Picture-Valued Invariants, Groups G_n^k, and Applications of Virtual Knot Theory to Classical Knot Theory

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1 Non-Reidemeister Knot Theory and Its Applications to Geometry, Topology and Dynamical Systems

Classical knot theory deals with *diagrams* and *invariants.* By means of horizontal *trisecants*, we construct a new theory of classical braids with invariants valued in *pictures*.

The key idea in the simplest case is to consider braids as motions of points on the plane and mark those moments where three points are collinear by a generator.

These pictures are closely related to diagrams of the initial object. The main tool is the notion of *free k-braid group* G_n^k. In the simplest case, for free 2-braids, the word problem and the conjugacy problem can be solved by finding the minimal representative, which can be thought of as a graph, and is unique, as such. We prove a general theorem about invariants of dynamical systems which are valued in such groups and hence, in pictures. We describe various applications of the above theory: invariants of weavings (collections of skew lines in $\mathbb{R}^3$), and many other objects in geometry and topology. In general, provided that for some topological objects (considered up to isotopy, homotopy etc) some easy axioms (coming from some dimensional constraints) hold, one can construct similar dynamical systems and picture-valued invariants. These easy axioms correspond to some *codimension*-1 *condition*, which lead to generators of the group G_n^k.

This lecture is based on [1].

2 On Free k-Braid Groups G_n^k.

For two integers $n > k$, we define the group G_n^k as the group having the following $\binom{n}{k}$ generators a_m, where m runs the set of all unordered k-tuples $m_1, \ldots, m_k$, whereas each m_i are pairwise distinct numbers from $\{1, \ldots, n\}$.

For each $(k+1)$-tuple U of indices $u_1, \ldots, u_{k+1} \in \{1, \ldots, n\}$,consider the $k+1$ sets $m^j = U \backslash \{u_j\}, j = 1, \ldots, k+1$. With U, we associate the relation

$$a_{m^1} \cdot a_{m^2} \cdots a_{m^{k+1}} = a_{m^{k+1}} \cdots a_{m^2} \cdot a_{m^1}; \tag{1}$$

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for two tuples U and $\bar{U}$, which differ by order reversal, we get the same relation.

Thus, we totally have $\frac{(k+1)!\binom{n}{k+1}}{2}$ relations.

We shall call them the *tetrahedron relations*.

For k-tuples m, m' with $Card(m \cap m') < k-1$, consider the *far commutativity relation:*

$$a_m a_{m'} = a_{m'} a_m \qquad (2).$$

Note that the far commutativity relation can occur only if $n > k+1$.

Besides that, for all multiindices m, we write down the following relation:

$$a_m^2 = 1 \qquad (3)$$

Define G_n^k as the quotient group of the free group generated by all a_m for all multiindices m by relations (1), (2) and (3).

These groups turn out to be closely related to classical braid groups, virtual braid groups and many other problems in low-dimensional topology and group theory.

We shall address various algebraic aspects of the group G_n^k, how it is related to classical and virtual braid groups, and how picture-valued invariants come into play.

References

[1] V.O.Manturov, Non-Reidemeister Knot Theory and Its Applications in Dynamical Systems, Geometry, and Topology, *http://arxiv.org/abs/1501.05208*

[2] V.O.Manturov, *Parity in Knot Theory*, Mat. Sbornik, 201:5 (2010), pp. 65-110.

[3] V.O.Manturov, Parity and projection from virtual knots to classical knots, *J. Knot Theory Ramifications* , 22:9 (2013), 1350044, 20 pp.