Unbounded potential recovery with fixed energy in the plane

Keith Rogers

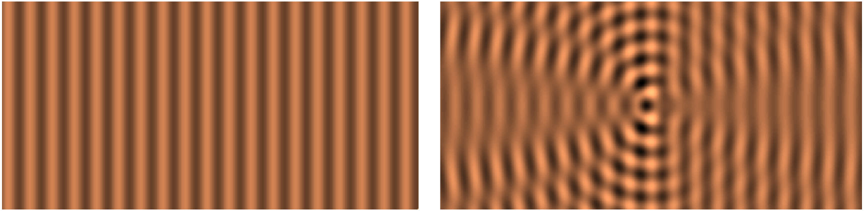
INSTITUTO DE CIENCIAS MATEMÁTICAS

CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS

Joint work with **Kari Astala** and **Daniel Faraco**

Inverse scattering at a fixed energy k^2

- For all $\theta \in \mathbb{S}^{d-1}$ we send plane waves $e^{ik\theta\cdot x}$ toward an unknown object.
- For all $\vartheta \in \mathbb{S}^{d-1}$ we measure the scattered waves.



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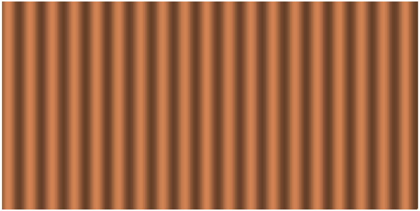

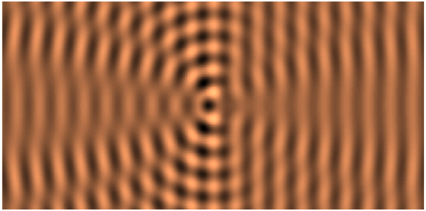

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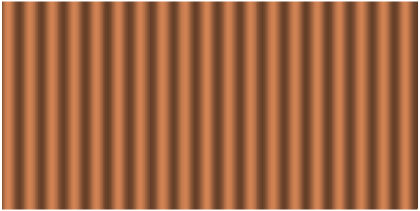

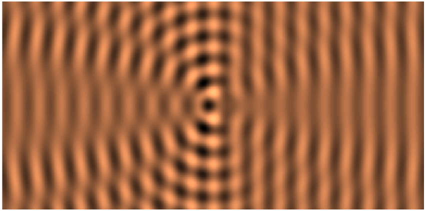

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- If we consider $e^{ik\theta\cdot x}$ to be the wavefunction of a beam of neutrons fired at a nucleus, the task is to recover the nuclear potential $V(x)$ at each x.

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- So from now on we consider only the quantum problem.

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- The challenge is then to recover V from A.

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- Tejero

On mathematical retreat

On the border between France and Spain (near Baztan, Navarra)

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Bukhgeim's solutions to Laplace's equation

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- The solutions grow exponentially but $|e^{i\psi_{n,x}} e^{i\overline{\psi}_{n,x}}| = 1$.

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$$\begin{aligned}\left\langle (\Lambda_V - \Lambda_0)[e^{i\psi_{n,x}}], e^{i\overline{\psi}_{n,x}} \right\rangle &= \int_\Omega V\, e^{i\psi_{n,x}} e^{i\overline{\psi}_{n,x}} \\ &= \int V(z)\, e^{i\frac{n}{4}\left((z_1-x_1)^2-(z_2-x_2)^2\right)}\, dz.\end{aligned}$$

Thus by the method of stationary phase,

$$\begin{aligned}\frac{n}{4\pi}\left\langle (\Lambda_V - \Lambda_0)[e^{i\psi_{n,x}}], e^{i\overline{\psi}_{n,x}} \right\rangle &= \int V(z)\, \frac{n}{4\pi}\, e^{i\frac{n}{4}\left((z_1-x_1)^2-(z_2-x_2)^2\right)}\, dz \\ &= V * K_n(x) \\ &\to V(x) \qquad \text{as } n \to \infty.\end{aligned}$$

