INVERSE PROBLEMS IN SPACETIME II: RECONSTRUCTION OF A LORENTZIAN MANIFOLD FROM LIGHT OBSERVATION SETS

YAROSLAV KURYLEV, MATTI LASSAS, GUNTHER UHLMANN

Abstract: *Let (M, g) be a globally hyperbolic Lorentzian manifold. The light observation set $\mathcal{P}_U(q)$ corresponding to the source point $q \in M$ and the open observation domain $U \subset M$ is the intersection of the future light cone $\mathcal{L}_q^+$ emanated from q and the domain U. Let p^- and p^+ be points in U such that there is a time-like curve $\mu \subset U$ from p^- to p^+. Also, let $V \subset M$ be a relatively compact and open set such that V in the causal past of p^+ but does not intersect the causal past p^-. Assume that one is given the set U, as a manifold, the conformal class of the metric $g|_U$, and the collection of the light observation sets $\mathcal{P}_U(q)$ for all $q \in V$. This corresponds to the observations in U that one obtains from point sources located in V. These data is shown to uniquely determine the topological and differentiable structures of V and the conformal class of the metric g on V.*

AMS classification: 53C50, 35J25, 83C05.

Keywords: Inverse problems, Lorentzian manifolds.

1. Introduction and main results

We consider the inverse problem of the reconstruction of a region of a smooth Lorentzian manifold (M, g) of type $(1, n-1)$, $n \geq 3$, from light observation sets. Physically, this corresponds to the case of a passive observer, who registers light or gravitational waves coming from rapidly varying or suddenly appearing sources in the Universe, like e.g. supernova, quasars, variable stars, see [12]. Due to the existence of the conjugate points (or physically speaking, gravitational lensing or Einstein rings) such observations can by strongly distorted. In particular, the observation times of light signals, that have been caused by the same event but have traveled along different routes through a gravitational lens, may differ more than a year [21]. In this paper we assume that such point sources form a dense set in a region V of M. Then, we first show that V can be reconstructed as a topological manifold from these data. After that, we show that the differentiable structure of V and the conformal class of $g|_V$ can be reconstructed. Though the main interest of this paper is on passive observations, its results are used in our related paper [15] where we consider inverse problems with active observers which produce special

Date: May 14, 2014.

sources in order to imitate points sources of the type considered in this paper. The results are also related to other inverse scattering problems in relativity, see e.g. [8, 9, 22].

To formulate the result, we first introduce some definitions. The metric signature of g is $(-,+,+,\dots,+)$. In this paper we assume that (M,g) is time-oriented so that we can define future and past pointing time-like and causal paths. For $p,q\in M$ we denote $p\ll q$ if $p\neq q$ and there is a future pointing time-like path from p to q. We denote $p<q$, if $p\neq q$ and there is a future pointing causal path from p to q and denote $p\leq q$ when either $p=q$ or $p<q$. By the chronological future of $p\in M$, $I^+(p)$, we mean the set of all points q, such that $p\ll q$. By the causal future of p, $J^+(p)$, we mean the set of all points q, such that $p\leq q$. Similarly, we introduce the chronological past, $I^-(p)$, and the causal past, $J^-(p)$, see [19]. Note that $I^\pm(p)$ are always open. In this paper, we assume that (M,g) is *globally hyperbolic*, see Sec. 2 below. In this case, $J^\pm(p)$ are closed and $\mathrm{cl}(I^\pm(p))=J^\pm(p)$. To the knowledge of the authors, it is not presently known if there exist homeomorphic 4-dimensional, globally hyperbolic, Lorentzian manifolds that are not diffeomorphic but they are known to exist in higher dimensions, see [6]. The topology of such manifolds, even with asymptotically flat vacuum spacetimes can be very complicated, [10, 11]. A map $\Psi:(V_1,g_1)\to(V_2,g_2)$ is a conformal diffeomorphism if $\Psi: V_1\to V_2$ is a diffeomorphism and $\Psi_*g_1=e^{2f(x)}g_2(x)$ with some scalar function $f(x)$. We say that a curve $\alpha([t_1,t_2])$ is a pre-geodesic if $\alpha(t)$ is C^1-smooth curve such that $\dot\alpha(t)\neq 0$ on $t\in[t_1,t_2]$, and $\alpha([t_1,t_2])$ can be re-parametrized so that it becomes a geodesic. A conformal diffeomorphism preserves the light-like pre-geodesics by [3, Th. 9.17]. Also, we say that $\Psi: V_1\to V_2$ preserves causality if $x<y$ implies that $\Psi(x)<\Psi(y)$.

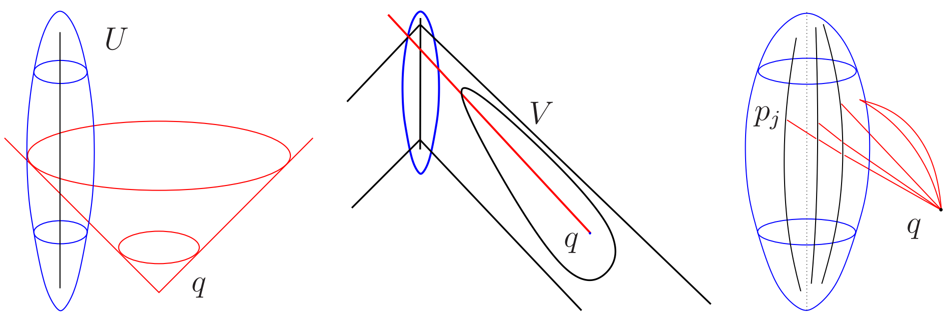

Figure 1. Left: *The future light cone $\mathcal{L}_q^+$ from the point q is shown as a red cone. The point q is the tip of the cone. The observation set U is shown in blue. The light observation point set $\mathcal{P}_U(q)$ with a point source at q is the intersection $\mathcal{L}_q^+\cap U$.* **Middle:** *In Thm. 1.2, we consider a set $V\subset I^-(p^+)\setminus J^-(p^-)$. The boundary of V is shown in the figure as a black curve. The red line is a light ray from a point $q\in V$ that is observed in the blue set U. These observations are shown to determine V as a differentiable manifold and the conformal class of the metric on it.*

Right: *The black curves are the paths μ_{a_j} and the red curves are light-like geodesics from q. Some light rays intersect μ_{a_j} at $p_j = \mu_{a_j}(f_{a_j}(q))$ before their cut points, but light-rays from q may intersect a geodesics μ_a after their cut points. For any $q_0 \in V$ we can find $a_j \in \mathcal{A}$, $j = 1, 2, \dots, n$ and a neighborhood of q_0 where the observation time functions $q \mapsto f_{a_j}(q)$ define a smooth coordinate system.*

Let $L_pM = \{\xi \in T_pM \setminus \{0\};\ g(\xi, \xi) = 0\}$. Also, $L_p^+M \subset L_pM$ and $L_p^-M \subset L_pM$ denote the future and the past light-like vectors in T_pM. For $W \subset M$, let $L^+W = \bigcup_{p\in W} L_p^+M \subset TM$. Let $\exp_q : T_qM \to M$ be the exponential map on (M, g). The geodesic starting at p in the direction $\xi \in T_pM \setminus \{0\}$ is the curve $\gamma_{p,\xi}(t) = \exp_p(\xi t)$, $t \geq 0$.

Let $\mu : [-1, 1] \to M$ be a smooth future pointing time-like path and $U \subset M$ be an open neighborhood of $\mu([-1, 1])$. Let $-1 < s_- < s_+ < 1$ and $p^\pm = \mu(s_\pm)$. Let $V \subset I^-(p^+) \setminus J^-(p^-)$ be a relatively compact open set, see Fig. 1 (Left and Middle).

Definition 1.1. *For $q \in V$, let $\mathcal{L}_q^+ = \exp_q\left(L_q^+M\right) \cup \{q\} = \{\gamma_{q,\xi}(t) \in M;\ \xi \in L_q^+M,\ t \geq 0\}$. The light observation set of q is*

$$\mathcal{P}_U(q) = \mathcal{L}_q^+ \cap U \in 2^U.$$

The collection of these sets is $\mathcal{P}_U(V) = \{\mathcal{P}_U(q);\ q \in V\} \subset 2^U$. Note $\mathcal{P}_U(V)$ is defined as an unindexed set, that is, for an element $\mathcal{P}_U(q) \in \mathcal{P}_U(V)$ we do not know what is the corresponding point q.

Above, $2^U = \{U';\ U' \subset U\}$ is the power set of U. Below, when $\Phi : U_1 \to U_2$ is a map, we say that the power set extension of Φ is the map $\widetilde{\Phi} : 2^{U_1} \to 2^{U_2}$ given by $\widetilde{\Phi}(U') = \{\Phi(z);\ z \in U'\}$ for $U' \subset U$.

Theorem 1.2. *Let (M_j, g_j), $j = 1, 2$, be two open, C^∞-smooth, time-oriented, globally hyperbolic Lorentzian manifolds of type $(1, n-1)$, $n \geq 3$. Let $\mu^{(j)} : [-1, 1] \to M_j$ be smooth time-like paths on (M_j, g_j) and U_j be open neighborhoods of $\mu^{(j)}([-1, 1])$. Assume that there exists a conformal diffeomorphism $\Phi : (U_1, g_1|_{U_1}) \to (U_2, g_2|_{U_2})$ satisfying $\Phi(\mu^{(1)}(s)) = \mu^{(2)}(s)$ for $s \in [-1, 1]$. Let $p_j^\pm = \mu^{(j)}(s_\pm)$, $-1 < s_- < s_+ < 1$, and V_j be open, relatively compact subsets of $I^-(p_j^+) \setminus J^-(p_j^-) \subset M_j$ such that the power set extension $\widetilde{\Phi}$ of Φ satisfies*

$$\widetilde{\Phi}(\mathcal{P}_{U_1}(V_1)) = \mathcal{P}_{U_2}(V_2). \tag{1}$$

Then there is a conformal diffeomorphism $\Psi : (V_1, g_1|_{V_1}) \to (V_2, g_2|_{V_2})$ that preserves the causality. Moreover, if $U_1 \cap V_1 \neq \emptyset$, then $\Phi_{U_1\cap V_1} = \Psi_{U_1\cap V_1}$.

