Inverse problems for non-linear hyperbolic equations and an inverse problem for the Einstein equation

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in collaboration with

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How one can determine the topology and metric of the space-time?

How one can determine the topology and metric of complicated structures in space-time with a radar-like device?

How the blue-prints of a "space-time radar" look like?

Figures: Anderson institute and Greenleaf-K.-L.-U.
Some results for hyperbolic inverse problems for linear equations:

- Kachalov-Kurylev 1998: Reconstruction with local data.
- Eskin 2008: Wave equation with time-depending (real-analytic) lower order terms.
- Helin-Lassas-Oksanen 2012: Combining several measurements for together for the wave equation.
Outline:

- Inverse problems in space-time for passive measurements
- Inverse problem for non-linear wave equation
- Einstein-scalar field equations
Inverse problems in space-time: Passive measurements

Can we determine structure of the space-time when we see light coming from many point sources that vary in time?
Definitions

Let \((M, g)\) be a Lorentzian manifold, where the metric \(g\) is semi-definite. \(\xi \in T_xM\) is light-like if \(g(\xi, \xi) = 0, \xi \neq 0\). \(\xi \in T_xM\) is time-like if \(g(\xi, \xi) < 0\). A curve \(\mu(s)\) is time-like if \(\dot{\mu}(s)\) is time-like.

Example: Minkowski space \(\mathbb{R}^{1+3}\). The metric at \((x^0, x^1, x^2, x^3) \in \mathbb{R}^{1+3}\) is

\[
ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.
\]
Definitions

Let \((M, g)\) be a Lorentzian manifold.

\[ L_q M = \{ \xi \in T_q M \setminus 0; \, g(\xi, \xi) = 0 \} , \]

\(L^+ q M \subset L_q M\) is the future light cone,

\[ J^+(q) = \{ x \in M; \, x \text{ is in causal future of } q \} , \]

\[ J^-(q) = \{ x \in M; \, x \text{ is in causal past of } q \} , \]

\(\gamma_{x, \xi}(t)\) is a geodesic with the initial point \((x, \xi)\).

\((M, g)\) is globally hyperbolic if

there are no closed causal curves and the set

\[ J^-(p_1) \cap J^+(p_2) \text{ is compact for all } p_1, p_2 \in M. \]

Then \(M\) can be represented as \(M = \mathbb{R} \times N\).
More definitions

Let $\mu = \mu((-1, 1)) \subset M$ be a time-like geodesics, $p^-, p^+ \in \mu$. We consider observations in a neighborhood $V \subset M$ of $\mu$.

Let $U \subset J^-(p^+) \setminus J^-(p^-)$ be an open, relatively compact set.

The light observation set $P_V(q)$ for $q \in U$ is the intersection of the future light cone of $q$ and $V$,

$$P_V(q) = \exp_q(L^+_q M) \cap V = \{\gamma_{q, \xi}(r) \in V; \; \xi \in L^+_q M, \; r \geq 0\}.$$
**Theorem**

Let $(M, g)$ be an open, globally hyperbolic Lorentzian manifold of dimension $n \geq 3$. Assume that $\mu$ is a time-like geodesic containing points $p^-$ and $p^+$, and $V \subset M$ is a neighborhood of $\mu$. Let $U \subset J^-(p^+) \setminus J^-(p^-)$ be a relatively compact open set. Then $(V, g|_V)$ and the collection of the light observation sets,

$$P_V(U) := \left\{ P_V(q) \subset V \mid q \in U \right\},$$

determine the set $U$, up to a change of coordinates, and the conformal class of the metric $g$ in $U$. 

![Diagram](image)
Reconstruction of the topological structure of $U$

Assume that $q_1, q_2 \in U$ are such that $P_V(q_1) = P_V(q_2)$.
Then all light-like geodesics from $q_1$ to $V$ go through $q_2$.
Let $x_1$ be the earliest point of $\mu \cap P_V(q_1)$. 
Reconstruction of the topological structure of $U$

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Let $x_1$ be the earliest point of $\mu \cap P_V(q_1)$. Using a short cut argument we see that there is a causal curve from $q_1$ to $x_1$ that is not a geodesic.
Reconstruction of the topological structure of \( U \)

Assume that \( q_1, q_2 \in U \) are such that \( P_V(q_1) = P_V(q_2) \).
Then all light-like geodesics from \( q_1 \) to \( V \) go through \( q_2 \).

