Singularities of the scattering kernel in the Lax-Phillips framework for the elastic wave equation

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$$\begin{cases} (\partial_t^2 - L(x,\partial_x))u(t,x) = 0 \quad \text{in } \mathbb{R}\times\Omega, \\ Nu|_{\partial\Omega} = 0 \qquad\quad \text{on } \mathbb{R}\times\partial\Omega \end{cases}$$

$$L_0(x,\partial_x) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_i}\partial_{x_j}, \ \ N = \sum_{i,j=1}^{n} a_{ij}(x)\nu_i\partial_{x_j}$$

(A.1) $a_{ij}(x) = {}^t a_{ji}(x)$ real $n\times n$-matrices, constant for large $|x|$

(A.2) $L_0(x,\xi)$ positive definite

(A.3) $\lambda_i(x,\xi)$ constant multiplicity

We formulate a scattering theory in the Lax-Phillips sense.

We would study relations between singularities of the scattering kernel and properties of the scatterer.

Formulation of the scattering theory

$$U(t) : \ {}^t(u(0,\cdot), \partial_t u(0,\cdot)) \longmapsto {}^t(u(t,\cdot), \partial_t u(t,\cdot))$$

a group of unitary operators on a Hilbert space H

We employ the equation in the free space (without the boundary):

$$(\partial_t^2 - L(x, \partial_x))u_0(t,x) = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^n.$$

We can obtain the wave operators $W_\pm : f_\pm \mapsto f$ such that

$$U(\pm\infty)f \ \simeq \ U_0(\pm\infty)f_\pm, \qquad W_\pm : H_0 \mapsto H \text{ is unitary.}$$

We make the translation representation $T_0^\pm$:

$$T_0^\pm : H_0 \longmapsto L^2(\mathbb{R}_s^1; N) \text{ is unitary } \ (N = L^2(S^{n-1})),$$

$$(T_0^\pm U_0(t) f)(s) = (T_0^\pm f)(s-t).$$

The scattering operator S is defined through $T_0^{\pm}$:

$$S = T_0^{+}(W_{+})^{-1}W_{-}(T_0^{-})^{-1}\colon L^2(\mathbb{R}_s^1; N) \to L^2(\mathbb{R}_s^1; N) \quad \text{unitary}$$

S is expressed with the kernel $S(s,\theta,\omega)$ (the scattering kernel):

$$(Sk)(s,\theta) = k(s,\theta) + \int\int S(s-\tilde{s},\theta,\tilde{\omega})k(\tilde{s},\tilde{\omega})d\tilde{s}d\tilde{\omega}$$

For the incident δ-function wave $\big($like $\delta(\text{const}\ \omega x - t)\big)$, the scattered wave is approximated by $\displaystyle\int S(\theta x - t, \theta, \omega)d\theta$.

$S(\theta x - t, \theta, \omega)$ means the scattered wave in the direction θ.

Represetation of $S(s, \theta, \omega)$

$\lambda_l(\omega)$: the eigen-value of $L_0(\omega)$
$P_l(\omega)$: the orthogonal projection to the eigen-space

$\delta(t - \lambda_l(\omega)^{-1/2}\omega x)P_l(\omega)$ is the plane wave of the λ_l-mode.

For the incident wave of this type let v_l be the scattered wave:

$$\left\{ \begin{array}{l} (\partial_t^2 - L_0(\partial_x))v_l(t, x; \omega) = 0 \quad \text{in } \mathbb{R} \times \Omega, \\ N v_l|_{\partial\Omega} = c_l(\omega) N \delta(t - \lambda_l(\omega)^{-1/2}\omega x)P_l(\omega), \\ v_l = 0 \ \text{ if } \quad t << 0. \end{array} \right.$$

We assume that every slowness surface is strictly convex:

(A.4) The Gauss curvature of $\{\xi : \lambda_l(\xi) = 1\}$ does not vanish.

Then $S(s, \theta, \omega)$ is represented with $v_l(t, x; \omega)$:

Theorem 1 (Soga: Osaka J. Math. 1992).