In addition to optical or X-ray astronomy, Theorem 1.2 may be applied to gravitational wave astronomy: If the events in the early Universe have created enough gravitational waves, so-called primordial gravitational waves [16], by Thm. 1.2 the structure of spacetime, could in principle be determined using gravitational wave observations, even before the recombination time when the Cosmic Microwave Background (CMB) emerged. By recent observations, it appears that measurements of such gravitational waves can be done by observing their effect on the CMB, [5].

When M_j, $j = 1, 2$, have significant Ricci-flat parts, Theorem 1.2 can be strengthened.

Corollary 1.3. *Assume that (M_j, g_j) and U_j, V_j, $j = 1, 2$ satisfy the conditions of Theorem 1.2 with the resulting conformal map $\Psi : V_1 \to V_2$ as in Theorem 1.2. Moreover, assume that $\Phi|_{U_1} : (U_1, g_1) \to (U_2, g_2)$ is an isometry and V_j are Ricci-flat. Also, assume that all components of V_j intersect U_j, $j = 1, 2$. Then the map Ψ is an isometry.*

2. Preliminary constructions

A Lorentzian manifold (M, g) of type $(1, n-1)$ is globally hyperbolic if (M, g) has no non-trivial closed causal curves and for any $q < p$, the set $J(q, p) = J^+(q) \cap J^-(p)$ is compact, see [4] and [19, Def. 14.20]. Throughout this paper we assume that (M, g) is globally hyperbolic.

In addition to the Lorentzian metric g, we introduce on M a smooth Riemannian metric g^+, see [18], which makes it possible to introduce a Riemannian distance on M and a Sasaki distance in TM.

Let $\mu_a^{(1)} : [-1, 1] \to U^{(1)}$ be a family of future pointing time-like paths indexed by $a \in \mathcal{A}$, where $\mathcal{A}$ is a metric space and $\mu^{(1)} = \mu_{a_0}^{(1)}$ with $a_0 \in \mathcal{A}$. We assume that $(a, s) \mapsto \mu_a^{(1)}(s)$ is a continuous and open map $\mathcal{A} \times [-1, 1] \to M_1$.

We define $\mu_a^{(2)}(s) = \Phi(\mu_a^{(1)}(s))$ for $a \in \mathcal{A}$ and $s \in [-1, 1]$. Then, by taking the sets U_1 and $U_2 = \Phi(U_1)$ smaller, we may assume that U_j are open sets of the from

$$U_j = \bigcup_{a \in \mathcal{A}} \mu_a^{(j)}([-1, 1]), \quad j = 1, 2. \tag{2}$$

Below, we assume that (2) is valid.

Remark 2.1. *Given a smooth time-like path $\mu^{(1)} : [-1, 1] \to U_1$, its neighborhood $U^{(1)}$, and the metric tensor g_1 in U_1, we can alway construct the paths $\mu_a^{(1)}$ satisfying the above assumptions and make then U smaller so that (2) is valid. Indeed, let $z_0 = \mu^{(1)}(-1)$ and $\eta_0 = \dot\mu^{(1)}(-1)$ and let Z_j, $j = 1, 2, \dots, n$ be a frame of vectors at $\mu(-1)$ and $Z_j(s)$, $s \in [-1, 1]$, be parallel translation of this frame along $\mu^{(1)}([-1, s])$. Moreover, let $\kappa^j(s)$ be such that $\nabla_{\dot\mu^{(1)}(s)} \dot\mu^{(1)}(s) = \kappa^j(s) Z_j(s)$. Let W be an ε_0-neighborhood of (z_0, η_0) in TM in the g^+-Sasaki metric and let $\mathcal{A}$ consists of $a = (z, \eta, (Y_j)_{j=1}^n)$ where $(z, \eta) \in W$ and $(Y_j)_{j=1}^n$ is a frame in $T_z M$ such that the g^+-Sasaki distance of (z, Y_j) to (z_0, Z_j) is less than ε_0. This space can be endowed with a structure that makes it an $n(n+2)$-dimensional Riemannian manifold. Then we can define $\mu_a^{(1)}$ to be the path with $\mu_a^{(1)}(-1) = z$, $\dot\mu_a^{(1)}(-1) = \eta$, and $\nabla_{\dot\mu_a^{(1)}(s)} \dot\mu_a^{(1)}(s) = \kappa^j(s) Y_j(s)$, where*

$Y_j(s)$ are the parallel translation of Y_j along the curve $\mu_a^{(1)}([-1,s])$. When $\varepsilon_0 > 0$ is small enough, we know that these curves exist and $\mu_a^{(1)}([-1,1]) \subset U_1$ for all $a \in \mathcal{A}$. Then we can replace the neighborhood U_1 of $\mu^{(1)}([-1,1])$ by the set defined by formula (2). Finally, note that if $\mu^{(1)}$ is a unit speed time-like geodesic, that is, a freely falling observer, then all above constructed $\mu_a^{(1)}$ are time-like geodesics.

To simplify notations, let us continue with the constructions on just one Lorentzian manifold, (M,g) and assume that we are given the data

$$\begin{aligned}&\text{the differentiable manifold } U, \text{ the conformal class of } g|_U,\\ &\text{the paths } \mu_a : [-1,1] \to U,\ a \in \mathcal{A}, \text{ and the set } \mathcal{P}_U(V),\end{aligned} \tag{3}$$

where $V \subset I^-(p^+)\setminus J^-(p^-)$ is a relatively compact open set. Note that the set U and the conformal class of $g|_U$ determines all light-like pre-geodesics in U by [3, Th. 9.17].

Let $s_{-2} \in (-1, s_-)$ and $s_{+2} \in (s_{+1}, 1)$ and $p_{\pm 2} = \mu_{a_0}(s_{\pm 2})$. By making the set $\mathcal{A}$ smaller if necessary, we may assume that for all $a \in \mathcal{A}$ we have

$$\mu_a(s_{-2}) \in I^+(\mu_{a_0}(-1)) \cap I^-(p^-),\ \ \mu_a(s_{+2}) \in I^-(\mu_{a_0}(1)) \cap I^+(p^+). \tag{4}$$

Let us consider points $x, y \in M$. For $x < y$, we define the time separation function $\tau(x,y) \in [0,\infty)$ to be the supremum of the lengths $L(\alpha) = \int_0^1 \sqrt{-g(\dot\alpha(s), \dot\alpha(s))}\, ds$ of the piecewise smooth causal paths $\alpha : [0,1] \to M$ from x to y. If the condition $x < y$ does not hold, we define $\tau(x,y) = 0$. We note that $\tau(x,y)$ satisfies the reverse triangle inequality

$$\tau(x,y) + \tau(y,z) \le \tau(x,z) \quad \text{for } x \le y \le z. \tag{5}$$

As M is globally hyperbolic, the time separation function $(x,y) \mapsto \tau(x,y)$ is continuous in $M \times M$ by [19, Lem. 14.21] and by [19, Lem. 14.22], the sets $J^\pm(q)$ are closed. For $q < p$ there is a causal geodesic $\gamma([0,1])$ with $\gamma(0) = q$ and $\gamma(1) = p$ such that $L(\gamma) = \tau(q,p)$, see [19, Lem. 14.19]. This geodesic is called a longest path from q to p.

When (x,ξ) is a non-zero vector, we define $\mathcal{T}(x,\xi) \in (0,\infty]$ to be the maximal value for which $\gamma_{x,\xi} : [0, \mathcal{T}(x,\xi)) \to M$ is defined.

For $(x,\xi) \in L^+M$, $x \in J^-(p^+)$, the value $T_{+2}(x,\xi) = \sup\{t \ge 0\,;\ \gamma_{x,\xi}(t) \in J^-(p_{+2})\}$ is finite by [19, Lem. 14.13]. Since $J^-(p_{+2})$ is closed and $\gamma_{x,\xi}$ are future-pointing curves, $T_{+2} : L^+W \to \mathbb{R}$, $W = J^-(p_{+2})$, is upper semicontinuous. Moreover, since the set

$$K = \{(x,\xi) \in L^+M;\ x \in \mathrm{cl}\,(V),\ \|\xi\|_{g^+} = 1\} \tag{6}$$

is compact, there is $c_0 \in \mathbb{R}_+$ such that $T_{+2}(x,\xi) \le c_0$ for all $(x,\xi) \in K$.

For $(x,\xi) \in L^+M$, we define the cut locus function

$$\rho(x,\xi) = \sup\{s \in [0, \mathcal{T}(x,\xi));\ \tau(x, \gamma_{x,\xi}(s)) = 0\}, \tag{7}$$

c.f. [3, Def. 9.32]. The points $x_1 = \gamma_{x,\xi}(t_1)$ and $x_2 = \gamma_{x,\xi}(t_2)$, $t_1, t_2 \in [0, t_0]$, $t_1 < t_2$, are cut points on $\gamma_{x,\xi}([0,t_0])$ if $t_2 - t_1 = \rho(x_1, \xi_1)$ where $\xi_1 = \dot\gamma_{x,\xi}(t_1)$. In particular, the point $p(x,\xi) = \gamma_{x,\xi}(s)|_{s=\rho(x,\xi)}$, if it exists, is called the first cut point on the geodesic $\gamma_{x,\xi}([0,\mathcal{T}(x,\xi))$. Using [3, Thm. 9.33], we see that the function $\rho(x,\xi)$ is lower semi-continuous on a globally hyperbolic Lorentzian manifold (M,g).

Recall that $\gamma_{x,\xi}(t)$ is a conjugate point on $\gamma_{x,\xi}([0,\mathcal{T}(x,\xi))$ if the differential of the map $\exp_x$ is not invertible at $t\xi$. By [3, Th. 9.15], on a globally hyperbolic manifold, $p(x,\xi)$ is either the first conjugate point along $\gamma_{x,\xi}$, or the first point on $\gamma_{x,\xi}$ where there is another light-like geodesic $\gamma_{x,\eta}$ from x to $p(x,\xi)$, $\eta \neq c\xi$.