Let \( x_1 \) be the earliest point of \( \mu \cap P_V(q_1) \).
Using a short cut argument we see that there is a causal curve from \( q_1 \) to \( x_1 \) that is not a geodesic.

This implies that \( q_1 \) can be observed on \( \mu \) before \( x_1 \).

The map \( P_V : \overline{U} \mapsto 2^{TV} \) is continuous and one-to-one.
As \( \overline{U} \) is compact, the map \( P_V : \overline{U} \to P_V(\overline{U}) \) is a homeomorphism.
Possible applications of the theorem

Left: Variable stars in Hertzsprung-Russell diagram on star types.
Right: Galaxy Arp 220 (Hubble Space Telescope)

Artistic impressions on matter falling into a black hole and Pan-STARRS1 telescope picture.
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“Can we image a wave using other waves?”
Inverse problem for non-linear wave equation

Let $M = \mathbb{R} \times N$, $\dim(M) = 4$. Consider the equation

$$\Box_g u(x) + a(x) u(x)^2 = f(x) \quad \text{on } M_1 = (-\infty, T) \times N,$$

$$u(x) = 0 \quad \text{for } x = (x^0, x^1, x^2, x^3) \in (-\infty, 0) \times N,$$

where $\text{supp}(f) \subset V$, $V \subset M_1$ is open,

$$\Box_g u = \sum_{p, q=0}^{3} |\det(g(x))|^{-\frac{1}{2}} \frac{\partial}{\partial x^p} \left( |\det(g(x))|^{\frac{1}{2}} g^{pq}(x) \frac{\partial}{\partial x^q} u(x) \right),$$

$f \in C^6_0(V)$ is a source, and $a(x)$ is a non-vanishing $C^\infty$-smooth function.

In a neighborhood $W \subset C^6_0(V)$ of the zero-function, define the measurement operator (source-to-solution operator) by

$$L_V : f \mapsto u|_V, \quad f \in W \subset C^6_0(V).$$
**Theorem**

Let \((M, g)\) be a globally hyperbolic Lorentzian manifold of dimension \((1 + 3)\). Let \(\mu\) be a time-like path containing \(p^−\) and \(p^+\), \(V \subset M\) be a neighborhood of \(\mu\), and \(a(x)\) be a non-vanishing function. Consider the non-linear wave equation

\[
\Box_g u(x) + a(x) u(x)^2 = f(x) \quad \text{on } M_1 = (-\infty, T) \times N,
\]

\[
u = 0 \quad \text{in } (-\infty, 0) \times N,
\]

where \(\text{supp}(f) \subset V\). Then \((V, g|_V)\) and the measurement operator \(L_V : f \mapsto u|_V\) determine the set \(J^+(p^-) \cap J^-(p^+) \subset M\), up to a change of coordinates, and the conformal class of \(g\) in the set \(J^+(p^-) \cap J^-(p^+)\).
Idea of the proof: Non-linear geometrical optics.

The non-linearity helps in solving the inverse problem.

Let \( u = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \varepsilon^4 w_4 + E_\varepsilon \) satisfy

\[
\Box_g u + a u^2 = f, \quad \text{on } M_1 = (-\infty, T) \times N,
\]

\[
\big| E_\varepsilon \big| \leq C \varepsilon^5.
\]

\[
u \big|_{(-\infty,0) \times N} = 0
\]

with \( f = \varepsilon f_1, \ v > 0 \).