Let $L(x, \partial_x) = L_0(\partial_x)$ and assume (A.1) $\sim$ (A.4). Then we have

$$S(s,\theta,\omega) = \sum_{i,j=1}^{n} \lambda_i(\xi)^{-n/4} \int_{\partial\Omega} \{P_i(\xi) \\ (\partial_t^{n-2} N v_j)(\lambda_i(\theta)^{-1/2}\theta x - s, x; \omega) \\ -\lambda_i(\xi)^{-1/2} P_i(\xi)({}^t N\theta x)(\partial_t^{n-1} v_j)(\lambda_i(\theta)^{-1/2}\theta x - s, x; \omega)\} dS_x.$$

This can be extended to the case of the variable coefficient if $L(x, \partial_x)$ has the unique continuation property.

The representation was obtained for the scalar-valued wave equation by Majda (CPAM 1977), Melrose (CPAM 1980), Soga (J. Kyoto U. 1983), ...

Constructing the asymptotic expansion of $v_j(t, x, ; \omega)$, we can examine sing supp $S(s, \theta, \omega)$ by Theorem 1.

This is the same approach as Majda (CPAM 1977) did for the scalar-valued wave equation.

Theorem 2 (Soga: Osaka J. Math. 1992).

Set $r(\omega) = \min_{x \in \partial\Omega} \omega x$. Then

(i) supp $P_i(\omega)S(s, -\omega, \omega)P_j(\omega) \subset (-\infty, -(\lambda_i(\omega)^{-1/2} + \lambda_j(\omega)^{-1/2})r(\omega)]$;

(ii) $P_i(\omega)S(s, -\omega, \omega)P_i(\omega)$ is singular at $s = -2\lambda_i(\omega)^{-1/2}r(\omega)$.

$\lambda_i(\omega)^{-1/2}r(\omega)$ is the time for which the λ_i-mode wave reaches $\partial\Omega$ from the origin $x = 0$.

The singularity of $P_i(\omega)S(s, -\omega, \omega)P_j(\omega)$ $(i \neq j)$ at $-(\lambda_i(\omega)^{-1/2} + \lambda_j(\omega)^{-1/2})r(\omega)]$ is very delicate.

This depends on the shape of $\partial\Omega$ (cf. M.Kawashita-Soga: J. Math. Soc. Japan 1990, etc.)

We can examine the asymptotic form of the singularity of $P_i(\omega)S(s, -\omega, \omega)P_i(\omega)$ at $s = -2\lambda_i(\omega)^{-1/2}r(\omega)$ in some cases.

Scattering of the surface waves (the Rayleigh wave, etc.)

We would take scattering of the surface waves into account.
Then we need to reconsider setting of the equation in the free space.

We select the half-space as the free space and perturb the flatness of the boundary.

In this selection we construct the scattering theory of the Lax-Phillips type and obtain the representation of the scattering kernel.

Furthermore, we reveal relation between the situation of the boundary and the scattering kernel for the Rayleigh wave.

New difficulties arise in these works because the surface waves do not have locality in the support.

Let Ω be the perturbed half-space ($\subset \mathbf{R}^3$) and L be the isotopic operator.

Then, there exist several kinds of waves, i.e., P-wave, S-wave, the Rayleigh wave, etc. and the phenomena are classified near $\partial\Omega$:

(P)	for an incident P-wave, P- and S-waves are reflected,
(SV)	for an incident S-wave, P- and S-waves are reflected,
(SH)	for an incident S-wave, only S-wave is reflected,
(SVO)	for an incident S-wave, S-waves is reflected totally,
(R)	there exists the Rayleigh wave.

The translation representation T_0 in the free space $\mathbb{R}^3_+$ can be constructed (M.Kawashita-W.Kawashita-Soga: Comm. PDE 2003):

$$T_0 : H_0 \longrightarrow L^2(\mathbb{R}^1; N) \quad \text{unitary},$$

$$N = \bigoplus_{\alpha = \Lambda} L^2(S_\alpha) \quad \Lambda = \{\text{P, SV, SH, SVO, R}\}.$$

S_α is the set of the directions of the incident waves.

