# A Riemann-Hilbert Problem in a Riemann Surface

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# Toda lattice

We begin with the doubly infinite periodic lattice

$$\dot{x}(n,t) = y(n,t),$$
$$\dot{y}(n,t) = exp(x(n-1,t) - x(n,t)) - exp(x(n,t) - x(n+1,t)).$$

Suppose we perturb a finite number of particles. What happens eventually, for large times?

# Flaschka transformation and the Lax pair

Setting

$$a_n(t) = \frac{1}{2} exp(\frac{1}{2}(x(n,t) - x(n+1,t))),$$

$$b_n(t) = -\frac{1}{2} y(n,t),$$

$$\text{we get}$$

$$\dot{b}_n(t) = 2(a_n(t)^2 - a_{n-1}(t)^2),$$

$$\dot{a}_n(t) = a_n(t)(b_{n+1}(t) - b_n(t)).$$

Jacobi self-adjoint operator: $L : \ell^2 \to \ell^2$.
Skew-self-adjoint operator: $B : \ell^2 \to \ell^2$.

$$(Lu)_n = a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1},$$

$$(Bu)_n = -b_{n-1}u_{n-1} + b_n u_{n+1},$$

Toda lattice becomes $dL/dt = LB - BL$

# Periodic Toda is explicitly solvable

Periodic Toda is explicitly solvable in terms of theta functions (Novikov, etc., Date-Tanaka 70s)

There exist real numbers $E_0 < E_1 < \dots\dots < E_{2g+1}$ s.t.$spec(L) = \cup_{j=0}^{g}[E_{2j}, E_{2j+1}]\}$ (generically $g+1=$ period of lattice).

Let $M$ be the Riemann surface associated with the function $\prod_{j=0}^{2g+1}(z - E_j), g \in \mathbb{N}$. $M$ is a compact, hyperelliptic Riemann surface of genus $g$. Then

$$a_{per}(n,t)^2 = \tilde{a}^2 \frac{\theta(\underline{z}(n+1,t))\theta(\underline{z}(n-1,t))}{\theta(\underline{z}(n,t))^2},$$

$$b_{per}(n,t) = \tilde{b} + \frac{1}{2}\frac{d}{dt}\log\Big(\frac{\theta(\underline{z}(n,t))}{\theta(\underline{z}(n-1,t))}\Big).$$

The constants $\tilde{a}$, $\tilde{b}$ depend only on the Riemann surface, i.e. on $E_0 < E_1 < \cdots < E_{2g+1}$.

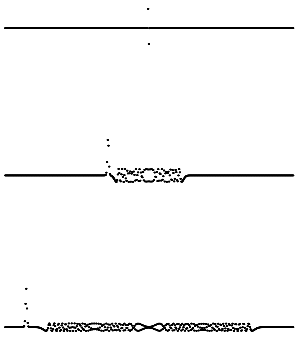
Here $\theta(z)$ is the Riemann theta function associated with $M$ and $z_\ell(n,t) = \alpha_\ell - n\beta_\ell + t\eta_\ell$, where again $\alpha_\ell, \beta_\ell, \eta_\ell$ depend only on the Riemann surface.

# Theta function

$$\theta(\underline{z}) = \sum_{\underline{m}\in\mathbb{Z}^g} \exp 2\pi\mathrm{i} \left( \underline{mz} + \frac{\underline{m\tau}\,\underline{m}}{2} \right), \qquad \underline{z} \in \mathbb{C}^g$$

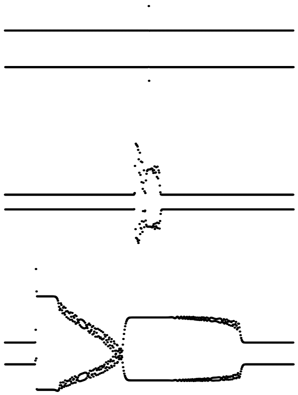
# Short Range Perturbation of Constant Background (K. 1993)

Plot of $a(., t)$ for three fixed consecutive times. The background is: $a_{st}(n, 0) = 1$, $b_{st}(n, 0) = 0$