Making the heuristic precise

As before, but with $e^{i\psi_{n,x}}$ replaced by $u = e^{i\psi_{n,x}}(1+w)$, As before, but with $e^{i\psi_{n,x}}$ replaced by $u = e^{i\psi_{n,x}}(1+w)$,

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We expect

$$\int V \frac{n}{4\pi} e^{i\psi_{n,x}} e^{i\overline{\psi}_{n,x}} \to V(x) \quad \text{as} \quad n\to\infty, \tag{conv}$$

so we also need to prove

$$\int V\, w \frac{n}{4\pi} e^{i\psi_{n,x}} e^{i\overline{\psi}_{n,x}} \to 0 \quad \text{as} \quad n\to\infty. \tag{remainder}$$

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Key estimate: $\|e^{i\psi_{n,x}}e^{i\overline{\psi}_{n,x}}F\|_{\dot H^{-s}} \leqslant Cn^{-s}\|F\|_{\dot H^s}.$

Connection with the time dependent nonelliptic equation

Writing $K_n(y) = \frac{n}{4\pi} e^{i\frac{n}{4}(y_1^2 - y_2^2)}$, it remains to prove

$$V * K_n(x) \to V(x) \quad \text{as} \quad n \to \infty. \tag{conv}$$

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Theorem

If $V \in H^{1/2}$ then (conv) holds for all $x \in \Omega\backslash E$ with $\dim_H(E) \leqslant 3/2$.

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By Frostman's lemma, it suffices to prove that

$$\mu\Big\{x \in \Omega \,:\, \limsup_{n\to\infty} |e^{i\frac{1}{n}\Box}V(x) - V(x)| \neq 0\Big\} = 0 \qquad (*)$$

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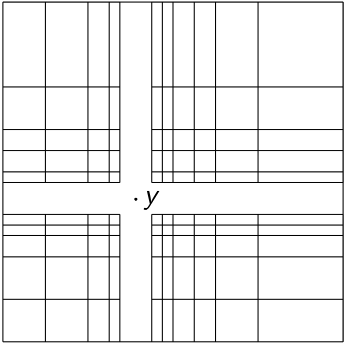
After factorising the problem into two one–dimensional problems and bounding an oscillatory integral, this follows from

$$\int_\Omega\int_\Omega\frac{d\mu(x)d\mu(y)}{|x_1-y_1|^{1/2}|x_2-y_2|^{1/2}}\leqslant C\|\mu\|c_\alpha(\mu),$$

which follows from a dyadic decomposition away from the singularities:

Sketch of Proof



$\square$

Potential Recovery

Corollary

- *For all compactly supported* $V \in H^{1/2}$,

$$V(x) = \lim_{n\to\infty} \frac{n}{4\pi i}\Big\langle (\Lambda_V - \Lambda_0)[u], e^{i\overline{\psi}_{n,x}} \Big\rangle, \quad \forall \;\; x \in \Omega\backslash E,$$

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- *There exist compactly supported* $V \in H^s$ *with* $s < 1/2$ *for which this recovery process fails completely.*

Theorem

For $V \in H^s$ *with* $s > 0$, *we can identify* $\Gamma : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$, *depending only on* $\psi_{n,x}$ *and* Λ_V, *such that*

$$u = (\mathrm{I} - \Gamma)^{-1}[e^{i\psi_{n,x}}] \qquad on \quad \partial\Omega.$$

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By writing $\Gamma[f] := T_r \circ \Big\langle (\Lambda_V - \Lambda_0)[f], G \Big\rangle$, we have

$$(\mathrm{I} - \Gamma)[u|_{\partial\Omega}] = e^{i\psi_{n,x}}|_{\partial\Omega},$$

and $\mathrm{I} - \Gamma$ can be inverted as before.

Numerical implementation

Original:

Recovered: