KAM for PDEs

Massimiliano Berti

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Joint work with Riccardo Montalto, University Zurich

1-d space-periodic Water Waves

Euler equations for an irrotational, incompressible fluid

$$\begin{cases} \partial_t \Phi + \frac{1}{2}|\nabla \Phi|^2 + g\eta = \kappa \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right) & \text{at } y = \eta(x)\,, \ \ x \in \mathbb{T} \\ \Delta \Phi = 0 & \text{in } y < \eta(x) \\ \nabla \Phi \to 0 & \text{as } y \to -\infty \\ \partial_t \eta = \partial_y \Phi - \partial_x \eta \cdot \partial_x \Phi & \text{at } \ \ y = \eta(x) \end{cases}$$

$$u = \nabla \Phi = \text{velocity field}, \ \ \mathrm{rot}\, u = 0 \text{ (irrotational)},$$
$$\mathrm{div}\, u = \Delta \Phi = 0 \text{ (uncompressible)}$$
$$g = \text{gravity}, \ \kappa = \text{surface tension coefficient}$$

Unknowns:

free surface $y = \eta(x)$ and the velocity potential $\Phi(x, y)$

Zakharov formulation

Infinite dimensional Hamiltonian system:

$$\partial_t u = J\nabla_u H(u) , \quad u := \begin{pmatrix} \eta \\ \psi \end{pmatrix} , \quad J := \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} ,$$

canonical Darboux coordinates:

$\eta(x)$ and $\psi(x) = \Phi(x, \eta(x))$ trace of velocity potential at $y = \eta(x)$

(η, ψ) uniquely determine Φ in the whole $\{y < \eta(x)\}$ solving the elliptic problem:

$$\Delta\Phi = 0 \quad \text{in } \{y < \eta(x)\}, \quad \Phi|_{y=\eta} = \psi, \quad \nabla\Phi \to 0 \text{ as } y \to -\infty$$

Zakharov-Craig-Sulem formulation

$$\begin{cases} \partial_t \eta = G(\eta)\psi = \nabla_\psi H(\eta,\psi) \\ \partial_t \psi = -g\eta - \frac{\psi_x^2}{2} + \frac{\left(G(\eta)\psi + \eta_x\psi_x\right)^2}{2(1+\eta_x^2)} + \frac{\kappa\eta_{xx}}{(1+\eta_x^2)^{3/2}} = -\nabla_\eta H(\eta,\psi) \end{cases}$$

Dirichlet–Neumann operator

$$G(\eta)\psi(x) := \sqrt{1+\eta_x^2}\,\partial_n\Phi|_{y=\eta(x)}$$

$G(\eta)$ is linear in ψ, self-adjoint and $G(\eta) \geq 0$, $\eta \mapsto G(\eta)$ nonlinear

Hamiltonian:

$$H(\eta,\psi) := \tfrac{1}{2}(\psi, G(\eta)\psi)_{L^2(\mathbb{T}_x)} + \int_{\mathbb{T}} g\frac{\eta^2}{2}\,dx + \kappa\sqrt{1+\eta_x^2}\,dx$$

kinetic energy + *potential energy* + *area surface integral*

Reversibility

$$H(\eta, -\psi) = H(\eta, \psi)$$

Involution

$$H \circ S = H, \quad S : (\eta, \psi) \to (\eta, -\psi), \quad S^2 = \mathrm{Id}$$

$$\Longrightarrow \eta(-t, x) = \eta(t, x)\,, \;\; \psi(-t, x) = -\psi(t, x)$$

Invariant subspace: *"Standing Waves"*

$$\eta(-x) = \eta(x)\,, \quad \psi(-x) = \psi(x)$$

Prime integral $\int_{\mathbb{T}} \eta(x)dx$ (Mass), $\partial_t \int_{\mathbb{T}} \psi(x)dx = -\int_{\mathbb{T}} \eta(x)dx$