# Short Range Perturbation of Periodic Background (K., Teschl 2007)

Periodic background: $a_{per}(n,0) = 1$, $b_{per}(n,0) = (-1)^n$



# Explicit formulae

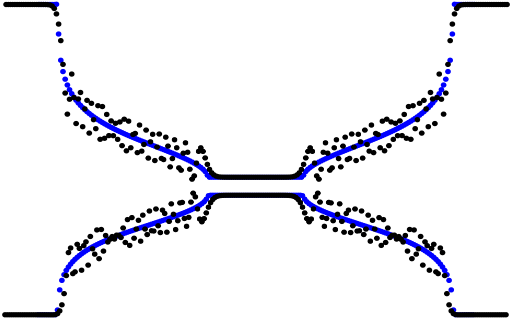
For long times the perturbed Toda lattice is asymptotically close to the following limiting lattice:

$$\prod_{j=n}^{\infty}\left(\frac{a_{lim}(j,t)}{a_{per}(j,t)}\right)^2 = \frac{\theta(\underline{z}(n,t))}{\theta(\underline{z}(n+1,t))}\frac{\theta(\underline{z}(n+1,t)+\underline{\delta}(n,t))}{\theta(\underline{z}(n,t)+\underline{\delta}(n,t))}\times$$
$$\times \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{C(n/t)}\log(1-|R|^2)\omega_{\infty_-\infty_+}\right),$$
$$\delta_\ell(n,t) = \frac{1}{2\pi\mathrm{i}}\int_{C(n/t)}\log(1-|R|^2)\zeta_\ell,$$
$$z_\ell(n,t) = \alpha_\ell - n\beta_\ell + t\eta_\ell,$$

where: $\theta$ Riemann theta function, $\zeta_\ell, \omega_{pq}$ Abelian differentials, $C(n/t)$ contour depending on the stationary phase points of the RHP, $\alpha_\ell, \beta_\ell, \eta_\ell$ are functions of the spectrum (via Abel integrals), and $R$ is the reflection coefficient of the perturbed lattice.

# Numerical Comparison

Numerical comparison for genus one case, which can be computed in terms of elliptic functions:



# Rieman-Hilbert factorization

The so-called *nonlinear stationary − phase − steepest − descent* method for the asymptotic analysis of Rieman-Hilbert factorization problems has been very successful in providing
(i) rigorous results on long time, long range and semiclassical asymptotics for solutions of completely integrable equations and correlation functions of exactly solvable models,
(ii) asymptotics for orthogonal polynomials of large degree,
(iii) the limiting eigenvalue distribution of random matrices of large dimension (and thus universality results, i.e. independence of the exact distribution of the original entries, under some conditions),
(iv) proofs of important results in combinatorial probability (e.g. the limiting distribution of the length of longest increasing subsequence of a permutation, under uniform distribution).

# Stationary phase

Even though the stationary phase idea was first applied to a Riemann-Hilbert problem and a nonlinear integrable equation by *Alexander Its* (1982) the method became systematic and rigorous in the work of *Deift and Zhou* (1993).
In analogy to the linear stationary-phase and steepest-descent methods, where one asymptotically reduces the given exponential integral to an exactly solvable one, in the nonlinear case one asymptotically reduces the given Riemann-Hilbert problem to an exactly solvable one.

# Non-self-adjointness

The term *nonlinear steepest descent method* is often used. In fact, there is a distinction between the stationary-phase idea and the steepest-descent idea: actual *steepest descent contours* appear in non-self-adjoint problems.
The distinction partly mirrors the
$self-adjoint/non-self-adjoint$ dichotomy of the underlying Lax operator;
see *Kamvissis K.McLaughlin Miller* (2003)
An extra feature appearing only in the nonlinear asymptotic theory is the $Lax-Levermore$ *variational problem*, discovered in 1979, before the work of Its, Deift and Zhou, but reappearing here in the guise of the so-called $g-function$ which is catalytic in the process of deforming Riemann-Hilbert factorization problems to exactly solvable ones.
Non-self-adjoint case: *Kamvissis Rakhmanov* (2005)

# THE LINEAR METHOD

Suppose one considers the Cauchy problem for, say, the linearized KdV
$u_t - u_{xxx} = 0$.
It can off course be solved via Fourier transforms. The end result of the Fourier method is an exponential integral. To understand the long time asymptotic behavior of the integral one needs to apply the stationary-phase method .
The underlying principle, going back to Stokes and Kelvin, is that the dominating contribution comes from the vicinity of the stationary phase points. Through a local change of variables at each stationary phase point and using integration by parts we can calculate each contributing integral asymptotically to all orders with exponential error. It is essential here that the phase $x\xi - \xi^3 t$ is real and that the stationary phase points are real.

