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Abstracts

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The shifted fourth moment of automorphic L-functions of prime power level

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An important subject in analytic number theory is the behavior of L-functions near the critical line. Questions of particular interest are subconvexity bounds and proportion of non-vanishing L-values. A possible way to analyse these problems is the method of moments.

In this talk, we prove an asymptotic formula for the shifted fourth moment of L-functions associated to primitive newforms of fixed weight k and level p^ν, where p is a fixed prime and $\nu \to \infty$. This is a continuation of the work of Rouymi, who computed the first three moments at prime power level, and a generalisation of results obtained for prime level by Duke, Friedlander & Iwaniec and Kowalski, Michel & Vanderkam. Furthermore, this proves a particular case of random matrix theory conjectures (including lower order terms) by Conrey, Farmer, Keating, Rubinstein and Snaith.

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The second moment of L-functions of holomorphic cusp forms of level N [1]

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Let $S_{2k}(N)$ be the space of holomorphic cusp forms of level N and weight $2k$ and let $O_{2k}(N)$ be an orthonormal basis of $S_{2k}(N)$. We prove new asymptotic formulae for

$$\sum_{f\in O_{2k}(N)} |L_f(1/2+it)|^2, \quad \int_0^T \sum_{f\in O_{2k}(N)} |L_f(1/2+it)|^2\, dt$$

with uniform in k, N, t error terms. In the case of weight 2 to overcome the lack of convergence of the series of Kloosterman sums in the Petersson formula we apply "Hecke's trick" and asymptotic formulae for convolutions of divisor functions.

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The mean-value theorem for arithmetical sums

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On periodic continued fractions with palindromic period

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Joint equidistribution of primitive integer points on spheres and the shape of their orthogonal complement

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A strengthening of Porter's result [1]

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Let $s\left(\frac{a}{b}\right)$ be the length of the standard continued fraction expansion of

$$\frac{a}{b} \in \mathbb{Q}, 0 < a \le b, \quad \frac{a}{b} = [0; d_1, d_2, \dots, d_s].$$

J.W. Porter (1975) proved that

$$\frac{1}{\varphi(b)} \sum_{\substack{1 \le a \le b \\ (a,b)=1}} s\left(\frac{a}{b}\right) = \frac{2\log 2}{\zeta(2)} \log b + C_P - 1 + O_\varepsilon\left(b^{-1/6+\varepsilon}\right), \qquad (1)$$

where C_P is the so-called Porter's constant. The first result of such kind, with error term $O(\log^4 \log b)$, is due to H.Heilbronn. Using the ideas of Heilbronn and Porter we prove the asymptotic formula (1) with error term

$$O_\varepsilon\left(b^{-1/6-1/27+\varepsilon}\right).$$

The proof is based on the new bounds, uniform in t aspect, for the error term in the additive divisor problem

$$\sum_{0<n<N} \sigma_{it}(n)\sigma_{-it}(N-n).$$

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On periodic continued fractions with palindromic period

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Randomness in Diophantine approximation

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We investigate statistical properties of counting functions for the number of solutions of Diophantine inequalities. Given an $(n \times m)$-matrix X with real coefficients, we consider the system of inequalities

$$\|Xq - p\| \leq c\,\|q\|^{-m/n} \quad \text{and} \quad \|q\| \leq R,$$

where $p \in \mathbb{Z}^n$ and $q \in \mathbb{Z}^m \backslash \{0\}$, and denote by $\psi_R(X)$ the number of solution of these inequalities. Asymptotic behaviour of this function depends very sensitively on Diophantine properties of the matrix X. Nonetheless, W. Schmidt showed that for almost all X it has universal asymptotics: $\psi_R(X) \sim v(R)$ as $R \to \infty$ for an explicit function $v(R) \to \infty$. We investigate finer statistical properties of the function ψ_R and, in particular, establish the following analogue of the central limit theorem.

Theorem 1. *There exists $\sigma = \sigma(n,m) > 0$ such that*

$$\mathrm{Vol}\left(\left\{X \in \ \mathrm{M}_{n,m}([0,1]) : \frac{\psi_R(X) - v(R)}{v(R)^{1/2}} \in (a,b)\right\}\right) \to \frac{1}{\sqrt{2\pi\sigma}} \int_a^b e^{-u^2/\sigma} du.$$

as $R \to \infty$.

The method that we develop can be also used to prove the law of iterated logarithms and the invariance principle for the function ψ_R.

