Inverse problems for non-linear hyperbolic equations and an inverse problem for the Einstein equation

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in collaboration with

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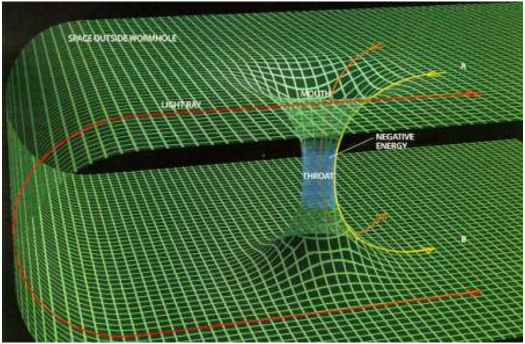
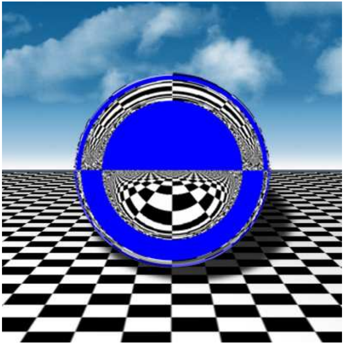
How one can determine the topology and metric of the space time?





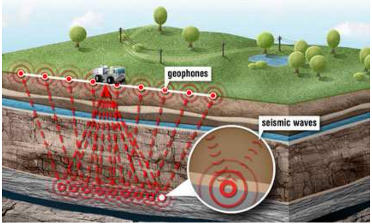
How one can determine the topology and metric of complicated structures in space-time with a radar-like device?

How the blue-prints of a "space-time radar" look like?

Figures: Anderson institute and Greenleaf-K.-L.-U.

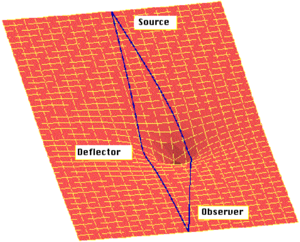
Some results for hyperbolic inverse problems for linear equations:

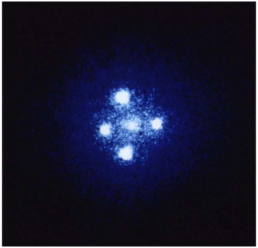
- Belishev-Kurylev 1992 and Tataru 1995: Reconstruction of a Riemannian manifold with time-indepedent metric.
  The used unique continuation fails for non-real-analytic time-depending coefficients (Alinhac 1983).
- Kachalov-Kurylev 1998: Reconstruction with local data.
- Eskin 2008: Wave equation with time-depending (real-analytic) lower order terms.
- Helin-Lassas-Oksanen 2012: Combining several measurements for together for the wave equation.

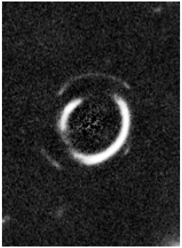


Outline:

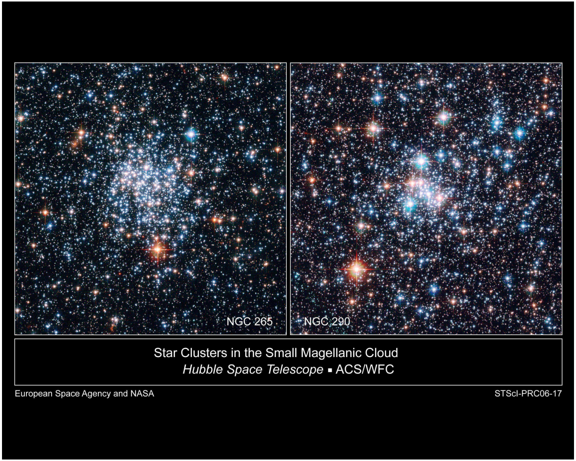
- Inverse problems in space-time for passive measurements
- Inverse problem for non-linear wave equation
- Einstein-scalar field equations





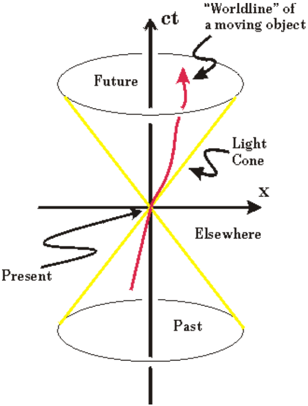


Inverse problems in space-time: Passive measurements



Can we determine structure of the space-time when we see light coming from many point sources that vary in time?