Returning to the longest paths, if $q<p$ but $\tau(q,p)=0$, then there is a light-like geodesic $\gamma_{q,\xi}([0,t])$ from q to p so that there are no cut points on $\gamma_{q,\xi}([0,t))$, see [19, Thm. 10.51 and Prop. 14.19]. Note that if $\gamma_{q,\xi}([0,t])$ is a light-like geodesic from q to $p=\gamma_{q,\xi}(t)$ such that there are cut-points on the geodesic $\gamma_{q,\xi}([0,t))$, (5) and (7) yield $\tau(q,p)>0$.

Moreover, it follows from [19, Prop. 10.46] that if q can be connected to p with a causal path which is not a light-like pre-geodesic then $\tau(q,p)>0$. Let us apply this fact to a path from q to p which is the union of the future pointing light-like pre-geodesics $\gamma_{q,\eta}([0,t_0])\subset M$ and $\gamma_{x_1,\theta}([0,t_1])\subset M$, where $x_1=\gamma_{q,\eta}(t_0)$, $p=\gamma_{x_1,\theta}(t_1)$ and $t_0,t_1>0$. Let $\xi=\dot\gamma_{q,\eta}(t_0)$. Then, if there is no $c>0$ such that $\xi=c\theta$, or equivalently, the union of these geodesic is not a light-like pre-geodesics, we have $\tau(q,p)>0$. In particular, this implies that there exists a time-like geodesic from q to p. In the following we call this kind of argument for a union of light-like geodesics a short-cut argument.

2.0.1. *Observation time functions.*

Definition 2.2. *Let $a\in\mathcal{A}$, and $q\in J^-(p^+)\setminus I^-(p^-)$. The observation time function $f_a:\ J^-(p^+)\setminus I^-(p^-)\to[-1,1]$ is defined by*

$$f_a(q)=\inf(\{s\in([-1,1];\ \mu_a(s)\in J^+(q)\}\cup\{1\}).$$

Moreover, let $\mathcal{E}_a(q)=\mu_a(f_a(q))$.

Above, $\mathcal{E}_a(q)$ is the earliest point on μ_a at which light from q is observed. The following lemma is a slight generalization of [15, Lemma 2.2]. We repeat its proof for the convenience of the reader.

Lemma 2.3. *Let $a\in\mathcal{A}$ and $q\in J^-(p^+)\setminus I^-(p^-)$. Then*

(i) It holds that $s_{-2}\le f_a(q)\le s_{+2}$.

(ii) We have $\mathcal{E}_a(q)\in J^+(q)$ and $\tau(q,\mathcal{E}_a(q))=0$. Moreover, the function $s\mapsto\tau(q,\mu_a(s))$ is continuous, non-decreasing on the interval $s\in[-1,1]$ and is strictly increasing on $[f_a(q),1]$.

(iii) Assume that $p\in U$. Then $p=\mathcal{E}_a(q)$ with some $a\in\mathcal{A}$ if and only if $p\in\mathcal{P}_U(q)$ and $\tau(q,p)=0$. Furthermore, these are equivalent to the fact that there are $\xi\in L^+_qM$ and $t\in[0,\rho(q,\xi)]$ such that $p=\gamma_{q,\xi}(t)$.

(iv) The function $q\mapsto f_a(q)$ is continuous on $J^-(p^+)\setminus I^-(p^-)$.

Proof. (i) This property follows from (4).

(ii) Since $J^+(q)$ is closed, $\mathcal{E}_a(q) \in J^+(q)$. The continuity of $\tau(q, \mu_a(s))$ follows from the continuity of $\tau(x, y)$ on $M \times M$.

If $\tau(q, \mu_a(f_a(q)))$ would be strictly positive, we would have $\mu_a(f_a(q)) \in I^+(q)$ and there would exist $s < f_a(q)$ such that $\mu_a(s) \in I^+(q)$. As this is not possible, we have $\tau(q, \mu_a(f_a(q))) = 0$.

Consider $s < s'$. Since μ_a is a time like-path, $\tau(\mu_a(s), \mu_a(s')) > 0$. Thus, when $s' > s \geq f_a(q)$, the inequality (5) yields $\tau(q, \mu_a(s)) < \tau(q, \mu_a(s'))$. For $s < f_a(q)$ we have $\mu_a(s) \notin J^+(q)$ and $\tau(q, \mu_a(s)) = 0$.

(iii) It is sufficient to prove the claim when $p \neq q$. First, assume that $p = \mathcal{E}_a(q)$. Then $p \in J^+(q)$ and by (ii), we have $\tau(q, p) = 0$. The existence of the light-like geodesic follows from the above.

Second, assume that $p \in J^+(q)$ and $\tau(q, p) = 0$. This implies by [19, Prop. 14.19] that there exists a light-like geodesic $\gamma_{q,\xi}([0, t])$ from q to p. If $\gamma_{q,\xi}([0, t))$ would have a cut-point, then $\tau(q, p) > 0$ which is not possible. Thus, $t \in [0, \rho(q, \xi)]$.

Third, assume that $p = \gamma_{q,\xi}(t)$ with $\xi \in L^+_q M$ and $0 \leq t \leq \rho(q, \xi)$. Then $\tau(q, p) = 0$. Let $a \in \mathcal{A}$ and $s_0 \in [-1, 1]$ be such that $p = \mu_a(s_0)$. By (i), $\tau(q, \mu_a(s)) > 0$ for $s > f_a(q)$ and thus $s_0 \leq f_a(q)$. However, $q \leq p = \mu_a(s_0)$ and thus $s_0 \geq f_a(q)$. Thus $s_0 = f_a(q)$ and $p = \mathcal{E}_a(q)$.

(iv) Assume that $x_j \to x$ in $J^-(p^+) \setminus I^-(p^-)$ as $j \to \infty$. Let $s_j = f_a(x_j)$ and $s = f_a(x)$. Since τ is continuous, for any $\varepsilon > 0$ we have $\lim_{j\to\infty} \tau(x_j, \mu_a(s + \varepsilon)) = \tau(x, \mu_a(s + \varepsilon)) > 0$ and thus for j large enough $s_j \leq s + \varepsilon$. Thus $\limsup_{j\to\infty} s_j \leq s$. Assume next that $\liminf_{j\to\infty} s_j = \widetilde{s} < s$ and denote $\varepsilon = \tau(\mu_a(\widetilde{s}), \mu_a(s)) > 0$. Then $\liminf_{j\to\infty} \tau(x_j, \mu_a(s)) \geq \liminf_{j\to\infty} \tau(\mu_a(s_j), \mu_a(s)) \geq \varepsilon$, and as τ is continuous in $M \times M$, we obtain $\tau(x, \mu_a(s)) \geq \varepsilon$, which is not possible as $s = f_a(x)$. This proves that $x \mapsto f_a(x)$ is continuous. $\square$

By Lemma 2.3 (iii), for any $q \in J^-(p^+) \setminus I^-(p^-)$ and $a \in \mathcal{A}$, we have $\mathcal{E}_a(q) \in \mathcal{P}_U(q)$. Since $\mathcal{P}_U(q) \subset J^+(q)$, we see using Def. 2.2 that the light observation set $\mathcal{P}_U(q)$ and the path μ_a determine the functions

$$f_a(q) = \min\{s \in [-1, 1];\ \mu_a(s) \in \mathcal{P}_U(q)\}, \quad \mathcal{E}_a(q) = \mu_a(f_a(q)). \tag{8}$$

2.0.2. *The set of the earliest observations.*

Definition 2.4. *Let $q \in J^-(p^+) \setminus I^-(p^-)$. Let*

$$\begin{aligned} \mathcal{D}_U(q) = \{(y, \eta) \in L^+U \quad ; \quad & y = \gamma_{q,\xi}(t) \in U,\ \eta = \dot\gamma_{q,\xi}(t), \\ & \textit{with some } \xi \in L^+_q M,\ 0 \leq t \leq \rho(x, \xi)\}, \\ \mathcal{D}^{reg}_U(q) = \{(y, \eta) \in L^+U \quad ; \quad & y = \gamma_{q,\xi}(t) \in U,\ \eta = \dot\gamma_{q,\xi}(t), \\ & \textit{with some } \xi \in L^+_q M,\ 0 < t < \rho(x, \xi)\}. \end{aligned} \tag{9}$$

We say that $\mathcal{D}_U(q)$ is the direction set of q and $\mathcal{D}^{reg}_U(q)$ is the regular direction set of q.

Let $\mathcal{E}_U(q) = \pi(\mathcal{D}_U(q))$ and $\mathcal{E}^{reg}_U(q) = \pi(\mathcal{D}^{reg}_U(q))$ where $\pi : TU \to U$ is the canonical projection, $\pi(y, \eta) = y$. We say that $\mathcal{E}_U(q)$ is the set of *the earliest observations of q in U and $\mathcal{E}_U^{reg}(q)$ is the set of regular earliest observation points of q. We denote the collection of the earliest observation sets by $\mathcal{E}_U(V) = \{\mathcal{E}_U(q) \in 2^U;\ q \in V\}$.*

Note that $\mathcal{E}_U(q) = \{\mathcal{E}_a(q);\ a \in \mathcal{A}\}$ and that the lower semicontinuity of $\rho(x,\xi)$ implies that $\mathcal{E}_U^{reg}(q) \subset U$ and $\mathcal{D}_U^{reg}(q) \subset TU$ are smooth $(n-1)$ dimensional submanifolds.