Our results generalise previous works of Leveque [1] and Philipp [2] that considered counting solutions for Diophantine approximation on the real line. Their approach uses fine properties of the continued fractions expansion which are not available in higher dimensions. To prove our results we develop a different new method that utilises techniques from the theory of dynamical systems on homogeneous spaces.

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On algebraic integers in short intervals and near the smooth curves

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Let $P(x) = a_nx^n + \ldots + a_1x + a_0$ be the irreducible polynomial with integer coefficients. The roots of this polynomial are the algebraic numbers α of the degree n and of the height $H(\alpha) = H(P)$. When $a_n = 1$ we speak about algebraic integer α of the degree n and of the height $H(\alpha) = H(P)$.

Consider the interval $I \subset \left[-\frac{1}{2}; \frac{1}{2}\right]$ of the length $c_1(n)Q^{-1}$. We have a question, under what conditions the interval I contains the algebraic numbers of the degree $\leq n$ and of the height $\leq Q$? The answer to this question was given in the paper of V.Bernik and F.Goetze [1] in 2014. In this paper it was shown, that for any integer $Q \geq 1$ there exists an interval I of the length $\mu I = \frac{1}{2}Q^{-1}$, which doesn't contain the algebraic numbers of any degree and of the height $\leq Q$. From the other hand, if constant $c_1(n)$ is sufficiently large then any interval I, $\mu I \geq c_1(n)Q^{-1}$ contains at least $c_2(n)Q^{n+1}\mu I$ of real algebraic numbers of the degree $\leq n$ and of the height $\leq Q$.

We obtain the analogous results in case of algebraic integers. It is easy to see, that there exists an interval I, $\mu I = \frac{1}{2}Q^{-1}$, which doesn't contain the algebraic integers of any degree and of the height $\leq Q$, as the set of algebraic integers is the subset of algebraic numbers.

Theorem 1. *For sufficiently large constant $c_3(n)$ and $Q > Q_0(n)$ any interval I of the length $\mu I \geq c_3(n)Q^{-1}$ contains at least $c_4(n)Q^n|I|$ of real algebraic integers α of the degree $\deg\alpha \leq n$, $n \geq 2$ and of the height $H(\alpha) \leq Q$.*

Analogous questions about the distribution of algebraic points on the plane were considered by V.Bernik, F.Goetze and O.Kukso in the paper [2]. The point (α, β) is an algebraic point if α and β are the roots of some polynomial $P \in \mathbb{Z}[x]$. We obtain the following results for algebraic integers.

Theorem 2. *For sufficiently large $Q > Q_0(n)$ any rectangle E of the square $\mu E > Q^{-\gamma}$, $0 \le \gamma < 1$ contains at least $c_7(n)Q^n\mu E$ of integer algebraic points (α, β) of the degree $\deg \alpha = \deg \beta \le n$, $n \ge 4$ and of the height $H(\alpha) = H(\beta) \le Q$.*

For proving of theorems 1 and 2 we use the method of Y. Bugeaud [3].

Interesting question is to study the distribution of algebraic points near the smooth curves. Recently the new results in estimating of the quantity of rational points near the smooth curves were obtained. In the paper [2] the lower estimate for the quantity of algebraic points of arbitrary degree near the smooth curves was obtained. We obtain the same results for the integer algebraic points.

Theorem 3. *Let $f(x)$ be a continuous function on the interval $J = [a, b]$ and let $L(Q, \lambda) = \{(x, y) : x \in J, |y - f(x)| < Q^{-\lambda}\}$, $0 < \lambda < \frac{1}{2}$. Then for $Q > Q_0(n, J, f)$ there are at least $c_8(n, J, f)Q^{n-\lambda}$ of integer algebraic points (α, β) of the degree $\deg \alpha = \deg \beta \le n$, $n \ge 4$ and of the height $H(\alpha) = H(\beta) \le Q$ such that $(\alpha, \beta) \in L(Q, \lambda)$.*

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Klein polyhedral

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The distribution of algebraic numbers on the complex plane [1]

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Let $p(x) = a_n x^n + \ldots + a_1 x + a_0$ be an integral polynomial of degree n, and let $H(p)$ be its height defined as $H(p) = \max_{0 \le i \le n} |a_i|$. For an algebraic number $\alpha \in \mathbb{C}$, its degree $\deg(\alpha)$ and its height $H(\alpha)$ are defined as the degree and the height of its minimal polynomial, i.e. the polynomial p of minimal degree with integral coprime coefficients such that $p(\alpha) = 0$.

In the talk, we will discuss how are algebraic numbers of an arbitrary fixed degree $n \ge 2$ distributed in the complex plane as the upper bound of their height tends to the infinity.

In 1999, V. Bernik and D. Vasil'ev showed [1] that complex algebraic numbers α form a regular system with a function $N(\alpha) = H(\alpha)^{-(n+1)/2}$, in other words, there exists a constant c_n depending on n only such that in any circle C contained in the unit circle $C_0 \subset \mathbb{C}$ for all sufficiently large $Q \ge Q_0(C)$ there exist at least $c_n Q^{n+1}|C|$ algebraic numbers $\alpha_1, \ldots, \alpha_k$ of degree at most n and height at most Q such that distances between them are at least $Q^{-(n+1)/2}$. However, this result describes in some sense only a regular "skeleton" of the set of complex algebraic numbers.

Now we formulate our result. For a region $\Omega \subset \mathbb{C}$, denote by $\Psi_Q(\Omega)$ the number of algebraic numbers in Ω of degree n and height at most Q. We assume that Ω does not intersect the real axis and that its boundary consists of a finite number of algebraic curves.

Theorem 1. ([2]) *The following asymptotic formula holds*

$$\Psi_Q(\Omega) = \frac{Q^{n+1}}{2\zeta(n+1)} \int_\Omega \psi_n(z)\nu(dz) + O\left(Q^n\right), \quad Q \to \infty, \qquad (1)$$

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where ν is the Lebesgue measure on the complex plane, $\zeta(\cdot)$ is the Riemann zeta function. The limit density ψ_n is given by the formula

$$\psi_n(z) = \frac{1}{|\Im z|} \int\limits_{D_n(z)} \left| \sum_{k=1}^{n-1} t_k \left((k+1)z^k - \frac{\Im z^{k+1}}{\Im z} \right) \right|^2 dt_1 \dots dt_{n-1},$$

where $\Im z$ denotes the imaginary part of $z \in \mathbb{C}$. The integration is performed over the region

$$D_n(z) = \Bigg\{ (t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1} : \max_{1 \le k \le n-1} |t_k| \le 1,$$
$$\left| z \sum_{k=1}^{n-1} t_k \left(z^k - \frac{\Im z^{k+1}}{\Im z} \right) \right| \le 1, \ \left| \frac{1}{\Im z} \sum_{k=1}^{n-1} t_k \Im z^{k+1} \right| \le 1 \Bigg\}.$$

The implicit constant in the big-O-notation in (1) depends only on the degree n, and on the maximal degree and the number of algebraic curves that form the boundary $\partial\Omega$.

The function ψ_n is positive on $\mathbb{C}$ and satisfies the following functional equations:

$$\psi_n(-z) = \psi_n(\bar{z}) = \psi_n(z), \qquad \psi_n\left(\frac{1}{z}\right) = |z|^4 \psi_n(z).$$

Let x_0 be a fixed real number. Then for real $y \to 0$

$$\psi_n(x_0 + iy) = A|y| \cdot (1 + O(y)),$$

where the constant A does not depend on y and can be written explicitly.

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On the smallest simultaneous powers nonresidue modulo a prime

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A theorem of the Elliott type for twists of L-functions of elliptic curves

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In [1] and [2] P.D.T.A. Elliott proved limit theorems for the modulus and argument of Dirichlet L-functions, respectively, with increasing modulus of a character. E. Stankus obtained [3] a limit theorem of the above type on a complex plane.

In a series of works, the author jointly with V. Garbaliauskienė considered limit theorems of the Elliott type for twists of L-functions of elliptic curves. Let E be an elliptic curve over the field of rational numbers with discriminant $\Delta \neq 0$. For each prime p, denote by E_p the reduction of E modulo p, which is a curve over the finite field $\mathbb{F}(p)$, and define the integer $\lambda(p)$ by $|E(\mathbb{F}_p)| = p+1-\lambda(p)$, where $|E(\mathbb{F}_p)|$ is the number of points of E_p. The L-function $L_E(s)$ of the curve E, for $\sigma > \frac{3}{2}$, is given by

$$L_E(s) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}.$$

Let χ be a Dirichlet character modulo q. Then the twist $L_E(s,\chi)$ of $L_E(s)$, for $\sigma > \frac{3}{2}$, is given by

$$L_E(s,\chi) = \prod_{p|\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta} \left(1 - \frac{\lambda(p)\chi(p)}{p^s} + \frac{\chi^2(p)}{p^{2s-1}}\right)^{-1},$$

and can be continued analytically to an entire function. For $Q \geq 2$, let $M_Q = \sum_{q\leq Q} \sum_{\substack{\chi=\chi(\text{mod } q) \\ \chi\neq\chi_0}} 1$, and $\mu_Q(\dots) = \frac{1}{M_Q} \sum_{q\leq Q} \sum_{\substack{\chi=\chi(\text{mod } q) \\ \chi\neq\chi_0 \\ \dots}} 1$, where in place of dots a condition satisfied by a pair $(q, \chi \ (\text{mod } q))$

is to be written. Suppose that $\sigma > \frac{3}{2}$. In the mentioned above works the weak convergence for $\mu_Q\left(|L_E(s,\chi)| \in A\right)$, $A \in \mathcal{B}(\mathbb{R})$, $\mu_Q\left(\exp\{i \arg L_E(s,\chi)\} \in A\right)$, $A \in \mathcal{B}(\gamma)$, and $\mu_Q(L_E(s,\chi) \in A)$, $A \in \mathcal{B}(\mathbb{C})$ was considered as $Q \to \infty$. Here $\mathcal{B}(X)$ denotes the Borel σ-field of the space X, and γ is the unit circle on the complex plane.

Let $H(D)$ be the space of analytic functions on $D = \{s \in \mathbb{C} : \sigma > 1\}$ equipped with the topology of uniform convergence on compacta. Our report is devoted to a limit theorem for $\mu_Q\left(L_E(s,\chi) \in A\right)$, $A \in \mathcal{B}(H(D))$, as $Q \to \infty$.

Define $\Omega = \prod_p \gamma_p$, where $\gamma_p = \gamma$ for all primes p. With the product topology and pointwise multiplication, the torus Ω is a compact topological group. Therefore on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H exists, and this gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to γ_p, and on $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $L_E(s,\omega)$ by the formula

$$L_E(s,\omega) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1}.$$

Let $P_{L_E}(A) = m_H\left\{\omega \in \Omega : L_E(s,\omega) \in A\right\}$, $A \in \mathcal{B}(H(D))$, be the distribution of the random element $L_E(s,\omega)$. Then we have the following statement.

Theorem. *Suppose that $Q \to \infty$. Then $\mu_Q\left(L_E(s,\chi) \in A\right)$, $A \in \mathcal{B}(H(D))$, converges weakly to P_{L_E}.*

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Joint approximation by shifts of zeta and L-functions

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Finite Dirichlet series with partially prescribed zeroes

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For a given finite set of arbitrary complex numbers

$$\{\rho_1, \dots, \rho_{n-1}\} \tag{1}$$

we can easily construct a finite Dirichlet series

$$1 + a_2 \cdot 2^{-s} + \dots + a_n \cdot n^{-s} \tag{2}$$

that vanishes on (1) (just by solving corresponding linear system). In 2011 the author began computer experiments selecting for the role of (1) initial non-trivial zeroes of the Riemann zeta function. Computations revealed several interesting phenomena.

I. When $n \to \infty$ and k is fixed, $a_k \to (-1)^{k+1}$ (while one might expect $a_k \to 1$ according to the Dirichlet series for the zeta function).

II. The values of (2) give very good approximations to the values of $(1-2\cdot 2^{-s})\zeta(s)$ in a large area lying to the left of the critical line $\Re(s) < 1$.

III. The series (2) has zeroes very close to the initial trivial zeroes of the zeta function and its initial non-trivial zeroes not used in (1).

IV. The coefficients a_k have certain number-theoretical meaning, in particular, they encode the sieve of Eratosthenes.

Item II can be interpreted as follows: *the non-trivial zeroes of the zeta function "know" about its pole* (and cancel it by the factor $1 - 2 \cdot 2^{-s}$).

Item III can be be interpreted as follows: *the initial non-trivial zeroes of the zeta function "know" about consequent non-trivial zeroes and about initial trivial zeroes.*

When the encoding of the sieve of Eratosthenes was discovered, it was supposed to be just yet another incarnation of the Euler product. Surprisingly, a variation of the sieve appears even if one takes for (1) zeroes of studied by H. Davenport and H. Heilbronn function defined by a Dirichlet series, having the functional equation but missing the Euler product (see [2]).