$$\int_{\mathbb{T}} \eta(x)\, dx = 0 = \int_{\mathbb{T}} \psi(x)\, dx$$

Quasi-periodic solution with n frequencies of $u_t = J\nabla H(u)$

Definition

$u(t,x) = U(\omega t, x)$ where $U(\varphi, x) : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$,
$\omega \in \mathbb{R}^n (= \textit{frequency vector})$ is irrational $\omega \cdot k \neq 0\,,\ \forall k \in \mathbb{Z}^n \setminus \{0\}$
$\Longrightarrow$ the linear flow $\{\omega t\}_{t\in\mathbb{R}}$ is DENSE on $\mathbb{T}^n$

The torus-manifold

$$\mathbb{T}^n \ni \varphi \mapsto U(\varphi, x) \in \text{phase space}$$

is invariant under the flow Φ_H^t of the PDE

$$\Phi_H^t \circ U = U \circ \Psi_\omega^t$$

$$\Psi_\omega^t : \mathbb{T}^n \ni \varphi \to \varphi + \omega t \in \mathbb{T}^n$$

Small amplitude solutions

Linearized system at $(\eta, \psi) = (0, 0)$

$$\begin{cases} \partial_t \eta = |D_x| \psi, \\ \partial_t \psi + \eta = \kappa \eta_{xx} \end{cases} \qquad G(0) = |D_x|$$

Solutions: **linear** standing waves

$$\eta(t, x) = \sum_{j \geq 1} \sqrt{\xi_j} \cos(\omega_j t) \cos(jx)$$

$$\psi(t, x) = -\sum_{j \geq 1} \sqrt{\xi_j} j^{-1} \omega_j \sin(\omega_j t) \cos(jx)$$

Linear frequencies of oscillations

$$\omega_j := \omega_j(\kappa) := \sqrt{j(1 + \kappa j^2)}, \; j \geq 1$$

Can be continued to solutions of the nonlinear Water Waves?

Fix finitely many indices $\mathbb{S} = \{\bar{\jmath}_1, \ldots, \bar{\jmath}_n\}$ (tangential sites)

Finite dimensional invariant tori for the linearized water-waves eq.

$$\eta(\varphi, x) = \sum_{j \in \mathbb{S}} \sqrt{\xi_j} \cos(\varphi_j) \cos(jx), \quad \xi_j > 0,$$
$$\psi(\varphi, x) = -\sum_{j \in \mathbb{S}} \sqrt{\xi_j} j^{-1} \omega_j(\kappa) \sin(\varphi_j) \cos(jx)$$

- ANGLES: $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{T}^n$
- FREQUENCIES: $\omega(\kappa) = (\omega_j(\kappa))_{j \in \mathbb{S}}$

- For all $\kappa \in [\kappa_1, \kappa_2]$ except a set of small measure $O(\gamma^a)$, $a > 0$, the vector $\omega(\kappa)$ is diophantine:
$$|\omega(\kappa) \cdot \ell| \geq \gamma \langle \ell \rangle^{-\tau}, \quad \forall \ell \in \mathbb{Z}^n \setminus \{0\}$$
- *Do they persist in the nonlinear Water Waves?*
- Linear system=*integrable system*, nonlinearity=*perturbation*

KAM theory is well established for 1*-d semi-linear PDEs*

$$u_t = L(u) + N(u)$$

- L = linear differential operator (ex. ∂_{xxx}, $\mathrm{i}\Delta$, ...)
- N = nonlinearity which depends on $u, u_x, \ldots, \partial_x^m u$

1. Semilinear PDE: order of L > order of $N = m$
2. Fully-linear PDE: order of L = order of N
3. Quasi-linear PDE: Fully-nonlinear and N linear in $\partial_x^m u$

Kuksin, Wayne, Craig, Poeschel, Bourgain, Eliasson, Chierchia, You, Kappeler, Grebert, Geng, Yuan, Biasco, Bolle, Procesi... also in $d \geq 2$

For quasi-linear or fully non-linear PDEs as Water-Waves eq?