# Airy integral

On the other hand, suppose we have something like the Airy exponential integral

$$Ai(z) = \frac{1}{\pi}\int_0^\infty cos(s^3/3 + zs)ds$$

and we are interested in $z \to \infty$. Set $s = z^{1/2}t$ and $x = z^{3/2}$. So $Ai(x^{2/3}) = \frac{x^{1/3}}{2\pi}\int_{-\infty}^{\infty} exp(ix(t^3/3 + t))dt$.
The phase is $h(t) = \frac{t^3}{3} + t$ and the zeros of $h'(t) = (t^2 + 1)$ are $\pm i$. As they are not real, before we apply any stationary-phase method, we have to deform the integral off the real line and along particular paths: these are the *steepest descent paths*. They are given by the simple characterization $Imh(t) = constant$. In our particular example, the curves of steepest descent are the imaginary axis and the two branches of a hyperbola. By deforming to one of these branches, we finally end up with Laplace type integrals and then apply the same method as above (local change of variables plus integration by parts) to recover valid asymptotics to all orders.

# THE NONLINEAR METHOD

The nonlinear method generalizes the ideas above, but also employs new ones.
The analog of the Fourier transform is the scattering coefficient $r(\xi)$ for the Jacobi operator: $L : \ell^2 \to \ell^2$.

$$(Lu)_n = a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1}$$

# Scattering for the Decaying Case

Joukowski transformation

$2\lambda = z + 1/z$

The continuous spectrum $[-1, 1]$ is mapped to the unit circle $|z| = 1$.
This is convenient but also essential!

# Scattering for the Decaying Case

$$(Lu)_n = a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1}$$
$$L\phi(n,z) = \lambda\phi(n,z); \quad \phi(n,z) \ \sim z^n, \ n \to \infty$$
$$L\psi(n,z) = \lambda\psi(n,z); \ \psi(n,z) \ \sim z^{-n}, \ n \to -\infty.$$
$$\textit{We can write } \phi(n,z) = A_n^+ z^n v_n(z) \textit{ and } \psi(n,z) = A_n^+ z^{-n} u_n(z)$$
$$\textit{where } A_n^+ = \Pi_{k=n}^{\infty}(2a_k)^{-1}, \ A_n^- = \Pi_{k=-\infty}^{n-1}(2a_k)^{-1} \ A = A_n^+ A_n^-.$$
$$\textit{Furthermore } v_n(z) = 1 + \Sigma_{n=1}^{\infty} v_{n,k} z^k, u_n(z) = 1 + \Sigma_{n=1}^{\infty} u_{n,k} z^k.$$

Both series converge uniformly in $|z| \leq 1$.

# Scattering for the Decaying Case

On the other hand, when $z \neq \pm 1$,

$$\phi(n,z), \psi(n,z) \quad \textit{and} \quad \phi(n,z^{-1}), \psi(n,z^{-1})$$

are two sets of independent solutions of the second order difference equation $L\chi = \lambda\chi$. Hence there exist $A(z), B(z), \alpha(z), \beta(z)$ such that

$$\begin{aligned} \phi(n,z) &= B(z)\psi(n,z) + A(z)\psi(n,z^{-1}) \\ \psi(n,z) &= \beta(z)\phi(n,z) + \alpha(z)\phi(n,z^{-1}) \\ & \qquad\qquad\qquad z \in C, z \neq \pm 1. \end{aligned}$$

We get

$$v_n(z) = \frac{A_n^-}{A_n^+} B(z) z^{-2n} u_n(z) + \frac{A_n^-}{A_n^+} A(z) u_n(z^{-1}).$$

# Scattering

Next define $U_1(n,z) = \psi(n,z)$, $U_2(n,z) = \frac{\phi(n,z)}{A(z)} = T(z)\phi(n,z)$. Since $\phi(n,z) \sim A_n^+ z^n$ and $\psi(n,z) \sim A_n^- z^{-n}$ near $z = 0$, we have $U_1(n,z) \sim A_n^- z^{-n}$, $U_2(n,z) \sim (A_n^-)^{-1} z^n$, near $z = 0$.