Assume that $x = \mu_a(s) \in \mathcal{P}_U(q)$. Then, if there is $y \in \mathcal{P}_U(q)$ such that $y \ll x$ then (5) implies that $\tau(q,x) \geq \tau(q,y) + \tau(y,x) > 0$. On the other hand, if there is no $y \ll x$, $y \in \mathcal{P}_U(q)$, it follows from the proof of Lemma 2.3(iii) that $y \in \mathcal{E}_U(q)$. Hence

$$\mathcal{E}_U(q) = \{x \in \mathcal{P}_U(q);\ \text{there are no } y \in \mathcal{P}_U(q) \text{ such that } y \ll x\}. \tag{10}$$

3. Constructive solution of the inverse problem

Formulas (8) and $\mathcal{E}_U(q) = \{\mathcal{E}_a(q);\ a \in \mathcal{A}\}$ imply that $\mathcal{P}_U(V)$ determines the set $\mathcal{E}_U(V)$. Hence to prove Theorem 1.2 it is enough to prove the following result:

Theorem 3.1. *Assume the conditions of Theorem 1.2 are valid except for condition (1) which is replaced by*

$$\widetilde{\Phi}(\mathcal{E}_{U_1}(V_2)) = \mathcal{E}_{U_2}(V_2).$$

Then the conclusions of Theorem 1.2 remain valid.

Next we aim to prove this result. To simplify the notations, we return to the case when we have only one manifold (M,g) and assume that we are given data (3). We need the following auxiliary result:

Proposition 3.2. **(i)** *Let $y \in U$, $\eta \in L_y^+M$, $r_1 > 0$, and $q \in V$ be such that $q \notin \gamma_{y,\eta}([-r_1,0])$ and $\gamma_{y,\eta}([-r_1,0]) \subset U$. Then $(y,\eta) \in \mathcal{D}_U(q)$ if and only if $\gamma_{y,\eta}([-r_1,0]) \in \mathcal{E}_U(q)$.*

(ii) *Let $y \in U$, $\eta \in L_y^+M$, and $\widehat{t} > 0$ be the largest number such that the geodesic $\gamma_{y,\eta}((-\widehat{t},0])$ is defined and has no cut points. Then for $q \in V$ we have $q \in \gamma_{y,\eta}((-\widehat{t},0))$ if and only if $(y,\eta) \in \mathcal{D}_U^{reg}(q)$.*

Proof. (i) Suppose $(y,\eta) \in \mathcal{D}_U(q)$. Then $y \in \mathcal{E}_U(q)$ and $\tau(q,y) = 0$. Since $q \notin \gamma_{y,\eta}([-r_1,0]) \subset U$, there is $t > r_1$ such that $\gamma_{y,\eta}(-t) = q$ and for $\xi = \dot\gamma_{y,\eta}(-t)$ we have $\gamma_{y,\eta}([-r_1,0]) = \gamma_{q,\xi}([t-r_1,t]) \subset \mathcal{P}_U(q)$. If there would be $y_1 \in \gamma_{y,\eta}([-r_1,0])$ such that $y_1 \notin \mathcal{E}_U(q)$, it follows from (10) that there is $z \in \mathcal{P}_U(q)$ such that $z \ll y_1$. Then we would have $z \ll y_1 \leq y$ and $y \in \mathcal{E}_U(q)$ which is not possible by (10). This shows that $\gamma_{y,\eta}([-r_1,0]) \subset \mathcal{E}_U(q)$.

On the other hand, assume that $\gamma_{y,\eta}([-r_1,0]) \subset \mathcal{E}_U(q)$. Then Lemma 2.3(ii) implies $\tau(q,y) = 0$. Denote $y_1 = \gamma_{y,\eta}(-r_1)$. Since $y_1 \in \mathcal{E}_U(q)$ and $y_1 \neq q$, there is $\xi \in L_q^+M$ and $t_1 > 0$ such that $\gamma_{q,\xi}(t_1) = y_1$. Then the union of the geodesics $\gamma_{q,\xi}([0,t_1])$ and $\gamma_{y,\eta}([-r_1,0])$ form a causal path from q to y. Using short cut arguments, we see that if the union of these geodesics do not form one light-like pre-geodesic, we have $\tau(q,y) > 0$, that is not possible. Hence $\gamma_{y,\eta}([-r_1,0])$ lies in the continuation of $\gamma_{q,\xi}([0,t_1])$, that is, there is $t > 0$ such that $\gamma_{y,\eta}([-r_1,0]) \subset \gamma_{q,\xi}([0,t])$ and $y = \gamma_{q,\xi}(t)$. Then, there is $c > 0$ such that $\eta = c\dot\gamma_{q,\xi}(t)$. Moreover, if $\gamma_{q,\xi}([0,t))$ would contain cut points then [19, Prop. 10.46] implies $\tau(q,y) > 0$, and this would lead to a contradiction with $y \in \mathcal{E}_U(q)$. Hence, $\gamma_{q,\xi}([0,t))$ contains no cut points. Thus, we have shown that $t \le \rho(q,\xi)$, $y = \gamma_{q,\xi}(t)$, and $\eta = c\dot\gamma_{q,\xi}(t)$. These imply that $(y,\eta) \in \mathcal{D}_U(q)$.

(ii) Let $(y,\eta) \in L^+U$ and $\widehat{t} > 0$ be as in the claim and $q \in V$.

First, assume that $q \in \gamma_{y,\eta}(-t_1)$, $t_1 \in (0,\widehat{t})$. Then, due to the symmetry of cut points, $\tau(q,y) = 0$ and thus for $\xi = \dot\gamma_{y,\eta}(-t_1)$ we have $y = \gamma_{q,\xi}(t_1)$ and $t_1 < \rho(q,\xi)$. Thus $(y,\eta) \in \mathcal{D}_U^{reg}(q)$.

Second, assume that $(y,\eta) \in \mathcal{D}_U^{reg}(q)$. Again, we see that there is $t_1 > 0$ such that $q = \gamma_{y,\eta}(-t_1)$ and for $\xi = \dot\gamma_{y,\eta}(-t_1)$ we have $y = \gamma_{q,\xi}(t_1)$ and $t_1 < R_1 := \rho(q,\xi)$. Since ρ is a lower semi-continuous, we see that when $\varepsilon \in (0,(R_1-t_1)/2)$ is small enough, the point $x_1 = \gamma_{q,\xi}(-\varepsilon)$ and $\xi_1 = \dot\gamma_{q,\xi}(-\varepsilon)$ satisfy $\rho(x_1,\xi_1) > R_1 - \varepsilon > t_1 + \varepsilon$ and hence $\tau(x_1,y) = 0$. This yields that $\widehat{t} > t_1$. Thus $q \in \gamma_{y,\eta}((-\widehat{t},0))$. □

Using this result we determine the direction sets $\mathcal{D}_U(q)$ from $\mathcal{E}_U(V)$:

Proposition 3.3. *Assume that we are given the data (3). Then*

(i) *For any $y \in U$, we can identify from the set $\mathcal{E}_U(V)$ the element $\mathcal{E}_U(q)$ for which $q = y$, if it exists. For such elements $L_y^+M \subset \mathcal{D}_U(q)$.*

(ii) *Let $q \in V$ and $(y,\eta) \in L^+U$. Then $(y,\eta) \in \mathcal{D}_U^{reg}(q)$ if and only if there exists a light-like pre-geodesic $\alpha([t_1,t_2]) \subset U$ such that $y = \alpha(t)$, $\eta = \dot\alpha(t)$, $t_1 < t < t_2$, and $\alpha([t_1,t_2]) \subset \mathcal{E}_U(q)$.*

(iii) *When $\mathcal{E}_U(q) \in \mathcal{E}_U(V)$ is given, one can determine the sets $\mathcal{D}_U(q)$, $\mathcal{D}_U^{reg}(q)$, and $\mathcal{E}_U^{reg}(q)$.*

Proof. (i) We observe that $q = y$ if and only if for $y \in \mathcal{E}_U(q)$ there are no $\eta \in L_y^+M$ and $t_0 > 0$ such that $\gamma_{y,\eta}([-t_0,0]) \subset \mathcal{E}_U(q)$. Claim (i) follows from this observation.

(ii) Let $q \in V$ and $\xi \in L_q^+V$ and $(y,\eta) = (\gamma_{q,\xi}(1), \dot\gamma_{q,\xi}(1))$. Using Definition 2.4 we see that $(y,\eta) \in \mathcal{D}_U^{reg}(q)$ if and only if $\gamma_{q,\xi}(1) \in U$ and $\rho(q,\xi) > 1$. This is equivalent to the fact that there are $t_1 \in (0,1)$ and $t_2 > 1$ such that $\gamma_{q,\xi}([t_1,t_2]) \subset U$ and $(\gamma_{q,\xi}(t_2), \dot\gamma_{q,\xi}(t_2)) \in \mathcal{D}_U(q)$. Also, by Lemma 3.2 (i) this is equivalent to the fact that there are $t_1 \in (0,1)$ and $t_2 > 1$ such that $\gamma_{q,\xi}([t_1,t_2]) \subset \mathcal{E}_U(q)$. This proves (ii).

(iii) Let $\mathcal{E}_U(q)$ be given. Since the conformal class of $g|_U$ is given, we can identify all light-like pre-geodesics in U. Thus by using (ii), we can verify for any $(y,\eta) \in L^+U$ whether it holds that $(y,\eta) \in \mathcal{D}_U^{reg}(q)$ or not. Thus we can determine the set $\mathcal{D}_U^{reg}(q)$. Then the set $\mathcal{D}_U(q)$ can be determined as the closure of the set $\mathcal{D}_U^{reg}(q)$ in TU. Finally, the set $\mathcal{E}_U^{reg}(q) = \pi(\mathcal{D}_U^{reg}(q))$ can be constructed using the map $\pi : TU \to U$. □

3.0.1. *Construction of V as a topological and differentiable manifold.* For $q \in J^-(p^+) \setminus I^-(p^-)$ we define the function $F_q : \mathcal{A} \to \mathbb{R}$ by $F_q(a) = f_a(q)$. Also, we denote by $\mathcal{F} : J^-(p^+) \setminus I^-(p^-) \to \mathbb{R}^{\mathcal{A}}$ the function $\mathcal{F}(q) = F_q$, that maps q to the function $F_q : \mathcal{A} \to \mathbb{R}$. We endow the set $\mathbb{R}^{\mathcal{A}}$ with the product topology.