- First KAM results:
  1. **quasi-linear/fully nonlinear perturbations of KdV**
  2. **Water Waves**
- GENERAL STRATEGY TO DEVELOP KAM THEORY FOR 1-D QUASI-LINEAR/ FULLY NONLINEAR PDES

developed with Pietro Baldi, Riccardo Montalto

Quasi-linear perturbations of NLS, Feola-Procesi

Water Waves: small amplitude periodic solutions

- **Plotnikov-Toland**: '01 Gravity Water Waves with Finite depth Standing waves, Lyapunov-Schmidt + Nash-Moser
- **Iooss-Plotnikov-Toland** '04, **Iooss-Plotnikov** '05
  Gravity Water Waves with Infinite depth
  Completely resonant, infinite dimensional bifurcation equation
- 3 D-Travelling waves
  Craig-Nicholls 2000: with surface tension (no small divisors),
  Iooss-Plotnikov '09, '11 : no surface tension (small divisors),
- **Alazard-Baldi** '14, Capillary-gravity water waves with infinite depth Standing waves

  No information about their linear stability
- QUESTION: WHAT ABOUT QUASI-PERIODIC SOLUTIONS?

KAM theory for unbounded perturbations: literature

Kuksin '98, Kappeler-Pöschel '03

$$u_t + u_{xxx} + uu_x + \varepsilon \partial_x f(x,u) = 0, \ x \in \mathbb{T}$$

Liu-Yuan '10 for Hamiltonian DNLS (and Benjamin-Ono)

$$\mathrm{i}u_t - u_{xx} + M_\sigma u + \mathrm{i}\varepsilon f(u,\bar{u})u_x = 0$$

Zhang-Gao-Yuan '11 Reversible DNLS

$$\mathrm{i}u_t + u_{xx} = |u_x|^2 u$$

Bourgain '96, Derivative NLW, periodic solutions

$$y_{tt} - y_{xx} + my + y_t^2 = 0\,, \qquad m \neq 0$$

Berti-Biasco-Procesi '12, '13, KAM, reversible DNLW

$$y_{tt} - y_{xx} + my = g(x,y,y_x,y_t), \qquad x \in \mathbb{T}$$

KAM for quasi-linear KdV, Baldi-Berti-Montalto '13-'14

$$\partial_t u + u_{xxx} - 3\partial_x u^2 + \partial_x \nabla_{L^2} P = 0\,, \quad x \in \mathbb{T}$$

$$u_t = \partial_x \nabla_{L^2} H(u)\,, \quad H = \int_{\mathbb{T}} \frac{u_x^2}{2} + u^3 + f(x,u,u_x)dx$$

Quasi-linear Hamiltonian perturbation

$$P(u) = \int_{\mathbb{T}} f(x,u,u_x)\, dx\,, \;\; f(x,u,u_x) = O(|u|^5 + |u_x|^5)$$

$$\begin{aligned}\partial_x \nabla_{L^2} P &:= -\partial_x\{(\partial_u f)(x,u,u_x)\} + \partial_{xx}\{(\partial_{u_x} f)(x,u,u_x)\} \\ &= \quad a_0(x,u,u_x,u_{xx}) + a_1(x,u,u_x,u_{xx})u_{xxx}\end{aligned}$$

There exist small amplitude quasi-periodic solutions:

$$u = \sum_{j\in\mathbb{S}} \sqrt{\xi_j} 2\cos(\omega_j^\infty(\xi)\, t + jx) + o(\sqrt{\xi}), \omega_j^\infty(\xi) = j^3 + O(|\xi|)$$

for a Cantor set of $\xi \in \mathbb{R}^n$ with density 1 at $\xi = 0$

KAM for Water Waves

Look for small amplitude quasi-periodic solutions $(\eta(t,x),\psi(t,x)=(\eta(\tilde\omega t,x),\psi(\tilde\omega t,x))$ of

Water Waves equations

$$\begin{cases} \partial_t \eta = G(\eta)\psi \\ \partial_t \psi = -\eta - \dfrac{\psi_x^2}{2} + \dfrac{\left(G(\eta)\psi + \eta_x \psi_x\right)^2}{2(1+\eta_x^2)} + \dfrac{\kappa \eta_{xx}}{(1+\eta_x^2)^{3/2}} \end{cases}$$

with frequencies $\tilde\omega_j$ (to be found) close to

Linear frequencies

$$\omega_j(\kappa) = \sqrt{j(1+\kappa j^2)}$$

Surface tension

$$\kappa \in [\kappa_1, \kappa_2], \qquad \kappa_1 > 0$$

Theorem (KAM for capillary-gravity water waves, B.-Montalto '15)

For every choice of the tangential sites $\mathbb{S} \subset \mathbb{N} \setminus \{0\}$, *there exists* $\bar{s} > \frac{|\mathbb{S}|+1}{2}$, $\varepsilon_0 \in (0,1)$ *such that: for all* $\xi_j \in (0, \varepsilon_0)$, $j \in \mathbb{S}$, $\exists$ *a Cantor like set* $\mathcal{G} := \mathcal{G}_\xi \subset [\kappa_1, \kappa_2]$ *with asymptotically full measure as* $\xi \to 0$, *i.e.* $\lim_{\xi \to 0} |\mathcal{G}_\xi| = (\kappa_2 - \kappa_1)$, *such that, for any surface tension coefficient* $\kappa \in \mathcal{G}_\xi$, *the* CAPILLARY-GRAVITY WATER WAVES EQUATION *has a quasi-periodic standing wave solution* $(\eta, \psi) \in H^{\bar{s}}$, *even in* x, *of the form*

$$\eta(t,x) = \sum_{j \in \mathbb{S}} \sqrt{\xi_j} \cos(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{\xi})$$

$$\psi(t,x) = -\sum_{j \in \mathbb{S}} \sqrt{\xi_j} j^{-1} \tilde{\omega}_j \sin(\tilde{\omega}_j t) \cos(jx) + o(\sqrt{\xi})$$

with frequency vector $\tilde{\omega} \in \mathbb{R}^{\mathbb{S}}$ *satisfying* $\tilde{\omega} - \omega(\kappa) \to 0$ *as* $\xi \to 0$. *The solutions are linearly* **stable**.

Remarks

1. The restriction of $\mathcal{C}_\varepsilon$ is not technical! Outside could be: "Chaos", "homoclinic/heteroclinic solutions", "Arnold Diffusion", ...
   Craig-Workfolk '94, **Zakharov** '95: the $5-th$ order (formal) Birkhoff normal form system is not integrable (with no surface tension)
2. There are no results about global in time existence of the water waves equations with periodic boundary conditions: **the previous theorem selects initial conditions which give rise to smooth solutions defined for all times**

Linear stability -reducibility

(L): linearized equation $\partial_t h = J\partial_u \nabla H(u(\omega t, x))h$

$$\mathcal{L} = \partial_t + \begin{pmatrix} \partial_x V + G(\eta)B & -G(\eta) \\ (1 + BV_x) + BG(\eta)B - \kappa\partial_x c\partial_x & V\partial_x - BG(\eta) \end{pmatrix}$$

$$u(\omega t, x) = (\eta, \psi)(\omega t, x)\,, \quad (V, B) = \nabla_{x,y}\Phi, \quad c := (1 + \eta_x^2)^{-3/2}$$