The ongoing study of such finite Dirichlet series can be followed on [1].

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The Eichler–Shimura relations for theta functions [1]

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Consider the space of meromorphic functions

$$f(z_1, z_2, z_3, z_4) = f\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

with a right group action of 2×2 matrices

$$\left(f\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = f\left(\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right).$$

Let

$$H(z, w; q) = \frac{\vartheta_1(z+w)\vartheta_1'}{\vartheta_1(z)\vartheta_1(w)} = \cot z + \cot w + 4\sum_{m,n=1}^{\infty} \sin(2mz + 2nw)q^{2mn}.$$

Then

$$G\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = H(z_1, z_4; q)H(z_2, -z_3; q)$$

fulfils the Eichlera–Shimura relations

$$G + G \circ S = 0, \qquad G + G \circ V + G \circ V^2 = 0,$$

where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

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Rational approximations to Catalans constant

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Some numerical observation on distribution of zeros of L-functions

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Consider functions L defined by Dirichlet series

$$L(s) = \sum_n \frac{c_n}{n^s}\,, \quad c_n \in \mathbb{C},$$

for $s \in \mathbb{C}$ with sufficiently large $\operatorname{Re} s$, which are meromorphic functions on $\mathbb{C}$ and satisfy Riemann's type functional equation

$$R^s\,\Omega(s)\,L(s) = R^{p-s}\,\Omega(p-s)\,L(p-s),$$

where R and p are real positive numbers, and $\Omega(s)$ is the product of terms like $\Gamma(\alpha s + \beta)$ with some real $\alpha > 0$ and $\beta \in \mathbb{C}$. If such function L admits Euler expansion as a product of local factors (i.e. can be written as a product over prime numbers similar to that for the Riemann ζ function) then one expects the zeros ρ are placed on the critical line $\operatorname{Re} s = p/2$ (excepting the trivial zeros ρ, which are placed on the real line $\mathbb{R}$). This is the general Riemann hypothesis. On the other hand, if L admits no Euler product then distribution of real parts of its zeros is entirely different.

We have studied numerically some of L-functions aiming to understand possible distribution of real parts of their zeros. The L-functions considered admits no Euler product. That are the L-function attached to the cubic theta function, the zeta functions of some quadratic forms and some others.

For numerical study of L-functions one needs expressions for $L(s)$ which are valued for all $s \in \mathbb{C}$ and not only for s with large $\operatorname{Re} s$. As that, we have used 'functional equations with free parameters' which are known also as 'smoothed functional equations'.

We have found the following phenomena:

The off-line zeros of L-functions are concentrated mainly near the semi-critical lines.

This observation is based just on numerical computation of zeros and we have no rigorous proof. By 'off-line zeros' we mean non-trivial zeros located out of the critical line. By 'semi-critical lines' we mean the lines $\mathrm{Re}\, s = \omega$ and $\mathrm{Re}\, s = p - \omega$, where ω is defined to be the infimum of the set of all σ such that

$$\int_{-T}^{T} |L(\sigma + it)|^2 dt = O(T) \quad \text{as} \quad T \to \infty.$$

There are some other interesting observations as well.

Short exponential sums and their applications

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On Selberg's mollification method in the theory of L-functions.

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It is well known that for a given L-function Selberg's positive proportion theorem about non-trivial zeros lying on the critical line follows from certain mean-value estimates of the product of L-function and a so-called "mollifier" (which we can choose in different ways). We shall discuss a special choice of the "mollifier" and a certain mean-value estimate associated with the product of L-function and its "mollifier", which imply Selberg's positive proportion theorem together with Selberg's density theorem for L-function. Also some problems connected with the two mentioned Selberg's theorem will appear in our talk.

Multiplicative subgroups and sum-products

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We will give a survey on sumsets of multiplicative subgroups and the connection of the circle of problems with the sum-products phenomenon in finite fields.