Definitions



Let (M, g) be a Lorentzian manifold,

where the metric g is semi-definite.

$\xi \in T_x M$ is light-like if $g(\xi, \xi) = 0$, $\xi \neq 0$.

$\xi \in T_x M$ is time-like if $g(\xi, \xi) < 0$.

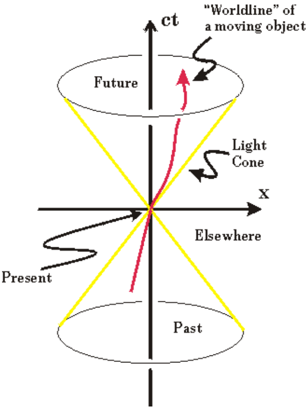
A curve $\mu(s)$ is time-like if $\dot{\mu}(s)$ is time-like.

Example: Minkowski space $\mathbb{R}^{1+3}$. The metric at

$(x^0, x^1, x^2, x^3) \in \mathbb{R}^{1+3}$ is

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$

Definitions



Let (M, g) be a Lorentzian manifold.

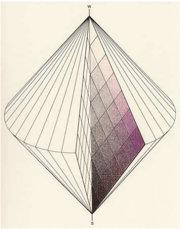
$L_q M = \{\xi \in T_q M \setminus 0;\ g(\xi, \xi) = 0\}$,

$L_q^+ M \subset L_q M$ is the future light cone,

$J^+(q) = \{x \in M;\ x \text{ is in causal future of } q\}$,

$J^-(q) = \{x \in M;\ x \text{ is in causal past of } q\}$,

$\gamma_{x,\xi}(t)$ is a geodesic with the initial point (x, ξ).



(M, g) is globally hyperbolic if

there are no closed causal curves and the set

$J^-(p_1) \cap J^+(p_2)$ is compact for all $p_1, p_2 \in M$.

Then M can be represented as $M = \mathbb{R} \times N$.

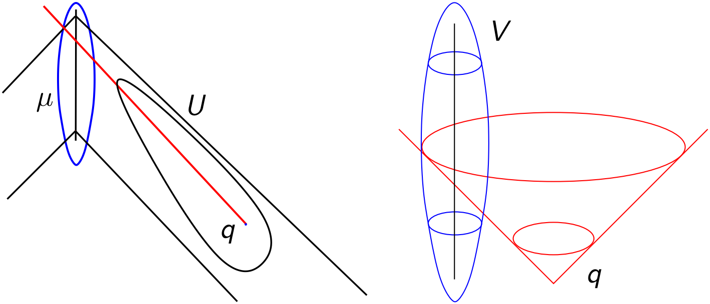
More definitions

Let $\mu = \mu((-1,1)) \subset M$ be a time-like geodesics, $p^-, p^+ \in \mu$. We consider observations in a neighborhood $V \subset M$ of μ.

Let $U \subset J^-(p^+) \setminus J^-(p^-)$ be an open, relatively compact set.

