Low energy spectral and scattering theory for relativistic Schrödinger operators

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Inverse problems and related topics
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- $H := \sqrt{-\Delta} + V, \quad H_0 := \sqrt{-\Delta}$

Relativistic Schrödinger operators

- $|V(x)| \leq \text{Const.} \langle x \rangle^{-\sigma}, \quad \sigma > 1, \quad x \in \mathbb{R}^3$

$\Longrightarrow$

- H & H_0 self-adjoint in L^2 $(= \mathsf{L}^2(\mathbb{R}^3))$

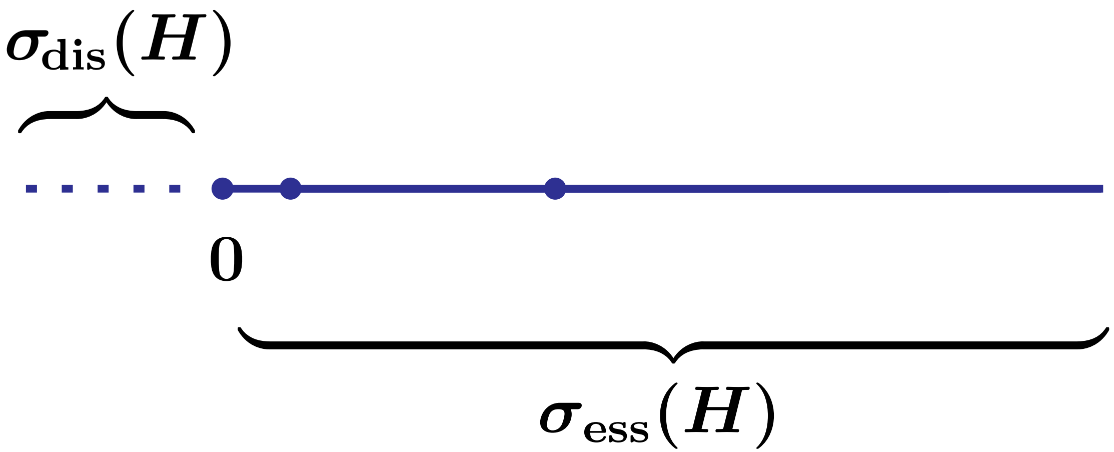
$$\text{Dom}(H) = \text{Dom}(H_0) = \mathsf{H}^1 (= \mathsf{H}^1(\mathbb{R}^3))$$

Sobolev space of order 1

- $\sigma_{\text{ess}}(H) = [0, \infty)$

NB. $\langle x \rangle = \sqrt{1 + |x|^2}$

Picture of $\sigma(H)$



♠ **0-energy plays an important role in the low energy asymptotics!**

0-energy eigenvalue

- **It is rare that $0 \in \sigma_p(H)$!**

0-energy eigenvalue

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Theorem 1 $|V(x)| \le \text{Const.}\langle x\rangle^{-\sigma}, \ \sigma > 1$

$\Longrightarrow$

$0 \notin \sigma_p(H_0 + aV)$ for $\forall a \in \mathbb{R}$

except for a discrete subset of $\mathbb{R}$

0-energy eigenvalue

- **Generalization**

  Need a class of potentials: $\mathsf{L}^3(\mathbb{R}^3;\mathbb{R})$

  $(\dagger)$ $\quad V \in \mathsf{L}^3(\mathbb{R}^3;\mathbb{R}) \Rightarrow V$ **is** H_0**-bound 0**

0-energy eigenvalue

- **Generalization**

  Need a class of potentials: $\mathsf{L}^3(\mathbb{R}^3;\mathbb{R})$

  ($\dagger$) $V \in \mathsf{L}^3(\mathbb{R}^3;\mathbb{R}) \Rightarrow V$ **is** H_0**-bound 0**