By (8), the set $\mathcal{E}_U(q)$ determines $F_q = \mathcal{F}(q)$ via the formula

$$F_q(a) = s, \quad \text{where } s \in [-1,1] \text{ is such that } \mu_a(s) \in \mathcal{E}_U(q),\ a \in \mathcal{A}. \tag{11}$$

Also, $F_q = \mathcal{F}(q)$ determines $\mathcal{E}_U(q)$ via the formula

$$\mathcal{E}_U(q) = \{\mu_a(F_q(a));\ a \in \mathcal{A}\}. \tag{12}$$

Below, we consider the sets $\mathcal{F}(V) = \{\mathcal{F}(q);\ q \in V\} \subset \mathbb{R}^{\mathcal{A}}$ and $\mathcal{E}_U(V) = \{\mathcal{E}_U(q);\ q \in V\} \subset 2^U$ as two representations for V. We will construct the topological and differentiable structure of V using $\mathcal{F}(V)$ and the conformal class of the metric $g|_V$ using $\mathcal{E}_U(V)$.

Lemma 3.4. *Let $V \subset J^-(p^+) \setminus I^-(p^-)$ be a relatively compact open set. Then the map $\mathcal{F} : V \to \mathcal{F}(V)$ is a homeomorphism.*

Proof. By Lemma 2.3 (iv), the map $\mathcal{F} : J^-(p^+) \setminus I^-(p^-) \to \mathbb{R}^{\mathcal{A}}$ is continuous. Below, let $\overline{V} = \mathrm{cl}\,(V)$ be the closure of V in M.

Next we show that the map $\mathcal{F} : \overline{V} \to \mathcal{F}(\overline{V}) = \overline{\mathcal{F}(V)}$ is injective. Since $\mathcal{F}(q)$ determines the set $\mathcal{E}_U(q)$ uniquely by (12), it is enough to show that the map $\mathcal{E}_U : \overline{V} \to \mathcal{E}_U(\overline{V})$ is injective. To prove this, we assume the opposite: Assume that there are $q_1 \neq q_2$ that satisfy $\mathcal{E}_U(q_1) = \mathcal{E}_U(q_2)$. By Prop. 3.3 (iii), this implies

$$\mathcal{D}_U(q_1) = \mathcal{D}_U(q_2). \tag{13}$$

Choose $a \in \mathcal{A}$ such that $q_i \notin \mu_a$, $i = 1, 2$. Let $(p, \eta) \in \mathcal{D}_U(q_i)$ with $p = \mathcal{E}_a(q_i)$. Then there are $t_i > 0$ such that $q_i = \gamma_{p,\eta}(-t_i)$. Since $q_1 \neq q_2$, we have $t_1 \neq t_2$, and let us assume that $t_2 > t_1$. Then, we see there are $\xi_i \in L^+_{q_i}(M)$ such that

$$(p, \eta) = (\gamma_{q_i,\xi_i}(t_i), \dot\gamma_{q_i,\xi_i}(t_i)), \quad (q_1, \xi_1) = (\gamma_{q_2,\xi_2}(t_2 - t_1), \dot\gamma_{q_2,\xi_2}(t_2 - t_1)).$$

Since $\rho(q, \xi)$ is lower semicontinuous, for any $\delta_1 > 0$ there is $\delta_2 > 0$ such that $\rho(q_2, \xi_2') > \rho(q_2, \xi_2) - \delta_1$ when $\xi_2' \in T_qM$ satisfies $\|\xi_2' - \xi_2\| < \delta_2$. Choosing δ_1 and δ_2 to be sufficiently small, we have that there is $\xi_2' \in T_{q_2}M$ that is not parallel to ξ_2, $\|\xi_2' - \xi_2\| < \delta_2$, and $t_2' \in (t_2 - 2\delta_1, t_2 - \delta_1)$ such that $p' = \gamma_{q_2,\xi_2'}(t_2') \in U$, $p' \neq q_1$, and $t_2' < \rho(q_2, \xi_2')$. Then for $\eta' = \dot\gamma_{q_2,\xi_2'}(t_2')$ we have $(p', \eta') \in \mathcal{D}_U(q_2)$. By (13), $(p', \eta') \in \mathcal{D}_U(q_1)$, and hence there is $t_1' > 0$ such that $q_1 = \gamma_{p',\eta'}(-t_1')$.

Observe that $\xi_1' = \dot\gamma_{p',\eta'}(-t_1')$ and ξ_1 are not parallel. Then, we see that the union of the geodesic $\gamma_{q_2,\xi_2}([0, t_2 - t_1])$ and the geodesic $\gamma_{p',\eta'}([0, -t_1'])$, oriented in the opposite direction, form a causal curve from q_2 to p' that is not a light-like pre-geodesic, and hence $\tau(q_2, p') > 0$. This is not possible as $p' \in \mathcal{E}_U(q_2)$. This contradiction proves that $\mathcal{E}_U : \overline{V} \to \mathcal{E}_U(\overline{V})$ is injective.

Since $\mathbb{R}^{\mathcal{A}}$ is a Hausdorff space, $\overline{V}$ is a compact set, and the map $\mathcal{F}:\overline{V}\to\mathcal{F}(\overline{V})$ is continuous and injective, we have that $\mathcal{F}:\overline{V}\to\mathcal{F}(\overline{V})$ is a homeomorphism. Thus $\mathcal{F}:V\to\mathcal{F}(V)$ is a homeomorphism. □

Remark 3.5. *The above construction is similar to the Kuratowski embedding, $K:x\mapsto dist(x,\cdot)$, from the metric space N to space $C(N)$. Also, in several inverse problems for Riemannian manifolds with boundary, a homeomorphic image of the compact manifold N has been obtained by using the embedding $R:x\mapsto dist(x,\cdot)$, $R:N\to C(\partial N)$, see* [1, 13, 14].

Our aim is to introduce coordinates in $\mathcal{F}(V)$ near any $\mathcal{F}(q)$ that make $\mathcal{F}(V)$ diffeomorphic to V.

Let $\mathcal{Z}=\{(q,p)\in V\times U;\ p\in\mathcal{E}_U^{reg}(q)\}$. Then for every $(q,p)\in\mathcal{Z}$ we there is a unique $\xi\in L_q^+M$ such that $\gamma_{q,\xi}(1)=p$ and $\rho(q,\xi)>1$. We will denote $\Theta(q,p)=(q,\xi)$ that defines a map by $\Theta:\mathcal{Z}\to L^+V$. Below, let $\mathcal{W}_\varepsilon(q_0,\xi_0)\subset TM$ be an ε-neighborhood of (q_0,ξ_0) with respect to the Sasaki-metric induced by g^+ on TM.

Lemma 3.6. *Let $(q_0,p_0)\in\mathcal{Z}$ and $(q_0,\xi_0)=\Theta(q_0,p_0)$. When $\varepsilon>0$ is small enough, the map*

$$X:\mathcal{W}_\varepsilon(q_0,\xi_0)\to M\times M,\quad X(q,\xi)=(q,\exp_q(\xi)) \tag{14}$$

is open and defines a diffeomorphism $X:\mathcal{W}_\varepsilon(q_0,\xi_0)\to\mathcal{U}_\varepsilon(q_0,p_0):=X(\mathcal{W}_\varepsilon(q_0,\xi_0))$. When ε is small enough, Θ coincides in $\mathcal{Z}\cap\mathcal{U}_\varepsilon(q_0,p_0)$ with the inverse map of X. Moreover, $\mathcal{Z}$ is a $(2n-1)$-dimensional manifold and the map $\Theta:\mathcal{Z}\to L^+M$ is C^∞-smooth.

Proof. Since the geodesic $\gamma_{q_0,\xi_0}([0,1])$ does not contain cut points and thus conjugate points, we see that when $\varepsilon>0$ is small enough, the set $\mathcal{U}_\varepsilon(q_0,p_0)=X(\mathcal{W}_\varepsilon(q_0,\xi_0))\subset M\times M$ is open and the map $X:\mathcal{W}_\varepsilon(q_0,\xi_0)\to\mathcal{U}_\varepsilon(q_0,p_0)$ has a C^∞-smooth inverse map $X^{-1}:\mathcal{U}_\varepsilon(q_0,p_0)\to\mathcal{W}_\varepsilon(q_0,\xi_0)$. Thus $X:\mathcal{W}_\varepsilon(q_0,\xi_0)\to\mathcal{U}_\varepsilon(q_0,p_0)$ is a diffeomorphism. Note that $X^{-1}(q_0,p_0)=(q_0,\xi_0)$.

If $\Theta:\mathcal{Z}\to L^+V$ would not be continuous at $(q_0,p_0)\in\mathcal{Z}$, there would exists a sequence $(q_k,p_k)\in\mathcal{Z}$ converging to (q_0,p_0) as $k\to\infty$, such that $\Theta(q_k,p_k)\in L^+M$ does not converge to $(q_0,\xi_0)=\Theta(q_0,p_0)$. Since $p_k\in J^-(p_{+2})$ and the function T_{+2} is bounded by $c_0\in\mathbb{R}_+$ in the set K given in (6), the sequence $\|\Theta(q_k,p_k)\|_{g^+}$ is uniformly bounded, and by considering a subsequence we may assume that $\Theta(q_k,p_k)\to(q_0,\eta)\in L^+M$ as $k\to\infty$ and $\eta\neq\xi_0$. In this case the geodesics $\gamma_{q_0,\xi_0}([0,1])$ and $\gamma_{q_0,\eta}([0,1])$ would be two light-like geodesics connecting q_0 to p_0 so that $\rho(q_0,\xi_0)\leq 1$. This would be in contradiction with the assumption that $p_0\in\mathcal{E}_U^{reg}(q_0)$. This shows that $\Theta:\mathcal{Z}\to L^+V$ is continuous at (q_0,p_0).