The light observation set $P_V(q)$ for $q \in U$ is the intersection of the future light cone of q and V,

$$P_V(q) = \exp_q(\overline{L_q^+ M}) \cap V = \{\gamma_{q,\xi}(r) \in V;\ \xi \in L_q^+ M,\ r \geq 0\}.$$

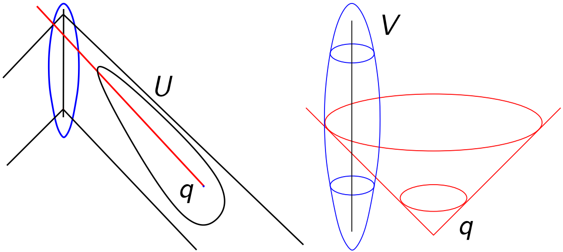


Theorem

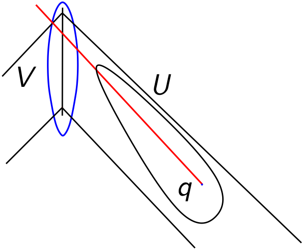
Let (M, g) be an open, globally hyperbolic Lorentzian manifold of dimension $n \geq 3$. Assume that μ is a time-like geodesic containing points p^- and p^+, and $V \subset M$ is a neighborhood of μ.
Let $U \subset J^-(p^+) \setminus J^-(p^-)$ be a relatively compact open set.
Then $(V, g|_V)$ and the collection of the light observation sets,

$$P_V(U) := \left\{ P_V(q) \subset V \;\middle|\; q \in U \right\},$$

determine the set U, up to a change of coordinates, and the conformal class of the metric g in U.

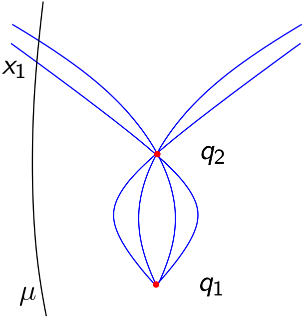


Reconstruction of the topological structure of U



Assume that $q_1, q_2 \in U$ are such that $P_V(q_1) = P_V(q_2)$. Then all light-like geodesics from q_1 to V go through q_2.

Let x_1 be the earliest point of $\mu \cap P_V(q_1)$.



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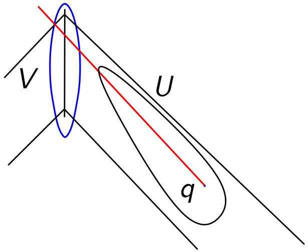

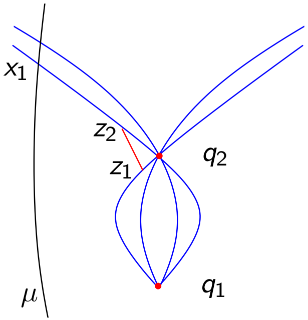

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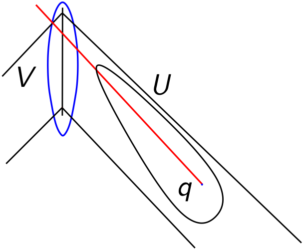
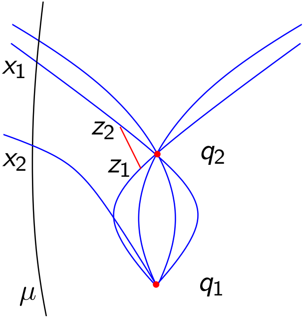
Let x_1 be the earliest point of $\mu \cap P_V(q_1)$. Using a short cut argument we see that there is a causal curve from q_1 to x_1 that is not a geodesic.

Reconstruction of the topological structure of U





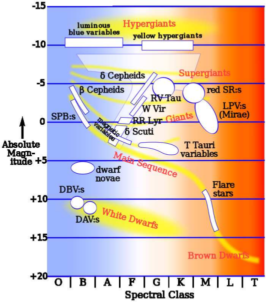
Assume that $q_1, q_2 \in U$ are such that $P_V(q_1) = P_V(q_2)$. Then all light-like geodesics from q_1 to V go through q_2.