With $H_V = H_0 + V$

Theorem 2 The set $\{V \in \mathsf{L}^3(\mathbb{R}^3;\mathbb{R}) \mid 0 \notin \sigma_p(H_V)\}$ contains an open and dense subset of $\mathsf{L}^3(\mathbb{R}^3;\mathbb{R})$

NB. $|V(x)| \leq \text{Const.}\langle x\rangle^{-\sigma},\ \sigma > 1 \quad \Rightarrow \quad V \in \mathsf{L}^3(\mathbb{R}^3;\mathbb{R})$

0-energy eigenvalue and resonance

Theorem 3 $|V(x)| \leq \text{Const.}\langle x\rangle^{-\sigma}, \quad \sigma > 2$

$0 \notin \sigma_p(H)$

$\Longrightarrow$

$\exists\, (I + uG_0v)^{-1} \in \mathcal{B}(\mathsf{L}^2)$

Notation

$$u := |V|^{1/2}\mathrm{sgn}(V), \quad v := |V|^{1/2} \qquad (V = uv)$$

$$[G_0 f](x) := \frac{1}{2\pi^2}\int_{\mathbb{R}^3} \frac{1}{|x-y|^2} f(y)\,\mathrm{d}y \qquad (G_0 = 1/\sqrt{-\Delta}\,)$$

Implication of Theorem 3

$0 \notin \sigma_p(H)$

$\Longrightarrow$

H has no 0-energy resonance (in a formal sense)

0-energy resonance

'Def.' f a 0-energy resonance of H

$\iff$ $f \in$ a space 'slightly' bigger than L^2

$f \notin \mathsf{L}^2$

$Hf = 0$ in distribution

0-energy resonance

'Def.' f a 0-energy resonance of H

$\Longleftrightarrow$ $f \in$ a space 'slightly' bigger than L^2

$f \notin \mathsf{L}^2$

$Hf = 0$ in distribution

Slightly bigger space

$$\mathsf{L}^2_{-s} := \{\, f \mid \big\| \langle x \rangle^{-s} f \big\|_{\mathsf{L}^2} < \infty \,\}, \quad 0 < s \leq 3/2$$

Implication of Theorem 3

$0 \notin \sigma_p(H)$

$\Longrightarrow$

H has no 0-energy resonance (in a formal sense)

$\because$) Formal arguments!

$$(I + uG_0v)f = 0 \quad (f \in \mathsf{L}^2)$$

$\Longleftrightarrow$

$$vf + \underbrace{vu}_{(=V)} G_0 vf = 0$$

$\Longleftrightarrow$

$$(\sqrt{-\Delta} + V)g = 0 \qquad (g := G_0 vf \in \mathsf{L}^2_{-\sigma/2};\ vf = \sqrt{-\Delta}g)$$

(†) If $\exists (I + uG_0v)^{-1} \in \mathcal{B}(\mathsf{L}^2)$, then $f = 0$, hence $g = 0$. $\square$

0-energy resonance

Theorem 4 $|V(x)| \leq \text{Const.}\langle x\rangle^{-\sigma}$, $\sigma > 3/2$

$f \in \mathsf{L}^2_{-s}$ for some $s \in (0, \min\{\sigma - 3/2, 1/2\})$

$Hf = 0$ in distribution

$\Longrightarrow$

$f \in \text{Dom}(H) = \mathsf{H}^1$ (0-energy eigenfunction!)

- "0-energy resonances do not exist!"