We define the scattered waves $w_\alpha(t,x;\omega)$ corresponding to $v_l(t,x;\omega)$. For $\alpha =$ P, SV and SH, w_α are of the same type as v_l.

For $\alpha =$ SVO and R, the incident waves w_α^0 are different in the definitions, i.e. w_α^0 are of δ-function type only on the boundary and their supports are spread on the whole space.

We employ some solution $w_{\alpha,+}^0(t.x;\omega)$ in the perturbed space asymptotically equal to $w_\alpha^0(t,x;\omega)$ as $t \to -\infty$; $w_\alpha^0(t,x;\omega)$ is of the form on $\partial\Omega$ similar to $\delta(c_R t - \omega x)$.

Set $w_\alpha = w_{\alpha,+}^0 - w_\alpha^0$ ($\alpha =$ SVO, R); we regard w_α as the scattered wave for w_α^0.

Then the scattering kernel $S(s,\theta,\omega) = \big(S_{\alpha\beta}(s,\theta,\omega)\big)_{\alpha,\beta\in\Lambda}$ is represented with w_α and w_α^0:

Theorem 3 (M.Kawashita-W.Kawashita-Soga: Trans. AMS 2006).

$$S_{\alpha\beta}(s,\theta,\omega) = c_{\alpha\beta}\int_{\Omega\cap\mathbb{R}^3_+}\int_{\mathbb{R}}\partial_{\tilde{s}}w^0_\alpha(\tilde{s},y;\theta)\ (\partial_s^2 - L_0)w_\beta(\tilde{s}-s,y;\omega)d\tilde{s}dy$$

$$+c_{\alpha\beta}\int_{\partial(\Omega\cap\mathbb{R}^3_+)}\{\int_{\mathbb{R}}\partial_{\tilde{s}}w^0_\alpha(\tilde{s},y;\theta)\ N_0 w_\beta(\tilde{s}-s,y;\omega)d\tilde{s}$$

$$-\int_{\mathbb{R}}N_0\partial_{\tilde{s}}w^0_\alpha(\tilde{s},y;\theta)\ w_\beta(\tilde{s}-s,y;\omega)d\tilde{s}\}dS_y.$$

The proof is based on examination of the Green functions for the equation changed by the Laplace transformation in t.

In the cases $\alpha,\beta =$ P, SV, SH, using Theorem 3, we can obtain the result for sing supp $S_{\alpha\beta}(s,\theta,\omega)$ in the same way as Theorem 2.

We examine sing supp $S_{RR}(s,\theta,\omega)$. This means that we observe the Rayleigh wave scattered in the direction θ for the incident Rayleigh wave in the direction ω.

$S_{RR}(s,\theta,\omega)$ seems connected with the situations of the boundary.

The Dirichlet-Neumann operator has the hyperbolic part $H(x,\partial_t,\partial_x)$.
The Rayleigh wave comes from the part $H(x,\partial_t,\partial_x)$.
$H(x,\partial_t,\partial_x)$ is similar to $(\partial_t^2 - c_R^2\Delta)$ outside the perturbed region.

Let $(q(t,y';\omega),p(t,y';\omega))$ be the bicharacterisic curve for $H(x,\partial_t,\partial_x)$ with $(q(0,y';\omega),p(0,y';\omega)) = (y',c_R^{-1}\omega)$.

Assume that any of those curves is non-trapping.

$$M_\omega^+(\theta) = \{y' \in \partial\mathbf{R}^3_+;\ \lim_{t\to\infty} p(t-s,y';\omega) = c_R^{-1}\theta,\ c_R^{-1}\omega\, y' = s,\ \ s << 0\},$$
$$s^+(\theta,\omega) = \sup_{y'\in M_\omega^+(\theta)} \lim_{t\to\infty}(c_R^{-1}q(t-s,y';\omega)\cdot\theta - t).$$

Theorem 4. (M.Kawashita-Soga: Meth. Appl. Anal. 2010)

(i) sing supp $[S_{RR}(\cdot,\theta,\omega)] \subset (-\infty, s^+(\theta,\omega)]$.