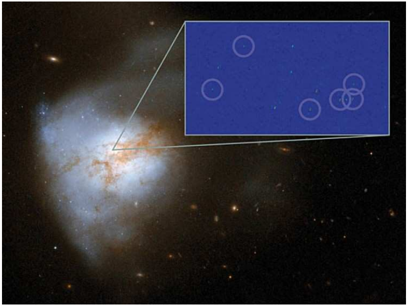
Let x_1 be the earliest point of $\mu \cap P_V(q_1)$. Using a short cut argument we see that there is a causal curve from q_1 to x_1 that is not a geodesic.

This implies that q_1 can be observed on μ before x_1.

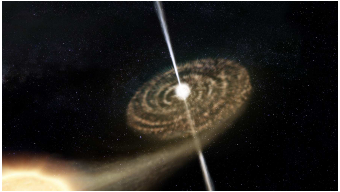
The map $P_V : \overline{U} \mapsto 2^{TV}$ is continuous and one-to-one.
As $\overline{U}$ is compact, the map $P_V : \overline{U} \to P_V(\overline{U})$ is a homeomorphism.

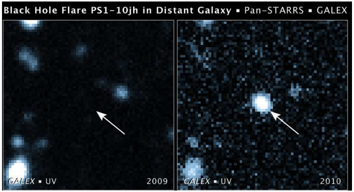
Possible applications of the theorem





Left: Variable stars in Hertzsprung-Russell diagram on star types.
Right: Galaxy Arp 220 (Hubble Space Telescope)





Artistic impressions on matter falling into a black hole and Pan-STARRS1 telescope picture.

Outline:

- Inverse problems in space-time for passive measurements
- Inverse problem for non-linear wave equation
- Einstein-scalar field equations

"Can we image a wave using other waves?"

Inverse problem for non-linear wave equation

Let $M = \mathbb{R} \times N$, $\dim(M) = 4$. Consider the equation

$$\Box_g u(x) + a(x)\, u(x)^2 = f(x) \quad \text{on } M_1 = (-\infty, T) \times N,$$
$$u(x) = 0 \quad \text{for } x = (x^0, x^1, x^2, x^3) \in (-\infty, 0) \times N,$$

where $\mathrm{supp}(f) \subset V$, $V \subset M_1$ is open,

$$\Box_g u = \sum_{p,q=0}^{3} |\det(g(x))|^{-\frac{1}{2}} \frac{\partial}{\partial x^p} \left(|\det(g(x))|^{\frac{1}{2}} g^{pq}(x) \frac{\partial}{\partial x^q} u(x) \right),$$

$f \in C_0^6(V)$ is a source, and $a(x)$ is a non-vanishing C^∞-smooth function.

In a neighborhood $\mathcal{W} \subset C_0^6(V)$ of the zero-function, define the measurement operator (source-to-solution operator) by

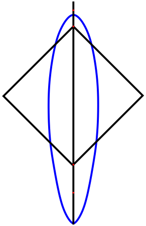
$$L_V : f \mapsto u|_V, \quad f \in \mathcal{W} \subset C_0^6(V).$$

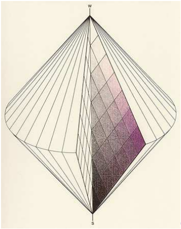
Theorem

Let (M, g) be a globally hyperbolic Lorentzian manifold of dimension $(1+3)$. Let μ be a time-like path containing p^- and p^+, $V \subset M$ be a neighborhood of μ, and $a(x)$ be a non-vanishing function. Consider the non-linear wave equation

$$\begin{aligned} \square_g u(x) + a(x)\, u(x)^2 &= f(x) \quad \text{on } M_1 = (-\infty, T) \times N, \\ u &= 0 \quad \text{in } (-\infty, 0) \times N, \end{aligned}$$

where $\mathrm{supp}(f) \subset V$. Then $(V, g|_V)$ and the measurement operator $L_V : f \mapsto u|_V$ determine the set $J^+(p^-) \cap J^-(p^+) \subset M$, up to a change of coordinates, and the conformal class of g in the set $J^+(p^-) \cap J^-(p^+)$.





Idea of the proof: Non-linear geometrical optics.

The non-linearity helps in solving the inverse problem.

Let $u = \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + \varepsilon^4 w_4 + E_\varepsilon$ satisfy

$$\begin{aligned} &\Box_g u + a u^2 = f, \quad \text{on } M_1 = (-\infty, T) \times N, \\ &u|_{(-\infty,0)\times N} = 0 \end{aligned}$$

with $f = \varepsilon f_1$, $\varepsilon > 0$.
When $Q = \Box_g^{-1}$, we have

$$\begin{aligned} w_1 &= Qf_1, \\ w_2 &= -Q(a\, w_1\, w_1), \\ w_3 &= 2Q(a\, w_1\, Q(a\, w_1\, w_1)), \\ w_4 &= -Q(a\, Q(a\, w_1\, w_1)\, Q(a\, w_1\, w_1)) \\ &\quad -4Q(a\, w_1\, Q(a\, w_1\, Q(a\, w_1\, w_1))), \\ \|E_\varepsilon\| &\le C\varepsilon^5. \end{aligned}$$

Interaction of waves in Minkowski space $\mathbb{R}^4$

Let x^j, $j = 1,2,3,4$ be coordinates such that $\{x^j = 0\}$ are light-like. We consider waves

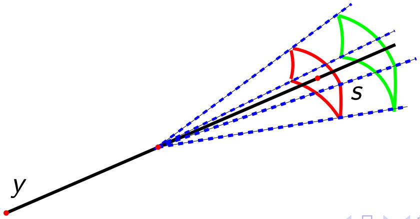
$$u_j(x) = v \cdot (x^j)_+^m, \quad (s)_+^m = |s|^m H(s), \quad v \in \mathbb{R},\ j = 1,2,3,4.$$

Waves u_j are conormal distributions, $u_j \in I^{m+1}(K_j)$, where

$$K_j \;=\; \{x^j = 0\} \subset \mathbb{R}^4, \quad j = 1,2,3,4.$$

The interaction of the waves $u_j(x)$ produce new sources on

$$\begin{aligned} K_{12} &= K_1 \cap K_2, \\ K_{123} &= K_1 \cap K_2 \cap K_3 = \text{line}, \\ K_{1234} &= K_1 \cap K_2 \cap K_3 \cap K_3 = \{q\} = \text{one point}. \end{aligned}$$



Interaction of two waves

If we consider sources $f_{\vec\varepsilon}(x) = \varepsilon_1 f_{(1)}(x) + \varepsilon_2 f_{(2)}(x)$, $\vec\varepsilon = (\varepsilon_1, \varepsilon_2)$, and the corresponding solution $u_{\vec\varepsilon}$ of the wave equation, we have

$$\begin{aligned} W_2(x) &= \frac{\partial}{\partial \varepsilon_1}\frac{\partial}{\partial \varepsilon_2} u_{\vec\varepsilon}(x)\Big|_{\vec\varepsilon=0} \\ &= Q(a\, u_{(1)} \cdot u_{(2)}), \end{aligned}$$

where $Q = \square_g^{-1}$ and

$$u_{(j)} = Qf_{(j)}.$$

Recall that $K_{12} = K_1 \cap K_2 = \{x^1 = x^2 = 0\}$. Since light-like co-vectors in the normal bundle N^*K_{12} are in $N^*K_1 \cup N^*K_2$,

$$\mathrm{singsupp}(W_2) \subset K_1 \cup K_2.$$

Thus no interesting singularities are produced by the interaction of two waves.