Proof of Theorem 4 (Formal)

Let $Hf = \sqrt{-\Delta} f + Vf = 0, \quad f \in \mathsf{L}^2_{-s}$

$(0 < s < \min\{\sigma - 3/2, 1/2\})$

$\Longrightarrow$

$\sqrt{-\Delta} f = -Vf \in \mathsf{L}^2_{\sigma - s}, \;\; \sigma - s > 3/2$

$(|V(x)| \le \text{Const.} \langle x \rangle^{-\sigma})$

$\Longrightarrow$

"$f = G_0 \sqrt{-\Delta} f$" $= -G_0 V f \in \mathsf{L}^2, \;\; \sqrt{-\Delta} f \in \mathsf{L}^2$

$(G_0 \in \mathcal{B}(\mathsf{L}^2_s; \mathsf{L}^2) \;\; \forall s > 3/2)$

$\Longrightarrow$

$f \in \mathsf{H}^1 = \text{Dom}(H) \quad \square$

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$\Longrightarrow$

$f \in \mathsf{H}^1 = \mathrm{Dom}(H) \quad \square$

($\natural$) Does $\sqrt{-\Delta} f$ make sense for $f \in \mathsf{L}^2_{-s}$?

($\sharp$) Can "$f = G_0 \sqrt{-\Delta} f$" be justified ?

$(\varphi = G_0 \sqrt{-\Delta} \varphi$ is trivial only for $\varphi \in \mathcal{S}$!)

Action of $\sqrt{-\Delta}$ on distributions

$$\langle \sqrt{-\Delta} f,\, \varphi \rangle := \langle f,\, \sqrt{-\Delta}\varphi \rangle \quad (f \in \mathcal{S}',\ \varphi \in \mathcal{S})$$

($\langle f,\, g\rangle$ is linear in f, anti-linear in g)

This definition does not make sense! ($\because \sqrt{-\Delta}\varphi \notin \mathcal{S}$)

(♢) $\sqrt{-\Delta} : \mathcal{S} \to \mathsf{L}^2_s$ continuous if $s < 5/2$ $(= 1 + 3/2)$

(♡) $\exists \varphi_0 \in \mathcal{S}$ such that $\sqrt{-\Delta}\varphi_0 \notin \mathsf{L}^2_s \quad \forall s \geq 5/2$

(♠) By duality, $\sqrt{-\Delta} f \in \mathcal{S}'$ if $f \in \mathsf{L}^2_{-s}$ for some $s < 5/2$

Key Lemma $0 \leq s < 5/2$, $f \in \mathsf{H}^1_{-s}$, $g \in \mathsf{L}^2_s$

$\Longrightarrow$

$$\langle \sqrt{-\Delta} f,\, g \,\rangle_{\mathsf{L}^2_{-s}, \mathsf{L}^2_s} = \langle f, \sqrt{-\Delta} g \,\rangle_{\mathsf{H}^1_{-s}, \mathsf{H}^{-1}_s}$$

Proof of "$f = G_0\sqrt{-\Delta}f$"

$\sqrt{-\Delta}f = -Vf \in \mathsf{L}^2_{\sigma-s}$ $\quad (f \in \mathsf{L}^2_{-s},\ 0 < s < \min\{\sigma - 3/2, 1/2\})$

$\Longrightarrow$

$f \in \mathsf{H}^1_{-s}$

$\Longrightarrow$

For $\forall\varphi \in \mathcal{S}$

$$\begin{aligned}
\langle \underbrace{G_0\sqrt{-\Delta}f}_{\in \mathsf{L}^2},\ \varphi \rangle_{\mathsf{L}^2} &= -\langle G_0 V f,\ \varphi \rangle_{\mathsf{L}^2} \\
&= -\langle V f,\ G_0\varphi \rangle_{\mathsf{L}^2} \quad \text{(change of integ'n order)} \\
&= \langle \sqrt{-\Delta}f,\ G_0\varphi \rangle_{\mathsf{L}^2} \quad (G_0\varphi \in \mathsf{L}^2_t \quad \forall t < 1/2) \\
&= \langle f,\ \sqrt{-\Delta}G_0\varphi \rangle_{\mathsf{H}^1_{-s},\mathsf{H}^{-1}_s} \quad \text{(by Key Lemma)} \\
&= \langle f,\ \varphi \rangle_{\mathsf{L}^2_{-s},\mathsf{L}^2_s}
\end{aligned}$$