We end up with a Riemann-Hilbert problem

$$\begin{pmatrix} U_2^+ & U_1^+ \end{pmatrix} = \begin{pmatrix} U_1^- & U_2^- \end{pmatrix} \begin{pmatrix} 1 - |r(z)|^2 & -\bar{r}(z) \\ r(z) & 1 \end{pmatrix}$$

with asymptotics $\begin{pmatrix} U_1 & U_2 \end{pmatrix} \sim \begin{pmatrix} A_n^- z^{-n} & (A_n^-)^{-1} z^n \end{pmatrix}$ at $z = 0$ and $\begin{pmatrix} U_1 & U_2 \end{pmatrix} \sim \begin{pmatrix} A_n^- z^n & (A_n^-)^{-1} z^{-n} \end{pmatrix}$ at infinity.

# Riemann-Hilbert problem

Defining
$y_1(n,z) = \frac{U_2(n,z)}{z^n}, \quad |z| < 1, \; y_1(n,z) = \frac{U_1(n,z)}{z^n}, \; |z| > 1,$
$y_2(n,z) = \frac{U_1(n,z)}{z^{-n}}, \; |z| < 1, \; y_2(n,z) = \frac{U_2(n,z)}{z^{-n}}, \; |z| > 1,$
and letting $Y = (y_1, y_2)$, we end up with the Riemann-Hilbert matrix factorization problem:

$$Y_+ = Y_- \begin{pmatrix} 1 - |r(z)|^2 & -\bar{r}(z) z^{2n} \\ r(z) z^{-2n} & 1 \end{pmatrix}$$

with $Y(\infty) = \begin{pmatrix} (A_n^-)^{-1} & A_n^- \end{pmatrix}$. Also $Y(0) = \begin{pmatrix} A_n^- & (A_n^-)^{-1} \end{pmatrix}$.

# Problem

There is a problem here. The normalization at infinity involves $A_n^+, A_n^-$ which is after all the solution. We can get rid of this by using a particular symmetry.

SYMMETRY: $Q(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (Q(0))^{-1} Q(z^{-1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

THEOREM. Let $Q$ be the solution of the RHP with same jump and converging to the identity at infinity. Then

$$Y = \begin{pmatrix} (\frac{1+\beta}{\alpha})^{1/2} & (\frac{\alpha}{1+\beta})^{1/2} \end{pmatrix} Q$$

where $Q(0) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

From $Y(0)$ one recovers the solution of the Toda lattice, since $Y(0) = \begin{pmatrix} A_n^- & (A_n^-)^{-1} \end{pmatrix}$.

# Riemann-Hilbert Problem for the decaying Toda Lattice

Let $r(z,0)$ be the reflection coefficient for the initial data.
Let $Q$ be analytic off the unit circle and $= I$ at infinity, with

$$Q_+ = Q_- \begin{pmatrix} 1-|r(z)|^2 & -\bar{r}(z)z^{2n} exp[(z-z^{-1})\frac{t}{2}] \\ r(z)z^{-2n} exp[-(z-z^{-1})\frac{t}{2}] & 1 \end{pmatrix}$$

Set

$$Y = \begin{pmatrix} (\frac{1+\beta}{\alpha})^{1/2} & (\frac{\alpha}{1+\beta})^{1/2} \end{pmatrix} Q$$

where $Q(0) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.
Then $Y(0) = \begin{pmatrix} A_n^- & (A_n^-)^{-1} \end{pmatrix}$, and one recovers $a_n$ from
$A_n^- = \Pi_{k=-\infty}^{n-1} (2a_k)^{-1}$.