There exists a quasi-periodic (Floquet) change of variable

$$H^s_x \ni h = \Phi(\omega t)(\psi, \eta, v)\,, \quad \psi \in \mathbb{T}^\nu\,, \eta \in \mathbb{R}^\nu\,, v \in H^s_x \cap L^2_{S^\perp}$$

which transforms (L) into the **constant coefficients** system

$$\begin{cases} \dot\psi = b\eta \\ \dot\eta = 0 \\ v_t = D_\infty v\,, \quad v = \sum_{j \notin \mathbb{S}} v_j e^{\mathrm{i}jx}\,,\ D_\infty := \mathrm{Op}(\mathrm{i}\mu_j)\,,\ \mu_j \in \mathbb{R} \end{cases}$$

$$\eta(t) = \eta_0, v_j(t) = v_j(0)e^{\mathrm{i}\mu_j t} \Longrightarrow \|v(t)\|_{H^s_x} = \|v(0)\|_{H^s_x} : \text{ stability}$$

1. Sharp **asymptotic expansion** of the eigenvalues

$$\mu_j = \lambda_3 j^{\frac{1}{2}} (1 + \kappa j^2)^{\frac{1}{2}} + \lambda_1 j^{\frac{1}{2}} + r_j(\omega)$$

where $\lambda_3 := \lambda_3(\omega)$, $\lambda_1 := \lambda_1(\omega)$ are constants (depending on $u(\omega t, x)$) satisfying

$$|\lambda_3 - 1| + |\lambda_1| + \sup_{j \in \mathbb{S}^c} |r_j| \leq C\varepsilon \,,$$

2. The map $\Phi(\varphi)$ satisfies *tame estimates* in Sobolev spaces:

$$\|\Phi h\|_s, \|\Phi^{-1} h\|_s \leq \|h\|_s + \|u\|_{s+\sigma} \|h\|_{s_0} \,, \ \forall s \geq s_0 \,.$$

Small-divisors problem. Look for $u(\varphi, x) := (\eta(\varphi, x), \psi(\varphi, x))$ zero of

$$\mathcal{F}(\omega, u) := \mathcal{F}(\omega, \eta, \psi) := \begin{pmatrix} \omega \cdot \partial_\varphi \eta - G(\eta)\psi \\ \omega \cdot \partial_\varphi \psi + \eta - \frac{\psi_x^2}{2} - \frac{\left(G(\eta)\psi + \eta_x \psi_x\right)^2}{2(1+\eta_x^2)} - \frac{\kappa \eta_{xx}}{(1+\eta_x^2)^{3/2}} \end{pmatrix}$$

Small amplitude solutions:

$$\mathcal{F}(\omega, 0) = 0 \quad D_{\eta,\psi}\mathcal{F}(\omega, 0) = \begin{pmatrix} \omega \cdot \partial_\varphi & -|D_x| \\ 1 - \kappa \partial_{xx} & \omega \cdot \partial_\varphi \end{pmatrix}$$

In Fourier basis

$$D_{\eta,\psi}\mathcal{F}(\omega, 0) = \mathrm{diag}_{\ell \in \mathbb{Z}^\nu, j \in \mathbb{Z}} \begin{pmatrix} \mathrm{i}\omega \cdot \ell & -|j| \\ 1 + \kappa j^2 & \mathrm{i}\omega \cdot \ell \end{pmatrix}$$

- QUESTION: is $D_u\mathcal{F}(\omega,0)$ invertible?

$$\det\begin{pmatrix} \mathrm{i}\omega\cdot\ell & -|j| \\ 1+\kappa j^2 & \mathrm{i}\omega\cdot\ell \end{pmatrix} = -(\omega\cdot\ell)^2+(1+\kappa j^2)|j| = -(\omega\cdot\ell)^2+\omega_j^2(\kappa)$$

Non-resonance condition:

$$\left|-(\omega\cdot\ell)^2+(1+\kappa j^2)|j|\right| \geq \frac{\gamma}{\langle\ell\rangle^\tau}, \forall(\ell,j)\in\mathbb{Z}^n\times\mathbb{Z}, (\ell,j)\neq 0, \ \tau>0$$

$\Longrightarrow D_u\mathcal{F}(\omega,0)$ is invertible, but the inverse is **unbounded**:

$$D_u\mathcal{F}(\omega,0)^{-1} : H^s \to H^{s-\tau}, \ \ \tau := \text{"}\textit{LOSS OF DERIVATIVES}\text{"}$$