ON THE STRUCTURE OF ARTIN'S L-FUNCTIONS

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Let p be a prime number, r, ν positive integers, $\mathbb{F}_p$ a prime finite field with p elements, $\mathbb{F}_q$ a finite extension of $\mathbb{F}_p$ of degree r, and $\mathbb{F}_{q^\nu}$ a finite extension of $\mathbb{F}_q$ of degree ν, so that

$$\mathbb{F}_p \subset \mathbb{F}_q \subset \mathbb{F}_{q^\nu} \ .$$

The Galois group G_ν of $\mathbb{F}_{q^\nu}$ over $\mathbb{F}_q$ is a cyclic group of order ν. If σ_ν is the generating element of G_ν then its action on an element $\alpha \in \mathbb{F}_{q^\nu}$ is defined by $\sigma_\nu(\alpha) = \alpha^q$. The map $tr_\nu : \mathbb{F}_{q^\nu} \to \mathbb{F}_q$, given by

$$tr_\nu(\alpha) = \alpha + \sigma_\nu(\alpha) + \cdots + \sigma_\nu^{\nu-1}(\alpha) = \alpha + \alpha^q + \cdots + \cdots + \alpha^{q^{\nu-1}} \ ,$$

is called the *relative trace* of $\alpha \in F_{q^\nu}$, and if $tr : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map from $\mathbb{F}_q$ to $\mathbb{F}_p$ then the map $tr(tr_\nu) : \mathbb{F}_{q^\nu} \to \mathbb{F}_p$, given by

$$tr(tr_\nu(\alpha)) = \alpha + \alpha^p + \cdots \alpha^{p^{r\nu-1}} \ ,$$

is called the *absolute trace* of the element x. Let now

$$\psi(a) = \exp(2\pi i \ tr(ca)/p), \qquad c \in F_q^*,$$

be a nontrivial additive character of the field F_q. Then

$$\psi_\nu(\alpha) = \psi(tr_\nu(\alpha))$$

is a nontrivial character of the field $\mathbb{F}_{q^\nu}$ which is called the *additive character induced by* ψ.

For a polynomial $f(x_1, \ldots, x_n) \in \mathbb{F}_q[x_1, \ldots, x_n]$ of degree $d \geq 1$ and a nontrivial additive character ψ_ν of the field $\mathbb{F}_{q^\nu}$ define the *character sum* $T_\nu = T_\nu(f)$ by

$$T_\nu = \sum_{x_1, \ldots, x_n \in \mathbb{F}_{q^\nu}} \psi_\nu(f(x_1, \ldots, x_n))$$

and consider the corresponding *Artin L - function*

$$L(z) = L(z, f) = \exp\left(\sum_{\nu=1}^{\infty} \frac{T_\nu}{\nu} z^\nu \right) \ ,$$

in the complex variable z. For $n = 1$ the classical result of A. Weil [5] says that $L(z)$ is a polynomial in $\mathbb{C}[z]$ of degree $d - 1$ with algebraic coefficients. A quite elementary proof of this result was given later by the author [4, Section 1.3]. For arbitrary $n \geq 1$, A. Grothendieck [3] proved by very deep methods of l-adic cohomology that $L(z)$ is always a rational function. E. Bombieri [1] conjectured that $L(z)$ has the special form

$$L(z) = P(z)^{(-1)^{n-1}}$$

with a polynomial $P(z)$, provided that $f(x_1, \dots, x_n)$ satisfies some kind of nonsingularity conditions. In his famous paper on the Weil conjectures, P. Deligne [2] among other results proved that Bombieri's conjecture is true if $d = \deg(f)$ is prime to the characteristic p of $\mathbb{F}_q$ and if the leading homogeneous part $f_d(x_1, \dots, x_n)$ of $f(x_1, \dots, x_n)$ defines, if $n \geq 2$, a smooth hypersurface in the projective space $\mathbb{P}^{n-1}$.

In this presentation we consider the case of polynomials $f(x_1, x_2) \in \mathbb{F}_q[x_1, x_2]$ in two variables x_1, x_2 and give an elementary proof of the Deligne result in the most interesting case when characteristic p of the field $\mathbb{F}_q$ is large enough.

Theorem. *Let $\mathbb{F}_q$ be a finite field of characteristic p and*

$$f(x_1, x_2) = \sum_{0 \leq d_1 + d_2 \leq d} a_{d_1, d_2} x_1^{d_1} x_2^{d_2} \in \mathbb{F}_q[x_1, x_2]$$

a polynomial of degree $d \geq 2$. If $p > d$ and the leading homogeneous part

$$f_d(x_1, x_2) = \sum_{d_1 + d_2 = d} a_{d_1, d_2} x_1^{d_1} x_2^{d_2}$$

of the polynomial $f(x_1, x_2)$ is nonsingular in the standard sense (i.e. there is no point over $\overline{\mathbb{F}}_q$ at which f, $\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$ vanish simultaneously) then the Artin L-function $L(z)$ is a rational function of the form

$$L(z) = P(z)^{-1} ,$$

where $P(z) \in \mathbb{C}[z]$ is a polynomial of degree $(d-1)^2$.