Let $\varepsilon_1\in(0,\varepsilon)$ and $\mathcal{Y}_{\varepsilon_1}=\mathcal{Z}\cap\mathcal{U}_{\varepsilon_1}(q_0,p_0)$ be a neighborhood of (q_0,p_0) in the relative topology of $\mathcal{Z}\subset V\times U$. When ε_1 is small enough, we have $\Theta(\mathcal{Y}_{\varepsilon_1})\subset\mathcal{W}_\varepsilon(q_0,\xi_0)$. Then for $(q,p)\in\mathcal{Y}_{\varepsilon_1}$ and $(q,\xi)=\Theta(q,p)\in\mathcal{W}_\varepsilon(q_0,\xi_0)$ we have $\exp_q(\xi)=p$, and hence $X(\Theta(q,p))=(q,p)$. Since

$\Theta(q,p)\in\mathcal{W}_\varepsilon(q_0,\xi_0)$, we have $\Theta(q,p)=X^{-1}(q,p)$. Thus for $(q,p)\in\mathcal{Y}_{\varepsilon_1}$ the function $\Theta:\mathcal{Y}_{\varepsilon_1}\to TM$ coincides with the smooth function $X^{-1}:\mathcal{Y}_{\varepsilon_1}\to TM$. Since $(q_0,p_0)\in\mathcal{Z}$ is arbitrary, this shows that $\mathcal{Z}$ is a $(2n-1)$-dimensional manifold and $\Theta:\mathcal{Z}\to L^+M$ is C^∞-smooth. □

Proposition 3.7. *Let $q_0\in I^-(p^+)\setminus J^-(p^-)$ and $(q_0,p_j)\in\mathcal{Z}$, $j=1,2,\dots,n$ and $\xi_j\in L^+_{q_0}M$ be such that $\gamma_{q_0,\xi_j}(1)=p_j$. Assume that ξ_j, $j=1,2,\dots,n$ are linearly independent. Then, if $a_j\in\mathcal{A}$ and $\vec a=(a_j)_{j=1}^n$ are such that $p_j\in\mu_{a_j}$, there is a neighborhood $V_1\subset M$ of q_0 such that the corresponding observation time functions*

$$\mathbf{f}_{\vec a}(q)=(f_{a_j}(q))_{j=1}^n$$

define a C^∞-smooth coordinates in V_1. Moreover, $\nabla f_{a_j}|_{q_0}$, the gradient of f_{a_j} with respect to q at q_0 satisfies $\nabla f_{a_j}|_{q_0}=c_j\xi_j$ with some $c_j\neq 0$.

Proof. Let $(q_0,p_0)\in\mathcal{Z}$ and $\xi_0\in L^+_{q_0}M$ such that $\gamma_{q_0,\xi_0}(1)=p_0$. Moreover, let $\varepsilon>0$ be so small that the map $X:\mathcal{W}_\varepsilon(q_0,\xi_0)\to\mathcal{U}_\varepsilon(q_0,p_0)$ has a C^∞-smooth inverse, see (14). We denote this inverse map by $X^{-1}(q,p)=(q,\xi(q,p))$ and $\mathcal{W}=\mathcal{W}_\varepsilon(q_0,\xi_0)$ and $\mathcal{U}=\mathcal{U}_\varepsilon(q_0,p_0)$.

We associate with any $(q,p)\in\mathcal{W}$ the energy $E(q,p)=E(\gamma_{q,\xi(q,p)}([0,1]))$ of the geodesic segment $\gamma_{q,\xi(q,p)}([0,1])$ from p to q. Here, the energy of a piecewise smooth curve $\alpha:[0,l]\to M$ is defined by

$$E(\alpha)=\frac12\int_0^l g(\dot\alpha(t),\dot\alpha(t))\,dt.$$

Observe that the sign of $E(q,p)$ depends on the causal nature of $\gamma_{q,p}$. In particular, $E(q,p)=0$ if and only if $\xi(q,p)$ is light-like. Moreover, since X^{-1} is C^∞-smooth on $\mathcal{U}$, also $E(q,p)$ is C^∞-smooth in $\mathcal{U}$.

Let us return to consider $(q_0,p_0)\in\mathcal{Z}$ and let $a\in\mathcal{A}$ be such that $p_0\in\mu_a$. Then $p_0=\mu_a(s_0)$ with $s_0=f_a(q)$.

Let $V_0\subset V$ be an open neighborhood of q_0 and $t_1,t_2\in(s_{-2},s_{+2})$, $t_1<s_0<t_2$ be such that $V_0\times\mu_a([t_1,t_2])\subset\mathcal{U}$. Then for $q\in V_0$ and $s\in(t_1,t_2)$ the function $\mathbf{E}_a(q,s):=E(q,\mu_a(s))$ is well defined and smooth. Using the first variation formula for $\mathbf{E}_a(q,s)$, see e.g. [19, Prop. 10.39], we obtain

$$\frac{\partial\mathbf{E}_a(q_0,s)}{\partial s}\Big|_{s=s_0}=g\left(\eta,\,\dot\mu_a(f_a(q_0))\right),\quad \nabla\mathbf{E}_a(q,s_0)\Big|_{q=q_0}=-\xi_0, \tag{15}$$

where $\xi_0=\xi(q_0,p_0)$ and $\eta=\dot\gamma_{q_0,\xi_0}(1)$. Since $\dot\mu_a(s)$ is time-like and future-pointing and η is light-like and future-pointing, $\frac{\partial\mathbf{E}_a}{\partial s}(q_0,s_0)<0$.

It follows from the implicit function theorem that there is an open neighborhood $V_a\subset V_0$ of q_0 and a smooth function $q\mapsto s(q,a)$ defined for $q\in V_a$ such that $s(q_0,a)=f_a(q_0)$ and $\mathbf{E}_a(q,s(q,a))=0$. Then $q\mapsto s(q,a)$ and $q\mapsto f_a(q)$ coincide in V_a, and it follows from (15) that

$$\nabla f_a(q)\Big|_{q=q_0}=\frac{1}{c(q_0,a)}\xi(q_0,p_0),\ \ c(q_0,a)=\frac{\partial\mathbf{E}_a}{\partial s}(q_0,s))\Big|_{s=f_a(q_0)}, \tag{16}$$

where we recall that $p_0 = \mu_a(s_0) = \mathcal{E}_a(q)$ and $s_0 = f_a(q)$.

Next we choose $p_1, p_2, \dots, p_n \in \mathcal{E}_U^{reg}(q_0)$ and let $\xi_1, \dots, \xi_n \in L^+_{q_0}(M)$ be such that $p_i = \gamma_{q_0,\xi_i}(1)$. We assume that $\xi_1, \dots, \xi_n \in L^+_{q_0}(M)$ are linearly independent. Moreover, let $a_j \in \mathcal{A}$ be such that $p_j \in \mu_{a_j}$ and $\vec{a} = (a_j)_{j=1}^n$. Finally, we denote by $q \mapsto s(q, a_j)$ the above constructed smooth functions that are defined in some neighborhoods $V_{a_j} \subset V$ of q_0

Let $V_{\vec{a}} = \bigcap_{j=1}^n V_{a_j}$ and consider the map

$$\mathbf{f}_{\vec{a}} : V_{\vec{a}} \to \mathbb{R}^n, \quad \mathbf{f}_{\vec{a}}(q) = (f_{a_1}(q), \dots, f_{a_n}(q)).$$

It follows from (16) that the map $\mathbf{f}_{\vec{a}}$ has a non-degenerate differential at q_0 and, therefore, the function $\mathbf{f}_{\vec{a}} : V_{\vec{a}} \to \mathbb{R}^n$ defines a C^∞-smooth coordinate system in some neighborhood of q_0. $\square$

Definition 3.8. *Let $\vec{a} = (a_j)_{j=1}^n \in \mathcal{A}^n$, $W \subset \mathcal{F}(V)$ be an open set, $s_{a_j} = f_{a_j} \circ \mathcal{F}^{-1}$, and $\mathbf{s}_{\vec{a}} = \mathbf{f}_{\vec{a}} \circ \mathcal{F}^{-1}$. We say that $(W, \mathbf{s}_{\vec{a}})$ are C^0-observation coordinates on $\mathcal{F}(V)$ if the map $\mathbf{s}_{\vec{a}} : W \to \mathbb{R}^n$ is an open and injective map. Also, we say that $(W, \mathbf{s}_{\vec{a}})$ are C^∞-observation coordinates on $\mathcal{F}(V)$ if $\mathbf{s}_{\vec{a}} \circ \mathcal{F} : \mathcal{F}^{-1}(W) \to \mathbb{R}^n$ are C^∞-smooth local coordinates on $V \subset M$, see Fig. 1(Right).*

Note that, by the invariance of domain theorem, the above $\mathbf{s}_{\vec{a}} : W \to \mathbb{R}^n$ is open if it is injective. Even though for a given $\vec{a} \in \mathbb{R}^{\mathcal{A}}$ there are several sets W for which $(W, \mathbf{s}_{\vec{a}})$ form C^0-observation coordinates, to clarify the notations, we sometimes denote the coordinates $(W, \mathbf{s}_{\vec{a}})$ by $(W_{\vec{a}}, \mathbf{s}_{\vec{a}})$.

Since $\mathcal{F} : V \to \mathcal{F}(V)$ is a homeomorphism, we can determine all C^0-observation coordinates on $\mathcal{F}(V)$ using data (3). Next we will consider $\mathcal{F}(V)$ as a topological manifold endowed with the C^0-observation coordinates and denote $\mathcal{F}(V) = \widehat{V}$. We denote the points of this manifold by $\widehat{q} = \mathcal{F}(q)$. Next we construct a differentiable structure on $\widehat{V}$ that is compatible with that of V.