When \( Q = \Box_g^{-1} \), we have

\[
w_1 = Qf_1,
\]

\[
w_2 = -Q(aw_1 w_1),
\]

\[
w_3 = 2Q(aw_1 Q(aw_1 w_1)),
\]

\[
w_4 = -Q(aw_1 Q(aw_1 w_1) Q(aw_1 w_1)) - 4Q(aw_1 Q(aw_1 Q(aw_1 w_1)) Q(aw_1 w_1))),
\]

\[
||E_\varepsilon|| \leq C\varepsilon^5.
\]
Interaction of waves in Minkowski space $\mathbb{R}^4$

Let $x^j, j = 1, 2, 3, 4$ be coordinates such that $\{x^j = 0\}$ are light-like. We consider waves

$$u_j(x) = v \cdot (x^j)^m_+, \quad (s)^m_+ = |s|^m H(s), \quad v \in \mathbb{R}, \; j = 1, 2, 3, 4.$$  

Waves $u_j$ are conormal distributions, $u_j \in I^{m+1}(K_j)$, where

$$K_j = \{x^j = 0\} \subset \mathbb{R}^4, \; j = 1, 2, 3, 4.$$  

The interaction of the waves $u_j(x)$ produce new sources on

$$K_{12} = K_1 \cap K_2,$$
$$K_{123} = K_1 \cap K_2 \cap K_3 = \text{line},$$
$$K_{1234} = K_1 \cap K_2 \cap K_3 \cap K_3 = \{q\} = \text{one point}.$$
Interaction of two waves

If we consider sources $f_{\vec{\varepsilon}}(x) = \varepsilon_1 f_1(x) + \varepsilon_2 f_2(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, and the corresponding solution $u_{\vec{\varepsilon}}$ of the wave equation, we have

$$W_2(x) = \frac{\partial}{\partial \varepsilon_1} \frac{\partial}{\partial \varepsilon_2} u_{\vec{\varepsilon}}(x) \bigg|_{\vec{\varepsilon} = 0}$$

$$= Q(a \, u_1 \cdot u_2),$$

where $Q = \Box^{-1}_g$ and

$$u(j) = Qf(j).$$

Recall that $K_{12} = K_1 \cap K_2 = \{x^1 = x^2 = 0\}$. Since light-like co-vectors in the normal bundle $N^*K_{12}$ are in $N^*K_1 \cup N^*K_2$,

$$\text{singsupp}(W_2) \subset K_1 \cup K_2.$$

Thus no interesting singularities are produced by the interaction of two waves.
Interaction of three waves

If we consider sources \( f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{3} \varepsilon_j f_j(x) \), \( \vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \), and the corresponding solution \( u_{\vec{\varepsilon}} \), we have

\[
W_3 = \partial_{\varepsilon_1} \partial_{\varepsilon_2} \partial_{\varepsilon_3} u_{\vec{\varepsilon}} \big|_{\vec{\varepsilon}=0} = Q(a u(1) \cdot Q(a u(2) \cdot u(3))) + \ldots,
\]

where \( Q = \Box_g^{-1} \). The interaction of the three waves happens on the line \( K_{123} = K_1 \cap K_2 \cap K_2 \).

The normal bundle \( N^* K_{123} \) contains light-like directions that are not in \( N^* K_1 \cup N^* K_2 \cup N^* K_3 \) and hence new singularities appear.

Using standard tools of microlocal analysis we can analyze \( Q(a u(1) \cdot a u(2)) \), but not the interaction of 3 waves. Let

\[
F_\tau(x) = e^{\tau(i x \cdot p_0 - (x-x_0)^2)} \phi(x).
\]

By studying

\[
\langle F_\tau, W_3 \rangle_{L^2(M)} = \langle Q^* F_\tau, a u(1) \cdot Q(a u(2) \cdot u(3)) \rangle_{L^2(M)} + \ldots,
\]

as \( \tau \to \infty \), we can detect if \( (x_0, p_0) \in WF(W_3) \).
Interaction of waves:

The non-linearity helps in solving the inverse problem. Artificial sources can be created by interaction of waves using the non-linearity of the wave equation.