$\Longrightarrow$

$f = G_0\sqrt{-\Delta}f \qquad \square$

$\sigma_p(H)$ at low energy

Theorem 5 $|V(x)| \leq \text{Const.}\langle x\rangle^{-\sigma}$, $\sigma > 5/2$

If $0 \notin \sigma_p(H)$

$\Longrightarrow$

$\exists\, \lambda_0 > 0$ s.t. $[0, \lambda_0) \cap \sigma_p(H) = \emptyset$

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In other words

Theorem 6 $|V(x)| \le \text{Const.}\langle x \rangle^{-\sigma},\ \sigma > 5/2$

If $\sigma_{\mathrm{p}}(H) \cap (0,\ +\infty)$ accumulate at 0

$\Longrightarrow$

$0 \in \sigma_p(H)$

The wave and scattering operators

(Known facts)

- $\exists \, W_\pm \equiv W_\pm(H, H_0)$

  $:= s - \lim_{t \to \pm\infty} \mathrm{e}^{itH} \mathrm{e}^{-itH_0}$ Isometries in L^2

  Wave operators

The wave and scattering operators

(Known facts)

- $\exists\, W_{\pm} \equiv W_{\pm}(H, H_0)$
  $$:= s - \lim_{t \to \pm\infty} \mathrm{e}^{itH}\, \mathrm{e}^{-itH_0}$$ Isometries in L^2

  Wave operators

- $\mathrm{Ran}(W_-) = \mathrm{Ran}(W_+) = \mathcal{H}_p(H)^{\perp}$

  Asymptotic completeness

- $S := W_+^* W_-$ unitary in L^2

  Scattering operator

- **Scattering matrices $S(\lambda)$**

$$S \cong \int_{\mathbb{R}_+}^{\oplus} S(\lambda)\, \mathrm{d}\lambda \quad \text{in the spectral representation}$$

($\cong$ means a unitary equivalence)

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$$S \cong \int_{\mathbb{R}_+}^{\oplus} S(\lambda)\, d\lambda \quad \text{in the spectral representation}$$

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Define $\mathcal{F}_0 : \mathsf{L}^2 \longrightarrow \mathsf{L}^2(\mathbb{R}_+, d\lambda\,; \mathfrak{h})$ unitary

by $[\mathcal{F}_0 f](\lambda) := \lambda[\mathcal{F} f](\lambda\,\cdot\,)$

where $\mathfrak{h} := \mathsf{L}^2(\mathbb{S}^2)$, $\mathcal{F}$ Fourier transform

$\Longrightarrow$

(†) $\big[[\mathcal{F}_0 S \mathcal{F}_0^*]\varphi\big](\lambda) = S(\lambda)\varphi(\lambda)$ for $\varphi \in \mathsf{L}^2(\mathbb{R}_+, d\lambda\,; \mathfrak{h})$

($\mathcal{B}(\mathfrak{h})$-valued multiplication operator)

($\diamond$) $\big([\mathcal{F}_0 S \mathcal{F}_0^*]\varphi,\ \psi\big)_{\mathsf{L}^2(\mathbb{R}_+, d\lambda\,;\mathfrak{h})} = \displaystyle\int_0^\infty (S(\lambda)\varphi(\lambda),\ \psi(\lambda))_{\mathfrak{h}}\, d\lambda$

Action of the dilation group

$$[U_\tau f](x) = \mathrm{e}^{3\tau/2}\, f(\mathrm{e}^\tau\, x) \text{ for } f \in \mathsf{L}^2,\ \tau \in \mathbb{R}$$

(NB. U_τ is unitary!)

Action of the dilation group

$$[U_\tau f](x) = \mathrm{e}^{3\tau/2} f(\mathrm{e}^\tau x) \text{ for } f \in \mathsf{L}^2, \tau \in \mathbb{R}$$

(NB. U_τ is unitary!)