(ii) If $M_\omega^+(\theta)$ is one point, $S_{RR}(s,\theta,\omega)$ is singular at $s = s^+(\theta,\omega)$.

Asymptotic solutions

Do the solutions exist for any boundary values?

The construction is connected with the reduced wave equation:

$$\begin{cases} (\sigma^2 - L_0(D_x))u(x', x_n) = 0 \quad \text{in} \quad \mathbb{R}^{n-1} \times \{x_n > 0\}, \\ u|_{x_n=0} = f(x') \qquad \text{on } \mathbb{R}^{n-1} \end{cases} \qquad (D_x = -i\partial_x).$$

$$u(x', x_n) = \int e^{ix'\xi'} e^{i|\sigma| x_n z_j(\xi'/|\sigma|)} \hat{f}(\xi') \bar{d}\xi', \qquad (\bar{d}\xi' = (2\pi)^{1-n})$$

$$\hat{f}(\xi') \in \text{Ker } [I - L_0(\xi'/|\sigma|, z_j)].$$

$\{z_j\}$ are the roots of det $[I - L_0(\xi'/|\sigma|, z)]$:

z_j real $\rightarrow$ the body waves
z_j non-real $\rightarrow$ the surface waves

z_j^+ $(j = 1, \ldots, k)$: the outgoing real roots $(\partial_{\xi_n}\lambda_i(\xi'/|\sigma|, z_+^j) > 0)$,
z_j^+ $(j = k+1, \ldots, d)$: the non-real roots with Im $z_j^+ > 0$.

$$\text{Ker } [I - L_0(\xi'/|\sigma|, z_1^+)] + \cdots + \text{Ker } [I - L_0(\xi'/|\sigma|, z_d^+)] \overset{?}{=} \mathbb{C}^n$$

$$\mathrm{Ker}\ [I - L_0(\xi'/|\sigma|, z_j)] = \int_{c(z_j)} \big(I - L_0(\xi'/|\sigma|, z)\big)^{-1}\, đz\ \mathbb{C}^n$$

$$(đz = (2\pi i)^{-1})$$

Theorem 5. Assume (A.1)∼(A.4) and for non-real z_j^+

(A.5) multiplicity of $z_j^+(\eta') = \dim \mathrm{Ker}\ [I - L_0(\eta', z_j^+)]$.

Then there exist matrices $h_j(\eta')$ such that

$$\sum_{j=1}^{d} \int_{c(z_j^+)} \big(I - L_0(\eta', z)\big)^{-1}\, đz\ h_j(\eta') = I.$$

This yields $\displaystyle\sum_{j=1}^{d} \mathrm{Ker}\ [I - L_0(\xi'/|\sigma|, z_j)] = \mathbb{C}^n$

The proof is based on complex analysis for $\big(I - L_0(\xi'/|\sigma|, z)\big)^{-1}$.

Using Theorem 5, we can make an (outgoing) asymptotic solution for non-glancing $f(x')$.

The principal part is of the form

$$\int e^{ix'\xi'}\Big\{\int_{C^+}\sum_{j=1}^{d} e^{i|\sigma|x_n(z+\varphi_j(x,\xi'/|\sigma|))}g_j(x,\xi'/|\sigma|,z)$$

$$\big(I-L_0(x,\xi'/|\sigma|,z)\big)^{-1}h_j(x,\xi'/|\sigma|)\,\bar{d}z\Big\}\chi(x',\xi')\widehat{f}(\xi')\,\bar{d}\xi'.$$

$$\varphi_j(x,\xi'/|\sigma|)|_{x_n=0}=0,$$
$$g_j(x,\xi'/|\sigma|,z)=0\ (z\neq z_j^+),$$
$$g_j(x,\xi'/|\sigma|,z)=1\ (z=z_j^+).$$

$\big(I-L_0(x,\xi'/|\sigma|,z)\big)^{-1}$ has a simple pole at $z=z_j^+$.