The interaction of 3 waves creates a point source in space that seems to move at a higher speed than light, that is, it appears like a tachyonic point source, and produces a new “shock wave” type singularity.
Interaction of three waves.
Interaction of four waves

Consider sources \( f_\vec{\varepsilon}(x) = \sum_{j=1}^{4} \varepsilon_j f(j)(x) \), \( \vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \), the corresponding solution \( u_\vec{\varepsilon} \), and

\[
W_4 = \partial_{\varepsilon_1} \partial_{\varepsilon_2} \partial_{\varepsilon_3} \partial_{\varepsilon_4} u_\vec{\varepsilon}(x) \bigg|_{\vec{\varepsilon}=0}.
\]

Since \( K_{1234} = \{ q \} \) we have \( N^* K_{1234} = T_q^* M \). Thus, when the conic waves intersect, an artificial point source appears. We have

\[
singsupp(W_4) \subset (\bigcup_{j=1}^{4} K_j) \cup \Sigma \cup L^+_q M,
\]

where \( \Sigma \) is the union of conic waves produced by 3-interactions. Above, \( L^+_q M = \exp_q(L^+_q M) \) is the union of future going light-like geodesics starting from the point \( q \).
Interaction of four waves.

The 3-interaction produces conic waves (only one is shown below).

The 4-interaction produces a spherical wave from the point \( q \) that determines the light observation set \( P_V(q) \).
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Einstein equations

The Einstein equation for the $(-, +, +, +)$-type Lorentzian metric $g_{jk}$ of the space time is

$$\text{Ein}_{jk}(g) = T_{jk},$$

where

$$\text{Ein}_{jk}(g) = \text{Ric}_{jk}(g) - \frac{1}{2} (g^{pq} \text{Ric}_{pq}(g)) g_{jk}.$$

In vacuum, $T = 0$. In wave map coordinates, the Einstein equation yields a quasilinear hyperbolic equation and a conservation law,

$$g^{pq}(x) \frac{\partial^2}{\partial x^p \partial x^q} g_{jk}(x) + B_{jk}(g(x), \partial g(x)) = T_{jk}(x),$$

$$\nabla_p (g^{pj} T_{jk}) = 0.$$
One can not do measurements in vacuum, so matter fields need to be added. We can consider the coupled Einstein and scalar field equations with sources,

\[ \text{Ein}(g) = T, \quad T = \mathbf{T}(\phi, g) + \mathcal{F}_1, \quad \text{on } (-\infty, T) \times N, \]
\[ \Box g \phi_\ell - m^2 \phi_\ell = \mathcal{F}_2^\ell, \quad \ell = 1, 2, \ldots, L, \]
\[ g|_{t<0} = \hat{g}, \quad \phi|_{t<0} = \hat{\phi}. \]

Here, \( \hat{g} \) and \( \hat{\phi} \) are \( C^\infty \)-smooth and satisfy equations (1) with the zero sources and

\[ T_{jk}(g, \phi) = \sum_{\ell=1}^L \partial_j \phi_\ell \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \partial_q \phi_\ell - \frac{1}{2} m^2 \phi_\ell^2 g_{jk}. \]

To obtain a physically meaningful model, the stress-energy tensor \( T \) needs to satisfy the conservation law

\[ \nabla_p (g^{pj} T_{jk}) = 0, \quad k = 1, 2, 3, 4. \]
Definition
Linearization stability (Choquet-Bruhat, Deser, Fischer, Marsden)
Let \( f = (f^1, f^2) \) satisfy the linearized conservation law

\[
\sum_{\ell=1}^{L} f^2_{\ell} \partial_j \hat{\phi}_\ell + \frac{1}{2} \hat{g}^{pk} \hat{\nabla}_p f^1_{kj} = 0, \quad j = 1, 2, 3, 4
\]  

and let \((\dot{g}, \dot{\phi})\) be the corresponding solution of the linearized Einstein equation. We say that \( f \) has the Linearization Stability (LS) property if there is \( \varepsilon_0 > 0 \) and families

