MODULI SPACES OF FLAT CONNECTIONS AND FREE LIE ALGEBRAS

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Let G be a compact connected Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra. Fix an invariant inner product on $\mathfrak{g}$. This choice defines an isomorphism of G-modules $\mathfrak{g}^* \cong \mathfrak{g}$. It also defines canonical Atiyah-Bott symplectic structures on moduli spaces of flat G-connections on orientable 2-surfaces.

Let $x, y, z \in \mathfrak{g}$. Their adjoint orbits $O_x, O_y, O_z \subset \mathfrak{g} \cong \mathfrak{g}^*$ can be identified with coadjoint orbits, and they carry Kirillov-Kostant-Souriau (KKS) symplectic structures. Then, one can define the multiplicity space

$$M(x, y, z) = O_x \times O_y \times O_z//G = \{(a, b, c) \in O_x \times O_y \times O_z; a + b + c = 0\}/G.$$

By the Marsden-Weinsetein Reduction Theorem, it carries a symplectic structure induced by the ones on the orbits. At the same time, one can assign to the triple (x, y, z) a moduli space of flat connections on a sphere with 3 marked points:

$$\mathcal{M}(e^x, e^y, e^z) = \{(A, B, C) \in C_x \times C_y \times C_z; ABC = 1\}/G,$$

where $C_x = \{ge^x g^{-1}; g \in G\}$ is the conjugacy class of $e^x \in G$. The moduli space $\mathcal{M}(e^x, e^y, e^z)$ carries the Atiyah-Bott symplectic structure. The following theorem is due to L. Jeffrey [5]:

Theorem. *For x, y, z small enough, the spaces $M(x, y, z)$ and $\mathcal{M}(e^x, e^y.e^z)$ are symplectomorphic.*

The purpose of this talk is to give an explicit formula for symplectomorphisms of the Jeffrey's theorem. In more detail, let $\mathfrak{lie}(x, y)$ be the free Lie algebra in two variables x and y and let F be an automorphism of $\mathfrak{lie}_2(x, y)$ such that

1) $F(x) \in O_x, F(y) \in O_y$, that is $F(x) = e^{\mathrm{ad}_\alpha} x, F(y) = e^{\mathrm{ad}_\beta} y$ for some $\alpha, \beta \in \mathfrak{lie}_2(x, y)$;
2) $F(x + y) = \log(e^x e^y)$.

We show that such automorphisms exist, and that every such automorphism induces a formal symplectomorphism between $M(x, y, z)$ and $\mathcal{M}(e^x, e^y.e^z)$. An estimate on coefficients of F is needed if one wants to prove the convergence of F for x, y, z small. This result is related to the Kashiwara-Vergne problem for quadratic Lie algebras (see [1] and [3]) and to the famous Drinfeld's Lemma (Proposition 5.7 in [4]).

The talk is based on a joint work with Florian Naef [2].

References

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