Interaction of three waves

If we consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{3} \varepsilon_j f_{(j)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, and the corresponding solution $u_{\vec{\varepsilon}}$, we have

$$\begin{aligned} W_3 &= \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3} u_{\vec{\varepsilon}}\big|_{\vec{\varepsilon}=0} \\ &= Q(a\, u_{(1)} \cdot Q(au_{(2)} \cdot u_{(3)})) + \ldots, \end{aligned}$$

where $Q = \Box_g^{-1}$. The interaction of the three waves happens on the line $K_{123} = K_1 \cap K_2 \cap K_2$.
The normal bundle N^*K_{123} contains light-like directions that are not in $N^*K_1 \cup N^*K_2 \cup N^*K_3$ and hence new singularities appear.

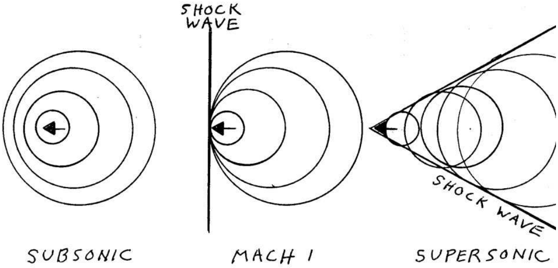
Using standard tools of microlocal analysis we can analyze $Q(au_{(1)} \cdot au_{(2)})$, but not the interaction of 3 waves. Let $F_\tau(x) = e^{\tau(i\, x\cdot p_0 - (x-x_0)^2)}\phi(x)$. By studying

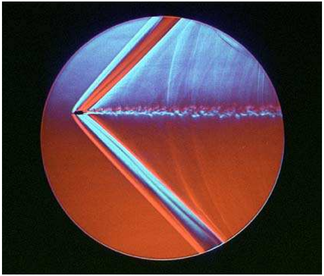
$$\langle F_\tau, W_3\rangle_{L^2(M)} = \langle Q^* F_\tau, a\, u_{(1)} \cdot Q(au_{(2)} \cdot u_{(3)})\rangle_{L^2(M)} + \ldots,$$

as $\tau \to \infty$, we can detect if $(x_0, p_0) \in \mathrm{WF}(W_3)$.

Interaction of waves:

The non-linearity helps in solving the inverse problem.
Artificial sources can be created by interaction of waves using the non-linearity of the wave equation.





The interaction of 3 waves creates a point source in space that seems to move at a higher speed than light, that is, it appears like a tachyonic point source, and produces a new "shock wave" type singularity.

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Interaction of three waves.

Interaction of four waves

Consider sources $f_{\vec{\varepsilon}}(x) = \sum_{j=1}^{4} \varepsilon_j f_{(j)}(x)$, $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$, the corresponding solution $u_{\vec{\varepsilon}}$, and

$$W_4 \quad = \quad \partial_{\varepsilon_1}\partial_{\varepsilon_2}\partial_{\varepsilon_3}\partial_{\varepsilon_4} u_{\vec{\varepsilon}}(x)\Big|_{\vec{\varepsilon}=0}.$$

Since $K_{1234} = \{q\}$ we have $N^* K_{1234} = T_q^* M$. Thus, when the conic waves intersect, an artificial point source appears. We have

$$\mathrm{singsupp}(W_4) \subset (\cup_{j=1}^{4} K_j) \cup \Sigma \cup \mathcal{L}_q^+ M,$$

where Σ is the union of conic waves produced by 3-interactions. Above, $\mathcal{L}_q^+ M = \exp_q(L_q^+ M)$ is the union of future going light-like geodesics starting from the point q.

Interaction of four waves.

The 3-interaction produces conic waves (only one is shown below).

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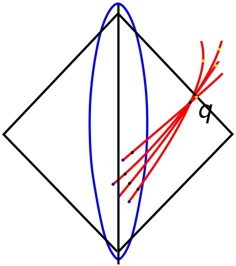
The 4-interaction produces a spherical wave from the point q that determines the light observation set $P_V(q)$.



