_2)\dots(n+j_1+\dots+j_{2k-1})| \\ - n(n+j_1)(n+j_1+j_2)\dots(n+j_1+\dots+j_{2k-1}) \Big].$$

We know $Z_k(a) = Z_k(b)$ for conformally equivalent functions a and b.

Proposition

The conformal invariance of $Z_k(a)$ is equivalent to the relations

$$Z_{j_1 \dots j_{2k}} = Z_{-j_1, \dots, -j_{2k}},$$

$$\sum_{\alpha=1}^{2k} (j_\alpha - 1) Z_{j_1, \dots, j_{\alpha-1}, j_\alpha+1, j_{\alpha+1}, \dots, j_{2k}} = 0.$$

4. Explicit formulas for Z_1 and Z_2

$$Z_1(a) = \sum_j Z_{j,-j}\, \hat{a}_j \hat{a}_{-j}, \quad Z_{j,-j} = \sum_n \Big(|n(n+j)| - n(n+j) \Big).$$

$$|n(n+j)| - n(n+j) = \begin{cases} -2n(n+j), & \text{if} \quad 0 < n < -j \text{ or } -j < n < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for a positive j,

$$Z_{j,-j} = -2 \sum_{n=-j}^{-1} n(n+j) = -2 \sum_{n=1}^{j} n^2 + 2j \sum_{n=1}^{j} n = \frac{1}{3}(j^3 - j).$$

Similarly, $Z_{j,-j} = \frac{1}{3}|j^3 - j|$ for a negative j.

Edward: $$Z_1(a) = \frac{1}{3} \sum_{j=-\infty}^{\infty} |j^3 - j|\, \hat{a}_j \hat{a}_{-j} = \frac{2}{3} \sum_{j=2}^{\infty} (j^3 - j)\, \hat{a}_j \hat{a}_{-j}.$$

Theorem

Coefficients of the form $Z_2(a) = \sum Z_{ijk\ell}\, \hat{a}_i \hat{a}_j \hat{a}_k \hat{a}_\ell$ *are completely determined by the following:*
(1) $Z_{ijk\ell} = 0$ *for* $i + j + k + \ell \neq 0$*;*
(2) $Z_{ijk\ell}$ *are symmetric and even:* $Z_{-i,-j,-k,-\ell} = Z_{i,j,k,\ell}$*;*
(3) $Z_{ijk,-i-j-k}$ *is the piece-wise polynomial function in* (i, j, k) *uniquely determined by*

$$Z_{ijk,-i-j-k} = \begin{cases} P_1(i,j,k) & \text{if} \quad i \geq 0,\ j \geq 0,\ k \geq 0; \\ P_2(i,j,k) & \text{if} \quad i \leq 0,\ j \geq 0,\ k \geq 0,\ i+j \leq 0,\ i+k \leq 0, \\ & \qquad i+j+k \geq 0 \end{cases}$$

$$P_1 = \frac{1}{15}\sigma_{(ijk)}\left(3i^5+15i^4j+10i^3j^2+10i^3jk-5i^3-25i^2j-10ijk+2i\right),$$

$$P_2 = \frac{1}{45}\sigma_{(jk)}\Big(5i^5 + 25i^4j + 10i^3j^2 + 20i^3jk - 10i^2j^3 - 15ij^4 - 20ij^3k$$
$$- 4j^5 - 5j^4k + 10j^3k^2 - 5i^3 - 15i^2j + 5ij^2 - 5j^2k + 4j\Big)$$

5. Some open questions

Main problem: given a positive function $b \in C^\infty(\gamma)$, one has to find all positive functions $a \in C^\infty(\gamma)$ satisfying

$$\mathrm{Sp}\,(a\Lambda_e) = \mathrm{Sp}\,(b\Lambda_e).$$

Zeta-invariants allow us to write down the infinite system of equations

$$Z_k(a) = b_k \quad (k = 1, 2, \dots) \tag{1}$$

The principle question on zeta-invariants: are the invariants $Z_k(a)$ $(k = 1, 2, \dots)$ independent of each other, i.e., does system (1) give us infinitely many conditions on the Fourier coefficients of a function a?