In conclusion one should be noted that the proof of the above theorem can be extended to the case of polynomials $f(x_1, \dots, x_n)$ in an arbitrary number $n \geq 2$ of variables $x_1, \dots, x_n$.

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On statistical properties of 3-dimensional Voronoi-Minkowski continued fractions [1]

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There exist two geometric interpretations of classical continued fractions admitting a natural generalization to the multidimensional case. In one of these interpretations, which is due to Klein, a continued fraction is identified with the convex hull (the Klein polygon) of the set of integer lattice points belonging to two adjacent angles (1895–1896). The second interpretation, which was independently proposed by Voronoi and Minkowski, is based on local minima of lattices, minimal systems, and extremal parallelepipeds (1896). The vertices of Klein polygons in plane lattices can be identified with local minima; however, beginning with the dimension 3, the Klein and VoronoiMinkowski geometric constructions become different.

The constructions of Voronoi and Minkowski is simpler from the computational point of view. In particular, they make it possible to design efficient algorithms for determining fundamental units in cubic fields. In both Voronois and Minkowskis approaches, the three-dimensional theory of continued fractions is based on interesting theorems of the geometry of numbers.

Analytical approach based on the method of trigonometric sums and estimates of Kloosterman sums allows to solve different problems concerned with classical continued fractions. The key idea is a uniform distribution of points (x, y) s.t. $xy + P \equiv 0 \pmod{a}$, or $\det\left(\begin{smallmatrix} a & x \\ y & * \end{smallmatrix}\right) = P$. The talk will be devoted to analogous 3D tool. It is also based on the estimates of Kloosterman sums and uses Linnik-Skubenko ideas from their work "Asymptotic distribution of integral matrices of third order" (1964). This tool, in particu-

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lar, allows to study statistical properties of Minkovski-Voronoi 3D continued fractions. The key idea here is a uniform distribution of points (x_1, x_2, y_1, y_2) s.t.

$$\det \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{\Large A}} & x_1 \\ & & x_2 \\ y_1 & y_2 & * \end{pmatrix} = P,$$

where A is a fixed 2×2 matrix with nonzero determinant.

Linnik's constant can be taken smaller than 5

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Induced bounded remainder sets [1]

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The Rauzy induced two dimensional tilings [1] are generalized on tilings of tori $\mathbb{T}^D = \mathbb{R}^D/\mathbb{Z}^D$ in any dimension D. For this purpose, the embedding method $T \overset{\text{em}}{\hookrightarrow} \mathbb{T}^D$of developments $T \subset \mathbb{R}^D$ of the torus $\mathbb{T}^D_L = \mathbb{R}^D/L$ for some lattices L. A feature of the developments T is that for a fixed shift $S : \mathbb{T}^D \longrightarrow \mathbb{T}^D$ its restriction $S|_T$ on the subset $T \subset \mathbb{T}^D$, i.e. the first-return map or the Poincare map, is equivalent to a rearrangement of subsets T_k generating some development splitting

$$T = T_0 \sqcup T_1 \sqcup \ldots \sqcup T_D.$$

In the considered case the induced map $S|_T$ is again isomorphic to a shift of the torus $\mathbb{T}^D_L$.

It is proved that all T_k are bounded remainder sets. By definition, it means the deviation $\delta_{T_k}(i, x_0)$ in the following formula

$$r_{T_k}(i, x_0) = a_{T_k} i + \delta_{T_k}(i, x_0),$$

where x_0 is arbitrary initial point on the torus $\mathbb{T}^D$, $r_{T_k}(i, x_0)$ is equal to the number of points $S^0(x_0), S^1(x_0), S^{i-1}(x_0)$ from the S-orbit of x_0 hit the set T_k, and the coefficient a_{T_k} is equal to the volume $\mathrm{vol}(T_k)$ of the set T_k, are limited. For these deviations $\delta_{T_k}(i, x_0)$ explicit estimates are proved.

The relationship between the induced map $S|_T$ and bounded remainder sets has been seen earlier in [2], [3].

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