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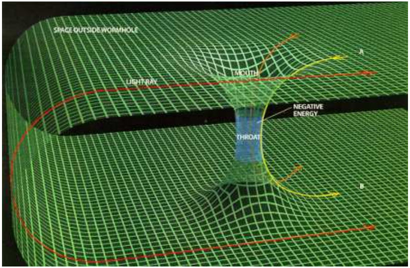
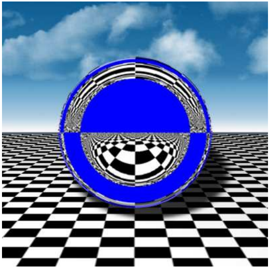




Einstein equations

The Einstein equation for the $(-,+,+,+)$-type Lorentzian metric g_{jk} of the space time is

$$\mathrm{Ein}_{jk}(g) = T_{jk},$$

where

$$\mathrm{Ein}_{jk}(g) = \mathrm{Ric}_{jk}(g) - \frac{1}{2}(g^{pq}\,\mathrm{Ric}_{pq}(g))g_{jk}.$$

In vacuum, $T = 0$. In wave map coordinates, the Einstein equation yields a quasilinear hyperbolic equation and a conservation law,

$$g^{pq}(x)\frac{\partial^2}{\partial x^p \partial x^q} g_{jk}(x) + B_{jk}(g(x), \partial g(x)) = T_{jk}(x),$$
$$\nabla_p(g^{pj}\,T_{jk}) = 0.$$

One can not do measurements in vacuum, so matter fields need to be added. We can consider the coupled Einstein and scalar field equations with sources,

$$\begin{aligned} &\mathrm{Ein}(g) = T, \quad T = \mathbf{T}(\phi, g) + \mathcal{F}_1, \quad \text{on } (-\infty, T) \times N, \\ &\square_g \phi_\ell - m^2 \phi_\ell = \mathcal{F}_2^\ell, \quad \ell = 1, 2, \dots, L, \\ &g|_{t<0} = \widehat{g}, \quad \phi|_{t<0} = \widehat{\phi}. \end{aligned} \tag{1}$$

Here, $\widehat{g}$ and $\widehat{\phi}$ are C^∞-smooth and satisfy equations (1) with the zero sources and

$$\mathbf{T}_{jk}(g, \phi) = \sum_{\ell=1}^{L} \partial_j \phi_\ell \, \partial_k \phi_\ell - \frac{1}{2} g_{jk} g^{pq} \partial_p \phi_\ell \, \partial_q \phi_\ell - \frac{1}{2} m^2 \phi_\ell^2 g_{jk}.$$

To obtain a physically meaningful model, the stress-energy tensor T needs to satisfy the conservation law

$$\nabla_p (g^{pj} T_{jk}) = 0, \quad k = 1, 2, 3, 4.$$

Definition

Linearization stability (Choquet-Bruhat, Deser, Fischer, Marsden)
Let $f = (f^1, f^2)$ satisfy the linearized conservation law

$$\sum_{\ell=1}^{L} f_\ell^2 \, \partial_j \widehat{\phi}_\ell + \tfrac{1}{2} \widehat{g}^{pk} \widehat{\nabla}_p f_{kj}^1 = 0, \quad j = 1,2,3,4 \tag{2}$$

and let $(\dot{g}, \dot{\phi})$ be the corresponding solution of the linearized Einstein equation. We say that f has the Linearization Stability (LS) property if there is $\varepsilon_0 > 0$ and families

$$\begin{aligned} \mathcal{F}_\varepsilon &= (\mathcal{F}_\varepsilon^1, \mathcal{F}_\varepsilon^2) = \varepsilon f + O(\varepsilon^2), \\ g_\varepsilon &= \widehat{g} + \varepsilon \dot{g} + O(\varepsilon^2), \\ \phi_\varepsilon &= \widehat{\phi} + \varepsilon \dot{\phi} + O(\varepsilon^2), \end{aligned}$$

where $\varepsilon \in [0, \varepsilon_0)$, such that $(g_\varepsilon, \phi_\varepsilon)$ solves the non-linear Einstein equations and the conservation law

