HOMOLOGY OF YANG-BAXTER OPERATORS AND SEARCH FOR (CO)CYCLE INVARIANTS OF KNOTS

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Our goal is to define Yang-Baxter homology and connect them to Khovanov homology for the appropriately chosen links and the Yang Baxter operator leading to the Jones polynomial.

We start from early history by describing Fox 3-colorings, introduced by Ralph Hartzler Fox (1913 -1973) around 1956 when he was explaining knot theory to undergraduate students at Haverford College.

We show how Fox colorings can be naturally generalized to Yang-Baxter weighted colorings and Yang-Baxter operators. This in turn can be used to define most of the quantum invariants of links, including the Jones, Hompflypt, and 2-variable Kauffman polynomials (as demonstrated by Jones and Turaev).

Then we define Khovanov homology following O.Viro very elementary approach, using the Kauffman bracket polynomial. We observe that Khovanov homology can be obtained as a homology of a small category with coefficient in the functor – the Khovanov functor - to k-modules (in Khovanov case the module, we use, should have a structure of Frobenius system).

The last part of the talk will be devoted to distributive structures coming from knot theory (racks and quandles), and their homology theory (parallel to group homology). I show how to generalize this homology to Yang-Baxter operators homology and speculate how to connect Yang-Baxter homology with Khovanov type homology.

The path I plan to take is as follows: Khovanov homology is the categorification of the Jones polynomial. Jones polynomial can be obtained using the specific Yang-Baxter operator. Yang-Baxter operator generalizes distributivity.

I do not assume a deep knowledge neither of knot theory nor homological algebra. We start from the basis, from Fox 3-coloring, Reidemeister moves, and Jones polynomial on one hand, and chain complexes, (pre)simplicial sets (and their geometric realizations) and chain homotopy from the homological algebra side. We should see, at my lecture, many open problems, which may become research problems for participants.

To have more details, in this extended abstract, let us remind how we define homology $H_n(\mathcal{C},\mathcal{F})$ for any small category $\mathcal{C}$ and a functor $\mathcal{F}$ from $\mathcal{C}$ to the category of modules, k-Mod over a commutative ring k.

We call the sequence of objects and functors, $x_0 \stackrel{f_0}{\to} x_1 \stackrel{f_1}{\to} \dots \stackrel{f_{n-1}}{\to} x_n$ an n-chain. We define the chain complex $C_*(\mathcal{C},\mathcal{F})$ as follows:

$$C_n(\mathcal{C},\mathcal{F}) = \bigoplus_{x_0 \stackrel{f_0}{\to} x_1 \stackrel{f_1}{\to} \dots \stackrel{f_{n-1}}{\to} x_n} \mathcal{F}(x_0)$$

where the sum is taken over all n-chains.

The boundary operation $\partial_n : C_n(\mathcal{C},\mathcal{F}) \to C_{n-1}(\mathcal{C},\mathcal{F})$ is given by:

$$\partial_n(\lambda; x_0 \stackrel{f_0}{\to} x_1 \stackrel{f_1}{\to} \dots \stackrel{f_{n-1}}{\to} x_n) = (\mathcal{F}(x_0 \stackrel{f_0}{\to} x_1)(\lambda); x_1 \stackrel{f_1}{\to} \dots \stackrel{f_{n-1}}{\to} x_n) +$$
$$\sum_{i=1}^{n} (-1)^i (\lambda; x_0 \stackrel{f_0}{\to} x_1 \stackrel{f_1}{\to} \dots \to x_{i-1} \stackrel{f_i f_{i-1}}{\to} x_{i+1} \to \dots \stackrel{f_{n-1}}{\to} x_n).$$

We denote by $H_n(\mathcal{C},\mathcal{F})$ the homology yielded by the above chain complex.

We notice that the boundary operation ∂_n can be written as an alternating sum $\sum_{i=0}^{n}(-1)^i d_i$ where d_i are called face maps and satisfy $d_i d_j = d_{j-1} d_i$ for $0 \leq i < j \leq n$. We say that

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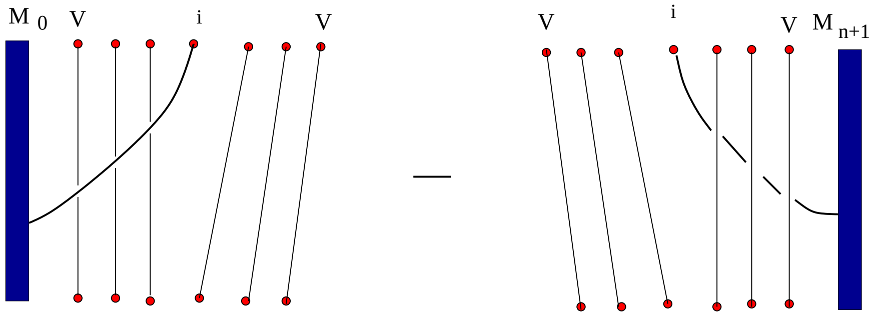
(C_n, d_i) is a presimplicial module. We can also define degeneracy maps $s_i : C_n \to C_{n+1}$, by doubling an object and putting identity between them; that is: $s_i(\lambda; x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{n-1}} x_n) = (\lambda; x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \ldots x_i \xrightarrow{Id} x_i \ldots \xrightarrow{f_{n-1}} x_n)$. We check that (C_n, d_i, s_i) is a simplicial module (complete semisimplicial system of Eilenberg).

The crucial role in our talk will have the following diagram visualizing ith face map in Yang-Baxter homology. Recall that the Yang-Baxter operator $R : V \otimes V \to V \otimes V$ is invertible and satisfies the following equation called the Yang-Baxter equation

$$(R \otimes Id)(Id \otimes R)(R \otimes Id) = (Id \otimes R)(R \otimes Id)(Id \otimes R).$$



References

[1] J. H. Przytycki, Knots and distributive homology: from arc colorings to Yang-Baxter homology, Chapter in: New Ideas in Low Dimensional Topology, World Scientific, Vol. 56, 413-488 (2015). e-print: `arXiv:1409.7044 [math.GT]`

[2] J. H. Przytycki, *Knot Theory and related with knots distributive structures; Thirteen Gdansk Lectures*, Gdansk University Press, in Polish (Teoria węzłów i związanych z nimi struktur dystrybutywnych), June, 2012, pp. 115. (Second, extended edition, in preparation).

[3] J. H. Przytycki, Distributivity versus associativity in the homology theory of algebraic structures, *Demonstratio Math.*, 44(4), 2011, 823-869;
e-print: `arXiv:1109.4850 [math.GT]`

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