\[
F_\varepsilon = (F^1_\varepsilon, F^2_\varepsilon) = \varepsilon f + O(\varepsilon^2),
\]

\[
g_\varepsilon = \hat{g} + \varepsilon \dot{g} + O(\varepsilon^2),
\]

\[
\phi_\varepsilon = \hat{\phi} + \varepsilon \dot{\phi} + O(\varepsilon^2),
\]

where \( \varepsilon \in [0, \varepsilon_0) \), such that \((g_\varepsilon, \phi_\varepsilon)\) solves the non-linear Einstein equations and the conservation law

\[
\nabla_j^{g_\varepsilon} (T^{jk}(g_\varepsilon, \phi_\varepsilon) + (F^1_\varepsilon)^{jk}) = 0, \quad k = 1, 2, 3, 4.
\]
Let $V_{\hat{g}} \subset M$ be a open set that is a union of freely falling geodesics that are near $\mu$, $L \geq 5$.

**Condition A:** Assume that at any $x \in V_{\hat{g}}$ the $5 \times 5$ matrix

$$[A_{j\ell}(x)]_{j,\ell \leq 5} = \begin{bmatrix} (\partial_j \hat{\phi}_\ell(x))_{\ell \leq 5, j \leq 4} \\ (\hat{\phi}_\ell(x))_{\ell \leq 5} \end{bmatrix}$$

is invertible.

Let $I^k(Y)$ be the space of conormal distributions for $Y \subset M$.

**Theorem**

Let condition A be valid, $W \subset V_{\hat{g}}$ be open, and $Y \subset W$ be a 2-dimensional space-like surface. Assume that $f = (f^1, f^2) \in I^k(Y)$ is supported in $W$ and $f^1$ has a principal symbol $a_{jk}(y, \eta)$ satisfying $\hat{g}^{lk}(y)\eta_l a_{jk}(y, \eta) = 0$ on $N^*Y$. Then there is a smoother correction term $f_{cor} \in I^{k-1}(Y)$ supported in $W$ such that $f + f_{cor}$ has a linearization stability property with a family $\mathcal{F}_\varepsilon$ supported in $W$. 
Idea of proof: We formulate the direct problem with adaptive source functions,

\[ \text{Ein}_{jk}(g) = P_{jk} - \sum_{\ell=1}^{L} (S_{\ell}\phi_{\ell} + \frac{1}{2} S_{\ell}^2) g_{jk} + T_{jk}(g, \phi), \]

\[ \Box g\phi_{\ell} - m^2 \phi_{\ell} = S_{\ell}, \quad \text{in } M_0, \quad \ell = 1, 2, 3, \ldots, L, \]

\[ S_{\ell} = Q_{\ell} + S_{\ell}^{2nd}(g, \phi, \nabla \phi, Q, \nabla Q, P, \nabla P), \]

\[ g = \hat{g}, \quad \phi_{\ell} = \hat{\phi}_{\ell}, \quad \text{in } (-\infty, 0) \times \mathcal{N}. \]

Here \( Q \) and \( P_{jk} \) are considered as the primary sources.

The functions \( S_{\ell}^{2nd} \) are constructed so that the conservation law is satisfied for all solutions \((g, \phi)\).
Let $V_{\hat{g}} \subset M$ be a neighborhood of the geodesic $\mu$ and $p^-, p^+ \in \mu$.

**Theorem**

Assume that the condition A is valid. Let

$$D = \{(V_g, g|_{V_g}, \phi|_{V_g}, \mathcal{F}|_{V_g}); \text{ } g \text{ and } \phi \text{ satisfy Einstein equations with a source } \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2), \text{ supp } (\mathcal{F}) \subset V_g, \text{ and } \nabla_j(T_{jk}(g, \phi) + \mathcal{F}_1^{jk}) = 0\}.$$ 

The data set $D$ determines uniquely the conformal type of the double cone $(J^+(p^-) \cap J^-(p^+), \hat{g})$. 

![Diagram of double cone](image-url)
Thank you for your attention!