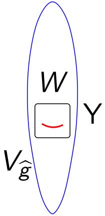
$$\nabla_j^{g_\varepsilon} (\mathbf{T}^{jk}(g_\varepsilon, \phi_\varepsilon) + (\mathcal{F}_\varepsilon^1)^{jk}) = 0, \quad k = 1,2,3,4.$$

Let $V_{\widehat{g}} \subset M$ be a open set that is a union of freely falling geodesics that are near μ, $L \geq 5$.

Condition A: Assume that at any $x \in V_{\widehat{g}}$ the 5×5 matrix

$$[A_{j\ell}(x)]_{j,\ell \leq 5} = \begin{bmatrix} (\,\partial_j \widehat{\phi}_\ell(x))_{\ell \leq 5,\ j \leq 4} \\ (\widehat{\phi}_\ell(x))_{\ell \leq 5} \end{bmatrix}$$

is invertible.



Let $I^k(Y)$ be the space of conormal distributions for $Y \subset M$.

Theorem

Let condition A be valid, $W \subset V_{\widehat{g}}$ be open, and $Y \subset W$ be a 2-dimensional space-like surface. Assume that $f = (f^1, f^2) \in I^k(Y)$ is supported in W and f^1 has a principal symbol $a_{jk}(y, \eta)$ satisfying $\widehat{g}^{lk}(y)\eta_l a_{jk}(y, \eta) = 0$ on N^*Y. Then there is a smoother correction term $f_{cor} \in I^{k-1}(Y)$ supported in W such that $f + f_{cor}$ has a linearization stability property with a family $\mathcal{F}_\varepsilon$ supported in W.

Idea of proof: We formulate the direct problem with adaptive source functions,

$$\mathrm{Ein}_{jk}(g) = P_{jk} - \sum_{\ell=1}^{L}(S_\ell\phi_\ell + \frac{1}{2}S_\ell^2)g_{jk} + \mathbf{T}_{jk}(g,\phi),$$

$$\square_g\phi_\ell - m^2\phi_\ell = S_\ell, \quad \text{in } M_0, \quad \ell = 1,2,3,\dots,L,$$

$$S_\ell = Q_\ell + \mathcal{S}_\ell^{2nd}(g,\phi,\nabla\phi,Q,\nabla Q,P,\nabla P),$$

$$g = \widehat{g}, \quad \phi_\ell = \widehat{\phi}_\ell, \quad \text{in } (-\infty,0)\times N.$$

Here Q and P_{jk} are considered as the primary sources.
The functions $\mathcal{S}_\ell^{2nd}$ are constructed so that the conservation law is satisfied for all solutions (g,ϕ).

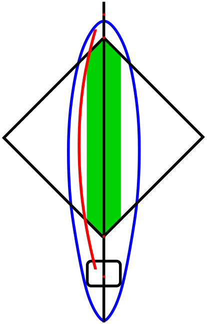
Let $V_{\widehat{g}} \subset M$ be a neighborhood of the geodesic μ and $p^-, p^+ \in \mu$.

Theorem

Assume that the condition A is valid. Let

$$\begin{aligned}\mathcal{D} \;=\; \{(V_g, g|_{V_g}, \phi|_{V_g}, \mathcal{F}|_{V_g});\ & g \textit{ and } \phi \textit{ satisfy Einstein equations} \\ & \textit{with a source } \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2),\ \textit{supp}\ (\mathcal{F}) \subset V_g, \textit{ and} \\ & \nabla_j(\mathbf{T}^{jk}(g,\phi) + \mathcal{F}_1^{jk}) = 0\}.\end{aligned}$$

The data set $\mathcal{D}$ determines uniquely the conformal type of the double cone $(J^+(p^-) \cap J^-(p^+), \widehat{g})$.



Thank you for your attention!