$Z_k(a) = 0$ $(k = 1, 2, \dots)$ for every function a belonging to the three-dimensional subspace

$$L = \{a \in C^\infty(\gamma) \mid a(\theta) = \hat{a}_0 + \hat{a}_1 e^{i\theta} + \hat{a}_{-1} e^{-i\theta}\}.$$

The converse statement is true in the case of $k = 1$ for real functions: if $Z_1(a) = 0$ for a real function $a \in C^\infty(\gamma)$, then $a \in L$. This is seen from Edward's formula that takes the following form in the case of a real function a:

$$Z_1(a) = \frac{2}{3} \sum_{n=2}^{\infty} (n^3 - n)\, |\hat{a}_n|^2.$$

How does the set of all (real) functions $a \in C^\infty(\mathbb{S})$ satisfying $Z_k(a) = 0$ for $k = 2, 3, \dots$ look like, can it be essentially different of L?

For a real function a,

$$Z_1(a) = \sum_{n\geq 2}(n^3 - n)|\hat{a}_n|^2 \geq c_1 \sum_{n\geq 2} n^3|\hat{a}_n|^2.$$

Problem

Does the inequality

$$Z_k(a) \geq c_k \sum_{n\geq 2} n^{2k+1}|\hat{a}_n|^{2k} \tag{1}$$

hold for every real function $a \in C^\infty(\mathbb{S})$ and for every $k = 2, 3, \dots$, where the coefficient $c_k > 0$ depends on k only? If the answer is "no", the same question can be asked for positive functions a.

The compactness theorem

The Hilbert space $H^s(\gamma)$ is the completion of $C^\infty(\gamma)$ with respect to the norm

$$\|a\|^2_{H^s(\gamma)} = \sum_n (1 + |n|^{2s})|\hat{a}_n|^2.$$

Theorem

Let $a^\nu \in C^\infty(\gamma)$ $(\nu = 1, 2, \dots)$ be a sequence of functions uniformly bounded from below by some positive constant

$$a^\nu(\theta) \geq c > 0.$$

Assume the Steklov spectrum $\mathrm{Sp}\,(a^\nu \Lambda_e)$ to be independent of ν. Then there exists a subsequence a^{ν_k} such that each a^{ν_k} is conformally equivalent to some function $b^{\nu_k} \in C^\infty(\gamma)$ and the sequence of norms $\|b^{\nu_k}\|_{H^{3/2}(\gamma)}$ is bounded. Hence, for every $s < 3/2$, the sequence b^{ν_k} contains a subsequence converging in $H^s(\gamma)$.

The scheme of the proof.

We can assume that $\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{a^\nu(\theta)}=1$ and $\hat{a}_1^\nu=\hat{a}_{-1}^\nu\geq 0$. The positiveness of a^ν implies $\quad \hat{a}_0^\nu\geq 1, \quad 0\leq \hat{a}_1^\nu\leq \hat{a}_0^\nu$. The first invariant

$$Z_1(a^\nu)=\frac{1}{3}\sum_{|n|\geq 2}|n^3-n|\,|\hat{a}_n^\nu|^2$$

is independent of ν. Therefore

$$\|\hat{a}^{\nu_k}\|^2_{H^{3/2}(\gamma)}\leq(\hat{a}_0^\nu)^2+(\hat{a}_1^\nu)^2+Z_1(a^\nu)\leq 2(\hat{a}_0^\nu)^2+Z_1(a^\nu).$$

If the sequence $\hat{a}_0^\nu$ was bounded, the statement of the theorem would follow.

On assuming $\hat{a}_0^\nu\to+\infty$, we look for a conformal transformation $\Phi_\nu: D\to D$ such that

$$\hat{b}_0^\nu\leq C \quad \text{for} \quad b^\nu=a^\nu\Phi_\nu.$$

It is possible to choose such Φ_ν in the form $\Phi_\nu(z)=\frac{z-\rho_\nu}{1+\rho_\nu z} \quad (-1<\rho_\nu<1)$.

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