# Stationary Phase Analysis

There are two stationary phase points $z_{sp}, z_{sp}^*$.
The Riemann-Hilbert Problem for the decaying Toda Lattice is reduced asymptotically to two factorization problems, their contours being small crosses, centered at $z_{sp}, z_{sp}^*$ respectively, that can be solved explicitly!!! in terms of parabolic cylinder functions.

cf. the linear stationary phase method, where the dominating contribution to an exponential integral comes from the vicinity of the stationary phase points.
In fact, after some rescaling and approximating the "local" integrals can be computed explicitly.
Similar things happen here, although more technical.
Approximating a Riemann-Hilbert Problem by another requires some harmonic analysis and introducing some singular equations, equivalent to the given RHPs.

# Important Observation

To make use of the symmetry one needs to use both copies of the spectral complex plane, that is both sheets of the underlying hyperelliptic curve. In the case of genus zero, they can be mapped to the complex plane via the Joukowski transformation. This is no more possible in thbe case of the periodic lattice with genus greater than zero. We therefore need to postulate the Riemann-Hilbert problem on a Riemann surface.

# Important Observation

The inverse scattering theory for the perturbed periodic lattice was constructed by Egorova-Michor-Teschl (2005). Jost functions are defined as solution of the eigenvalue problem with asymptotics defined in terms of the Baker-Akhiezer functions of genus g.

$$\lim_{n\to\pm\infty} w(z)^{\mp n}(\psi_{\pm}(z,n,t) - \psi_{q,\pm}(z,n,t)) = 0,$$

where $w(z)$ is the quasimomentum map

$$w(z) = \exp(\int_{E_0}^{p} \omega_{\infty_+,\infty_-}), \;\; p = (z,+).$$

# Factorization Problem on a Hyperelliptic Curve

The analogous problem on the hyperelliptic curve is

$$m_+(p,n,t) = m_-(p,n,t)J(p,n,t)$$

$$J(p,n,t) = \begin{pmatrix} 1-|R(p)|^2 & -\bar{R}(p)\bar{\Theta}(p,n,t)e^{-t\phi(p)} \\ R(p)\Theta(p,n,t)e^{t\phi(p)} & 1 \end{pmatrix},$$

where $\Theta(p,n,t) = \frac{\theta(\underline{z}(p,n,t))}{\theta(\underline{z}(p,0,0))}\frac{\theta(\underline{z}(p^*,0,0))}{\theta(\underline{z}(p^*,n,t))}$
and
$\phi(p,\frac{n}{t}) = 2\int_{E_0}^{p}\Omega_0 + 2\frac{n}{t}\int_{E_0}^{p}\omega_{\infty_+,\infty_-} \in \mathrm{i}\mathbb{R}$
for $p \in \Sigma$. Here $\omega_{\infty_+\infty_-}$ is the Abelian differential of the third kind with poles at $\infty_+$ and $\infty_-$ and $\Omega_0$ is the Abelian differential of the second kind with second order poles at $\infty_+, \infty_-$. All Abelian differentials are normalized to have vanishing $a_j$ periods.

# Divisor Conditions

But $m$ is NO MORE HOLOMORPHIC off the jump contour!
The appropriate divisor condition is that
$div(m_{j1}) \geq -div\mu(n,t), \quad div(m_{j2}) \geq -div\mu^*(n,t), \qquad j = 1,2$
where the poles are at the Dirichlet eigenvalues for the periodic problem.
Also we have bad conditions at the two infinities:
$m(\infty_+, n, t) = \begin{pmatrix} A_+(n,t) & \frac{1}{A_+(n,t)} \end{pmatrix}.$
$m(\infty_-, n, t) = \begin{pmatrix} \frac{1}{A_+(n,t)} & A_+(n,t) \end{pmatrix}.$
If we want to get rid of the bad condition at $\infty_+$ then we need a symmetry. Then we can consider the normalized RHP and extract the solution via
$A_+(n,t) = \sqrt{\frac{1+(m_{12}(\infty_-,n,t))}{(m_{11}(\infty_-,n,t))}}$

# Stationary Phase Analysis

There are $2(g+1)$ stationary phase points ($g+1$ on each sheet) but only at most 2(1 on each sheet) in a band. So the others do not contribute (except an exponentially small error).