Lemma 3.9. *Assume that we are given data (3). Then for any C^0-observation coordinates $(W_{\vec{a}}, \mathbf{s}_{\vec{a}})$ with $\vec{a} \in \mathcal{A}^n$ we can determine if $(W_{\vec{a}}, \mathbf{s}_{\vec{a}})$ are C^∞-observation coordinates on $\widehat{V}$. Moreover, for any $\widehat{q} \in \widehat{V}$ there exists C^∞-observation coordinates $(W_{\vec{a}}, \mathbf{s}_{\vec{a}})$ such that $\widehat{q} \in W_{\vec{a}}$.*

Proof. Let $q \in V$. We say that $p \in \mathcal{E}_U(q)$ and $a \in \mathcal{A}$ are associated if $p \in \mu_a$. Next, consider $p \in \mathcal{E}_U^{reg}(q)$ and $a \in \mathcal{A}$ that are associated. Note that then $q \notin \mu_a$. By (16), the function $f_a(q)$ satisfies

$$\nabla f_a(q) = c(q, a)\, \xi(q, \mathcal{E}_a(q)), \quad c(q, a) \neq 0.$$

Let

$$K(q) = \{(\xi_j)_{j=1}^n;\ \xi_j \in L^+_q M,\ \rho(q, \xi_j) > 1,\ \gamma_{q,\xi_j}(1) \in U\}$$

and $H : K(q) \to U^n$ be the map $H((\xi_j)_{j=1}^n) = (p_j)_{j=1}^n$, where $p_j = \gamma_{q,\xi_j}(1)$. Then $p_j \in \mathcal{E}_U^{reg}(q)$ and $\xi_j = \Theta(q, p_j)$. As ρ is lower semi-continuous, we see that $K(q) \subset (L^+_q M)^n$ is open. Clearly, H is continuous and as $\Theta : \mathcal{Z} \to$

L^+V is continuous and injective, we see that $H : K(q) \to H(K(q)) = (\mathcal{E}_U^{reg}(q))^n$ is a homeomorphism. We denote below $Y(q) = (\mathcal{E}_U^{reg}(q))^n$. Note that for all $\widehat{q} \in \widehat{V}$ the data (3) determine the set $Y(q) \subset U^n$, where $q = \mathcal{F}^{-1}(\widehat{q})$.

Let us consider the set

$$K_0(q) = \{(\xi_j)_{j=1}^n \in K(q);\ \xi_j,\ j = 1, \dots, n \text{ are linearly independent}\}.$$

Clearly, the set $K_0(q)$ is dense and open in $K(q)$, and hence $Y_0(q) := H(K_0(q))$ is open and dense in $Y(q)$.

Let $(W_{\vec{a}}, \mathbf{s}_{\vec{a}})$, $\vec{a} \in \mathcal{A}^n$ be C^0-observation coordinates on $\widehat{V}$, $\widehat{q} \in W_{\vec{a}}$, and $q = \mathcal{F}^{-1}(\widehat{q})$. Also, let $(p_j)_{j=1}^n \in Y(q)$ be such that p_j are associated with a_j. Similarly, let $(W_{\vec{b}}, \mathbf{s}_{\vec{b}})$, $\vec{b} \in \mathcal{A}^n$ be another C^0-observation coordinates on $\widehat{V}$ such that $\widehat{q} \in W_{\vec{b}}$, and let $(\widetilde{p}_j)_{j=1}^n \in Y(q)$ be such that $\widetilde{p}_j$ are associated with b_j. Note that then $p_j = \mathcal{E}_{a_j}^{reg}(q)$ and $\widetilde{p}_j = \mathcal{E}_{b_j}^{reg}(q)$.

In the case when $(\widetilde{p}_j)_{j=1}^n \in Y_0(q)$, q has a neighborhood $V_1 \subset V$ in which the function $\mathbf{f}_{\vec{b}} : V_1 \to \mathbb{R}^n$ give C^∞-smooth local coordinates. Thus, if $(\widetilde{p}_j)_{j=1}^n \in Y_0(q)$, then it holds that $(p_j)_{j=1}^n \in Y_0(q)$ if and only if

(i) Functions $s_{a_j} \circ \mathbf{s}_{\vec{b}}^{-1}$, $j = 1, 2, \dots, n$ are C^∞-smooth at $\mathbf{s}_{\vec{b}}(\widehat{q})$ and the Jacobian determinant $\det(D(\mathbf{s}_{\vec{a}} \circ \mathbf{s}_{\vec{b}}^{-1}))$ at $\mathbf{s}_{\vec{b}}(\widehat{q})$ is non-zero.

Denote $\vec{p} = (p_j)_{j=1}^n \in Y(q)$, and define $\mathcal{X}_{\vec{p}} \subset Y(q)$ to be the set of those $(\widetilde{p}_j)_{j=1}^n \in Y(q)$, for which there are $\vec{b} \in \mathcal{A}^n$ and C^0-observation coordinates $(W_{\vec{b}}, \mathbf{s}_{\vec{b}})$ such that $\widehat{q} = \mathcal{F}(q) \in W_{\vec{b}}$, $\widetilde{p}_j$ are associated with b_j for $j = 1, 2, \dots, n$, and the condition (i) is satisfied. If $\vec{p}$ is in $Y_0(q)$, we see that $Y_0(q) \subset \mathcal{X}_{\vec{p}}$. On the other hand, if $\vec{p}$ is not in $Y_0(q)$, we have $Y_0(q) \cap \mathcal{X}_{\vec{p}} = \emptyset$. Since the set $Y_0(q)$ is open and dense in $Y(q)$, we see that $\vec{p} \in Y_0(q)$ if and only if the interior of set $\mathcal{X}_{\vec{p}}$ is a dense subset of $Y(q)$. This in particular implies that using data (3) we can determine whether $(p_j)_{j=1}^n$ is in $Y_0(q)$ or not. The C^0-observation coordinates $(W_{\vec{a}}, \mathbf{s}_{\vec{a}})$, $\vec{a} \in \mathcal{A}^n$ are C^∞-observation coordinates on $\widehat{V}$ if and only if for all $\widehat{q} \in W_{\vec{a}}$, $q = \mathcal{F}^{-1}(\widehat{q})$, and $p_j = \mathcal{E}_{a_j}(q)$, $j = 1, 2, \dots, n$ we have $(p_j)_{j=1}^n \in Y_0(q)$. Thus we can determine all C^0-observation coordinates $(W_{\vec{a}}, \mathbf{s}_{\vec{a}})$ on $\widehat{V}$ that are C^∞-observation coordinates. Moreover, since for all $q \in V$ the set $Y_0(q)$ is non-empty, we see that any $\widehat{q} = \mathcal{F}(q) \in \widehat{V}$ belongs in the domain some C^∞-observation coordinates. □

We endow $\widehat{V} = \mathcal{F}(V)$ with the differentiable structure provided by all C^∞-observation coordinates on $\widehat{V}$. By Lemma 3.9 and [19, Lem. 1.42] the C^∞-observation coordinates make $\widehat{V}$ a differentiable manifold and its the differentiable structure is uniquely determined. Since the differentiable structure of V is determined by the functions $f_{\vec{a}}$ that are C^∞-smooth local coordinates, we see using Def. 3.8 that the map

$$\mathcal{F} : V \to \widehat{V} = \mathcal{F}(V) \tag{17}$$

is a diffeomorphism.

3.0.2. *Construction of the conformal type of the metric.* Let us denote by $\widehat g = \mathcal{F}_* g$ the metric on $\widehat V = \mathcal{F}(V)$ that makes $\mathcal{F}$ an isometry. Next we show that the set $\mathcal{F}(V)$, the paths μ_a and the conformal class of the metric g on U determine the conformal class of $\widehat g$ on $\widehat V$.

Lemma 3.10. *The data (3) determines a metric G on $\widehat V = \mathcal{F}(V)$ that is conformal to $\widehat g$ and the time orientation on $\widehat V$ that makes $\mathcal{F} : V \to \widehat V$ a causality preserving map.*

Proof. Let $(W_{\vec a}, \mathbf{s}_{\vec a})$ be C^∞-observation coordinates on $\widehat V$. Then by (16) the co-vectors $-ds_{a_1}|_{\widehat q}$ and $-ds_{a_2}|_{\widehat q}$ are non-parallel future-pointing light-like co-vectors. Thus their sum determines a future-pointing time-like co-vector field on $W_{\vec a}$. Using a suitable partition of unity we can construct future-pointing time-like co-vector field X on $\widehat V$.

Let $(W_{\vec a}, \mathbf{s}_{\vec a})$ be C^∞-observation coordinates on $\widehat V$. Let $\widehat q \in W_{\vec a}$ and $q \in V$ be such that $\widehat q = \mathcal{F}(q)$. Using the data (3), the function $F_q = \mathcal{F}(q) : \mathcal{A} \to \mathbb{R}$, and the formula (12), we can determine the set $\mathcal{E}_U(q) \subset U$. By Prop. 3.3 (iii), this further determines the set $\mathcal{D}_U^{reg}(q)$.

