Inverse problems for fourth order operators on the circle

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Second order operator

Consider the Schrödinger operator $-\partial^2 + q$ in $L^2(\mathbb{R})$ with the 1-periodic potential $q : \int_0^1 q(x)dx = 0$.

Let $\lambda_0^+ < \lambda_1^- \leqslant \lambda_1^+ < \lambda_2^- \leqslant ...$ be eigenvalues of the 2-periodic problem $f(0) = f(2), f'(0) = f'(2)$.

Let $\mu_1(t) < \mu_2(t) < ...$, $t \in [0,1]$, be eigenvalues of the Dirichlet problem $f(0) = f(1) = 0$ for the shifted potential $q(\cdot + t)$.

If $q \in C^3(\mathbb{T}), \mathbb{T} = \mathbb{R}/\mathbb{Z}$, then

$$\lambda_n^{\pm} = \pi n + O(n^{-2}), \quad n \to +\infty,$$

$$\mu_n(t) \in [\lambda_n^-, \lambda_n^+] \quad \forall \quad (n,t) \in \mathbb{N} \times [0,1] \tag{1}$$

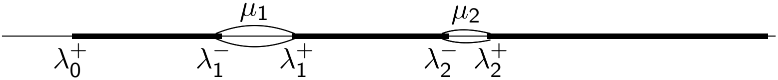
and the following trace formula holds

$$q(t) = \lambda_0^+ + \sum_{n=1}^{\infty} \Big(\lambda_n^+ + \lambda_n^- - 2\mu_n(t)\Big), \tag{2}$$

the series converges absolutely and uniformly on $t \in [0,1]$.

If the interval $[\lambda_n^-, \lambda_n^+]$ degenerates into a point, then $\mu_n(t) = \lambda_n^- = \lambda_n^+$ for all $t \in [0, 1]$.

If the interval $(\lambda_n^-, \lambda_n^+)$ is not empty and if t changes from 0 to 1, then $\mu_n(t)$ runs all interval $[\lambda_n^-, \lambda_n^+]$ making n complete revolutions.



Each of the function $\mu_n(t)$ satisfies the system of ordinary differential equations of the first order, so called Dubrovin equations. Due to (1) the initial problem for these equations has a unique solution. Then, using identity (2), we recover the potential q by the spectral dates $\lambda_0^+, \lambda_n^\pm, \mu_n(0), n \in \mathbb{N}$.

These results about the second order operator were obtained by Dubrovin (1975) and Its, Matveev (1975) (the finite band case), McKean, van Moerbeke (1975) (the smooth case), Trubowitz (1977) (the case $q, q' \in L^2(\mathbb{T})$), Korotyaev (1999) (the case $q \in L^2(\mathbb{T})$).

Fourth order operators

Let the 1-periodic functions p, q satisfy the conditions $p, p^{(4)}, q, q'' \in L^1(\mathbb{T}), \mathbb{T} = \mathbb{R}/\mathbb{Z}, \int_0^1 q(x)dx = 0$. Consider the operators

$$\partial^4 + 2\partial p \partial + q,$$

acting in $L^2(0,1)$ with the following boundary conditions:
the periodic boundary conditions

$$f^{(j)}(0) = f^{(j)}(1) \quad \forall \quad j = 0, 1, 2, 3,$$

the antiperiodic boundary conditions

$$f^{(j)}(0) = -f^{(j)}(1) \quad \forall \quad j = 0, 1, 2, 3,$$

and the Dirichlet boundary conditions

$$f(0) = f''(0) = f(1) = f''(1) = 0.$$

Spectra

Let

$$\lambda_0^+ \leqslant \lambda_2^- \leqslant \lambda_2^+ \leqslant \lambda_4^- \leqslant ...$$

be the eigenvalues of the operator with the periodic boundary conditions, let

$$\lambda_1^- \leqslant \lambda_1^+ \leqslant \lambda_3^- \leqslant ...$$

be the eigenvalues of the operator with the antiperiodic boundary conditions, and let

$$\mu_1(t) \leqslant \mu_2(t) \leqslant \mu_3(t) \leqslant ..., \quad t \in [0, 1],$$

be the eigenvalues of the operator with the shifted coefficients $p(\cdot + t), q(\cdot + t)$ and the Dirichlet boundary conditions.

Asymptotics

Theorem

i) The eigenvalues of the periodic and antiperiodic problems satisfy

$$\lambda_n^{\pm} = (\pi n)^4 - 2(\pi n)^2 p_0 - \frac{\|p\|^2 - p_0^2}{2} \pm |\widehat{V}_n| + o(n^{-\frac{3}{2}}), \qquad (3)$$

where $p_0 = \int_0^1 p(x)dx, \ \|p\|^2 = \int_0^1 |p(x)|^2 dx,$

$$\widehat{V}_n = \int_0^1 e^{-i2\pi nx}\Big(q(x) - \frac{p''(x)}{2}\Big)dx.$$

ii) The eigenvalues of the Dirichlet problem satisfy

$$\mu_n(t) = (\pi n)^4 - 2(\pi n)^2 p_0 - \frac{\|p\|^2 - p_0^2}{2} + O(n^{-2}) \qquad (4)$$

as $n \to +\infty$ *uniformly on* $t \in [0,1]$.

Trace formula

Theorem

The following trace formula is fulfilled

$$q(t) - \frac{p''(t)}{2} = \lambda_0^+ + \sum_{n=1}^{\infty} \big(\lambda_n^+ + \lambda_n^- - 2\mu_n(t)\big) \qquad \forall \quad t \in [0,1], \qquad (5)$$

the series converges absolutely and uniformly on t. If $p = \mathrm{const}$, then

$$q(t) = \lambda_0^+ + \sum_{n=1}^{\infty} \big(\lambda_n^+ + \lambda_n^- - 2\mu_n(t)\big) \qquad \forall \quad t \in [0,1], \qquad (6)$$

the series converges absolutely and uniformly on t.

Remark. 1) Identity (6) for the fourth order operator is similar to identity (2).

2) Using identity (5), we recover the function $q(t) - \frac{p''(t)}{2}$ by the spectral dates $\lambda_0^+, \lambda_n^\pm, \mu_n(t), n \in \mathbb{N}, t \in [0,1]$.

3) Formally we can write the equations for $\mu_n(t)$ analogous to the Dubrovin equations. However, we cannot prove the unique existence of solution of these equations, since we still have no the relations, similar to (1), for the fourth order operator.