Gaps correspond to a periodic asymptotics but bands correspond to a continuously modulated lattice.

The geometry of the lenses is more delicate because of the Riemann surface background.

Also the auxilliary scalar RHP has to be meromorphic, because of the Riemann-Roch theorem.

# A generalized Cauchy kernel

In the complex plane, the solution of a Riemann–Hilbert problem can be reduced to the solution of a singular integral equation. In our case the underlying space is a Riemann surface. Hence we have to replace the classical Cauchy kernel by a "generalized" Cauchy kernel appropriate to our Riemann surface. In order to get a single valued kernel we need again to admit $g$ poles. Allowing poles at the nonspecial divisor $div(\mu)$ the corresponding Cauchy kernel is given by

# A generalized Cauchy kernel

$\Omega_p^\mu = \omega_{p\infty_+} + \sum_{j=1}^g I_j(p)\zeta_j,$

where $I_j(p) = \sum_{l=1}^g c_{jl} \int_{\infty_+}^p \omega_{\mu_l,0},$

$\omega_{q,0}$ is the Abelian differential of second kind with second order pole at $q$ and s.t. $\int_{\alpha_k} \omega_{q,0} = 0, \quad \textit{all } k$
and $\zeta_j$ is a basis of holomorphic differentials.
Note that $I_j(p)$ has first order poles at the points $\mu_l$.
The constants $c_{jl}$ are chosen such that $\Omega_p$ is single valued.

# Connection with a Singular Integral Equation

THEOREM: Set

$\underline{\Omega}_p^\mu = \begin{pmatrix} \Omega_p^\mu & 0 \\ 0 & \Omega_p^{\mu^*} \end{pmatrix}$ and define the matrix operators as follows.

Given a 2x2 matrix $f$ defined on $\Sigma$ with Hölder continuous entries, let

$(Cf)(p) = \frac{1}{\pi}\int_\Sigma f\underline{\Omega}_p$, for $p \notin \Sigma$, and $(C_\pm f)(q) = \lim_{p\to q\in\Sigma}(Cf)(p)$ from the left and right of $\Sigma$ respectively (with respect to its orientation). Then:

1. The operators $C_\pm$ are given by the Plemelj formulas

$$(C_+ f)(q) - (C_- f)(q) = f(q),$$
$$(C_+ f)(q) + (C_- f)(q) = \frac{1}{\pi}\, PV \int_\Sigma f\underline{\Omega}_q^\mu,$$

and extend to bounded operators on $L^2(\Sigma)$.

2. $Cf$ is a meromorphic function off $\Sigma$, with divisor given by $((Cf)_{j1}) \geq -div(\mu)$ and $((Cf)_{j2}) \geq -div(\mu^*)$.

3. $(Cf)(\infty_+) = 0$.

# Connection with a Singular Integral Equation

Now, given any $b_-, b_+ \in L^\infty(\Sigma)$ with determinant equal to 1, let the operator $C_w : L^2(\Sigma) \to L^2(\Sigma)$ be defined by $C_w f = C_+(fw_-) + C_-(fw_+)$ for a $2 \times 2$ matrix valued $f$, where $w_+ = b_+ - I, \; and \; w_- = I - b_-$.

THEOREM. Assume that $\mu$ solves the singular integral equation $\mu = I + C_w\mu$ in $L^2(\Sigma)$. Let $Q$ be defined by the integral formulae

$$Q = I + C(\mu w), \quad on \quad M \setminus \Sigma,$$

where $w = w_+ + w_-$. Then $Q$ is a solution of the following meromorphic Riemann–Hilbert problem.

$$\begin{aligned} Q_+(p) = Q_-(p)b_-^{-1}(p)b_+(p), \; p \in \Sigma, \\ Q(\infty_+) = I, \\ (Q_{j1}) \geq -div(\mu), \quad (Q_{j2}) \geq -div(\mu^*). \end{aligned}$$

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