Then, let us fix a point $\widehat q = \mathcal{F}(q) \in W_{\vec a}$. Let $(y,\eta) \in \mathcal{D}_U^{reg}(q)$ and let $\widehat t > 0$ be the largest number such that the geodesic $\gamma_{y,\eta}((-\widehat t, 0]) \subset M$ is defined and has no cut points. For $q \in V$, Proposition 3.2 (ii) yields that $q \in \gamma_{y,\eta}((-\widehat t, 0))$ if and only if $(y,\eta) \in \mathcal{D}_U^{reg}(q)$. Hence (y,η) and the data (3) determine the set

$$\beta = \{\widehat q \in W_{\vec a};\ \widehat q = \mathcal{F}(q),\ \mathcal{D}_U^{reg}(q) \ni (y,\eta)\} = \mathcal{F}(\gamma_{y,\eta}((-\widehat t, 0))) \cap W_{\vec a}.$$

This implies that on $W_{\vec a} \subset \widehat V$ we can find the image, in the map $\mathcal{F}$, of the light-like geodesic segment $\gamma_{y,\eta}((-\widehat t, 0)) \cap \mathcal{F}^{-1}(W_{\vec a})$ that contains $q = \gamma_{y,\eta}(-t_1)$. Let $\alpha(s)$, $s \in (-s_0, s_0)$ be a smooth path on $W_{\vec a}$ such that $\partial_s \alpha(s)$ does not vanish, $\alpha((-s_0, s_0)) \subset \beta$, and $\alpha(0) = \widehat q$. Such smooth path $\alpha(s)$ can be obtained e.g. by parametrizing β by the arc-length with respect to some auxiliary smooth Riemannian metric on $W_{\vec a}$. Then $\widehat \xi = \partial_s \alpha(s)|_{s=0} \in T_{\widehat q} \widehat V$ has the form $\widehat \xi = c\mathcal{F}_*(\dot\gamma_{y,\eta}(t_1))$ where $c \neq 0$. Since we can do the above construction for all points $(y,\eta) \in \mathcal{D}_U^{reg}(q)$, we determine in the tangent space $T_{\widehat q}\widehat V$ the set $\Gamma = \mathcal{F}_*(\{c\xi \in L_q M;\ \exp_q(\xi) \in \mathcal{E}_U(q),\ c \in \mathbb{R},\ c \neq 0\})$, that is an open, non-empty subset of the light cone at $\widehat q$ associated to the metric $\widehat g$. Let us now consider the set Γ in the coordinates of $T_{\widehat q}\widehat V$ associated to $\mathbf{s}_{\vec a}$. Since the light cone is determined by a quadratic equation in the tangent space, having an open set Γ of the light cone we can uniquely determine the whole light cone. Using this construction with all points $\widehat q \in W_{\vec a}$, we can determine all light-like vectors in the tangent space $T_{\widehat q} W_{\vec a}$ for all $\widehat q \in W_{\vec a}$. The collections of light-like vectors at tangent spaces of $\widehat V$ determine uniquely the conformal class of the tensor $\widehat g = \mathcal{F}_* g$ in the manifold $\widehat V$, see [3, Thm. 2.3] (or [3, Lem. 2.1] for a constructive procedure).

The above shows that the data (3) determines the conformal class of the metric tensor $\widehat{g}$. In particular, we can construct a metric G on $\widehat{V}$ that is conformal to $\widehat{g}$ and satisfies $G(X,X)=-1$. □

We have shown that the data (3) determine the topological and the differentiable structures on $\widehat{V}=\mathcal{F}(V)$ and a metric G on it that makes the map $\mathcal{F}:(V,g|_V)\to(\widehat{V},G)$ a diffeomorphism and a conformal map. Moreover, we determine the time-orientation on $\widehat{V}$ that makes $\mathcal{F}$ a causality preserving map. This proves Theorem 3.1.

Finally, by Prop. 3.3 *(i)*, for any $y\in U$ we can identify if $y=q\in V$ and find the corresponding element $\mathcal{F}(q)\in\mathcal{F}(V)$. Thus we can find the set $\mathcal{F}(V\cap U)$ and the map $\mathcal{F}^{-1}:\mathcal{F}(V\cap U)\to V\cap U$. This yields the last claim of Thm. 1.2. Thus Theorem 1.2 is proven. □

Proof of Corollary 1.3. By Theorem 1.2, there is a conformal diffeomorphism $\Psi:(V_1,g^{(1)})\to(V_1,g^{(2)})$. By our assumptions, $\Phi:(U_1,g^{(1)})\to(U_1,g^{(2)})$ is an isometry, the Ricci curvature of $g^{(j)}$ is zero in V_j, and any point $x_1\in V_1$ can be connected to some point $y_1\in U_1\cap V_1$ with a piecewise smooth path $\mu_{x_1,y_1}([0,1])\subset V_1$. Note that then $\Psi(\mu_{x_1,y_1}([0,1]))\subset V_2$ connects $x_2=\Psi(x_1)$ to $y_2=\Psi(y_1)$.

To simplify notations we denote $\widehat{g}=g^{(1)}$ and $g=\Psi^*g^{(2)}$. Since Ψ is conformal, there is $f:V_1\to\mathbb{R}$ such that $\widehat{g}=e^{2f}g$ on V_1, and as $\Phi:U_1\to U_2$ is an isometry, $f=0$ in U_1. By [20, formula (2.73)], see also [17], the Ricci tensors $\mathrm{Ric}_{jk}(g)$ of g and $\mathrm{Ric}_{jk}(\widehat{g})$ of $\widehat{g}$ satisfy on V_1

$$\begin{aligned}0=\mathrm{Ric}_{jk}(\widehat{g}) &= \mathrm{Ric}_{jk}(g)-2\nabla_j\nabla_k f+2(\nabla_j f)(\nabla_k f)\\ &\quad -(g^{pq}\nabla_p\nabla_q f+2g^{pq}(\nabla_p f)(\nabla_q f))g_{jk}\end{aligned}$$

where $\nabla=\nabla^g$. For the scalar curvature this yields

$$0=e^{2f}\widehat{g}^{pq}\mathrm{Ric}_{pq}(\widehat{g})=g^{pq}\mathrm{Ric}_{pq}(g)-3g^{pq}\nabla_p\nabla_q f.$$

Combining the above with the fact that $\mathrm{Ric}_{jk}(g)=0$, we obtain

$$\nabla_j\nabla_k f-(\nabla_j f)(\nabla_k f)+g^{pq}(\nabla_p f)(\nabla_q f)g_{jk}=0.$$

This equation gives a system of first order ordinary differential equations for the vector field $Y=\nabla f$ along $\mu_{x_1,y_1}([0,1])$ with initial value $Y(x_1)=\nabla f(x_1)=0$, that has the unique solution $Y=0$.As $f(x_1)=0$, we obtain $f(\mu_{x_1,y_1}(t))=0$ for $t\in[0,1]$. Since all points $x\in V_1$ are connected in V_1 to the set U_1 by piecewise smooth paths, this shows that $f=0$. □

Acknowledgements The authors express their gratitude to MSRI, the Newton Institute, the Fields Institute and the Mittag-Leffler Institute, where parts of this work have been done.

YK was partly supported by EPSRC and AXA professorship at Mittag-Leffler institute. ML was partly supported by the Centre of Excellence in Inverse Problems. GU was partly supported by NSF, a Clay Senior Award at MSRI, a Chancellor Professorship at UC Berkeley, a Rothschild

Distinguished Visiting Fellowship at the Newton Institute, the Fondation de Sciences Mathématiques de Paris, and a Simons Fellowship.

References

[1] M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas, M. Taylor: Boundary regularity for the Ricci equation, Geometric Convergence, and Gel'fand's Inverse Boundary Problem, *Inventiones Mathematicae* 158 (2004), 261-321.

[2] C. Bär, N. Ginoux, F. Pfäffle: *Wave equations on Lorentzian manifolds and quantization*, ESI Lectures in Mathematics and Physics, 2007, 202 pp.

[3] J. Beem, P. Ehrlich, K. Easley: *Global Lorentzian geometry*, Pure and. Applied Mathematics, vol. 67, M. Dekker, 1981, pp. 270.

[4] A. Bernal, M. Sanchez: Globally hyperbolic spacetimes can be defined as "causal" instead of "strongly causal", *Class. Quant. Grav.* 24 (2007) 745-750.

[5] BICEP2 collaboration: Detection Of B-mode Polarization at Degree Angular Scales, preprint arXiv:1403.3985.

[6] V. Chernov, S. Nemirovski: Cosmic censorship of smooth structures. *Comm. Math. Phys.* 320 (2013) 469-473.

[7] C. Croke: Rigidity and the distance between boundary points. *J. Diff. Geometry,* 33 (1991), 445–464.

[8] T. Daude, F. Nicoleau: Inverse Scattering at Fixed Energy in de Sitter-Reissner-Nordström Black Holes. *Annales Henri Poincare* 12 (2011), 1-47.

[9] T. Daude, F. Nicoleau: Inverse scattering in de Sitter-Reissner-Nordström black hole spacetimes, *Rev. Math. Phys.* 22 (2010), 431-484.

[10] J. Isenberg, R. Mazzeo, D. Pollack: Gluing and wormholes for the Einstein constraint equations, *Comm. Math. Phys.* 231 (2002), 529-568.

[11] J. Isenberg, R. Mazzeo, D. Pollack: Topology of vacuum spacetimes, *Ann. Henri Poincare* 4 (2003), 369 - 383.

[12] C. Ibanogammalu (Ed.), *Variable Stars as Essential Astrophysical Tools*, Proceedings of the NATO Advanced Study Institute, 2000, 825 pp.

[13] A. Katchalov, Y. Kurylev, M. Lassas: *Inverse boundary spectral problems.* Chapman-Hall/CRC, 2001. xx+290 pp

[14] Y. Kurylev, M. Lassas, G. Uhlmann: Rigidity of broken geodesic flow and inverse problems. *Amer. J. Math.* 132 (2010), 529–562.

[15] Y. Kurylev, M. Lassas, G. Uhlmann: Inverse problems in spacetime I: Inverse problems for Einstein equations, preprint.

[16] L. Krauss, S. Dodelson, S. Meyer: Primordial Gravitational Waves and Cosmology, *Science* 328 (2010), 989-992.

[17] R. Mazzeo, F. Pacard: Poincare-Einstein metrics and the Schouten tensor, *Pacific J. Math.* 212 (2003), 169-185.

[18] K. Nomizu, H. Ozeki: The existence of complete Riemannian metrics. *Proc. Amer. Math. Soc.* **12** (1961) 889–891.

[19] B. O'Neill, *Semi-Riemannian geometry.* With applications to relativity. Pure and Applied Mathematics, 103. Academic Press, Inc., 1983. xiii+468 pp.

[20] A. Rendall: *Partial differential equations in general relativity.* Oxford University Press, 2008. xvi+279 pp.

[21] G. Rhee: An estimate of the Hubble constant from the gravitational lensing of quasar Q0957+561, *Nature* **350** (1991), 211-212.

[22] A. Vasy: Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by S. Dyatlov). *Invent. Math.* 194 (2013), 381-513.

Yaroslav Kurylev, UCL; Matti Lassas, University of Helsinki; Gunther Uhlmann, University of Washington, and University of Helsinki.
y.kurylev@ucl.ac.uk, Matti.Lassas@helsinki.fi, gunther@math.washington.edu