$\Longrightarrow$

- $U_{-\tau} W_\pm(H, H_0) U_\tau = W_\pm(H_0 + \mathrm{e}^{-\tau} V_\tau, H_0)$

($V_\tau(x) := V(e^{-\tau} x)$)

- $U_{-\tau} S U_\tau \cong \int_{\mathbb{R}_+}^{\oplus} S(\mathrm{e}^\tau \lambda) \, \mathrm{d}\lambda$

"$\tau \longrightarrow -\infty$" means "low energy limit"

Assumption 1

$\boldsymbol{\sigma}_{\mathbf{p}}(\boldsymbol{H}) \cap (\mathbf{0}, +\infty)$ **does not accumulate at 0.**

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$\sigma_{\mathrm{p}}(H) \cap (0, +\infty)$ does not accumulate at 0.

Let P be the projection on $\mathcal{G}_0 := \mathrm{Ker}(I + uG_0 v)$

$(u := |V|^{1/2}\mathrm{sgn}(V),\ v := |V|^{1/2})$

$$\left([G_0 f](x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} f(y)\,\mathrm{d}y \right)$$

$$[Q_0 f](x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} f(y)\,\mathrm{d}y$$

Assumption 2

The operator $PuQ_0 v\big|_{\mathcal{G}_0} : \mathcal{G}_0 \to \mathcal{G}_0$ is invertible.

Remark $|V(x)| \leq \text{Const.}\langle x\rangle^{-\sigma}$, $\sigma > 3$

Either (1) $0 \notin \sigma_p(H)$

or (2) $\|V\|_{L^\infty}$ is small enough

$\Longrightarrow$

Assumptions 1 & 2 hold

Low energy asymptotics of $W_\pm$ & S

Theorem 7 $|V(x)| \leq \text{Const.}\langle x\rangle^{-\sigma}$, $\sigma > 3$

Suppose that Assumptions 1 & 2 hold

$\Longrightarrow$

$$s - \lim_{\tau \to -\infty} U_{-\tau} W_\pm U_\tau = I$$

Low energy asymptotics of $W_\pm$ & S

Theorem 7 $|V(x)| \leq \text{Const.}\langle x \rangle^{-\sigma}$, $\sigma > 3$

Suppose that Assumptions 1 & 2 hold

$\Longrightarrow$

$$s - \lim_{\tau \to -\infty} U_{-\tau} W_\pm U_\tau = I$$

Corollary $|V(x)| \leq \text{Const.}\langle x \rangle^{-\sigma}$, $\sigma > 3$

Suppose that Assumptions 1 & 2 hold

$\Longrightarrow$

$$s - \lim_{\tau \to -\infty} U_{-\tau} S U_\tau = I$$

Low energy asymptotics of $S(\lambda)$

Theorem 8 $|V(x)| \le \text{Const.}\langle x\rangle^{-\sigma}$, $\sigma > 3$

Suppose that Assumptions 1 & 2 hold

$\Longrightarrow$

$$\lim_{\lambda \searrow 0} S(\lambda) = I \quad \text{in} \quad \mathcal{B}(\mathfrak{h}) \qquad (\text{NB. } \mathfrak{h} = \mathsf{L}^2(\mathbb{S}^2))$$

Comparison with the non-relativistic case

- **Jensen & Kato: Duke Math. J. (1979)**

  $-\Delta + V$ on $\mathbb{R}^3$

  $$S(\lambda) = \Sigma_0 + i\lambda^{1/2}\Sigma_1 + o(\lambda^{1/2}) \quad \text{in} \quad \mathcal{B}(\mathfrak{h})$$

  where

  $$\Sigma_0 = \begin{cases} I & \text{if } \nexists \text{ 0-resonances} \\ I + \{\text{rank one operator}\} & \text{if } \exists \text{ 0-resonances} \end{cases}$$

Спасибо!

有難うございました！