Nash-Moser Implicit Function Theorem

Newton tangent method for zeros of $\mathcal{F}(u) = 0$ + "smoothing":

$$u_{n+1} := u_n - S_n (D_u \mathcal{F})^{-1}(u_n)\mathcal{F}(u_n)$$

where S_n are regularizing operators (= "mollifiers")

- Advantage: QUADRATIC scheme

$$\|u_{n+1} - u_n\|_s \leq C(n)\|u_n - u_{n-1}\|_s^2$$

$\Longrightarrow$ convergent also if $C(n) \to +\infty$

- Difficulty: invert $\mathcal{L}(u) := (D_u\mathcal{F})(u)$ in a whole neighborhood of the expected solution with *tame* estimates of the inverse

$$\|\mathcal{L}(u)^{-1}h\|_s \leq \|h\|_{s+\sigma} + \|u\|_{s+\sigma_1}\|h\|_{s_0}\,, \quad \forall s \geq s_0$$

Difficulty: prove invertibility and tame estimates for the inverse of

Linearized operator at $u(\varphi, x) = (\eta, \psi)(\varphi, x)$

$$(D_u\mathcal{F})(u) = \omega \cdot \partial_\varphi + \begin{pmatrix} \partial_x V + G(\eta)B & -G(\eta) \\ (1 + BV_x) + BG(\eta)B - \kappa\partial_x c\partial_x & V\partial_x - BG(\eta) \end{pmatrix}$$

$$(V, B) = \nabla_{x,y}\Phi\,, \quad c := (1 + \eta_x^2)^{-3/2}$$

are smooth functions

$$G(\eta) = |D_x|\eta + R_\infty(\eta)\,, \quad R_\infty \in OPS^{-\infty}$$

is Dirichlet-Neumann operator

Ideas of Proof:

1. Nash-Moser implicit function theorem for a torus embedding $\varphi \mapsto i(\varphi)$ formulated like a "Théorème de conjugaison hypothétique" à la Herman
2. **Degenerate KAM theory: measure estimates**
3. **Analysis of linearized PDE on approximate solutions**
    - **Symplectic reduction of linearized operator to "normal" directions** developed with P. Bolle for NLW on $\mathbb{T}^d$:

      "*Existence of invariant torus* $\Longleftrightarrow$ *Normal form near the torus*"

      (Action-angle variables, more refined than Lyapunov-Schmidt)
    - Reduction of linearized PDE in normal directions:
        1. **Step 1. Pseudo-differential theory** in **original physical coordinates** (*not* in Fourier space).
           Advantage: pseudo-differential structure is more evident
           First steps similar to Alazard-Baldi + Egorov type analysis + more steps of decoupling
        2. **Step 2. KAM reducibility scheme**. Imply stability

A “Théorème de conjugaison hypothétique" and Degenerate KAM theory

- *A big issue in KAM theory: fullfill non-resonance conditions*
- Choose parameters
- Non-degeneracy conditions:
  1. Kolmogorov
  2. Arnold-Piartly
  3. ...
  4. Rüssmann (Herman-Fejoz for Celestial Mechanics)

weakean as much as possible the non-degeneracy conditions

Use κ (= surface tension) as a parameter

Small amplitude solutions: rescale $u \mapsto \varepsilon u$

$$\partial_t u = J\Omega u + \varepsilon J \nabla P_\varepsilon(u)\,, \quad \Omega := \Omega(\kappa) := \begin{pmatrix} 1 - \kappa\partial_{xx} & 0 \\ 0 & G(0) \end{pmatrix}$$

Tangential and normal dynamics

Decompose the phase space $u(x) = v(x) + z(x)$ as

$$H = H_{\mathbb{S}} \oplus H_{\mathbb{S}}^{\perp}\,, \quad H_{\mathbb{S}} := \Big\{ v := \sum_{j \in \mathbb{S}} \begin{pmatrix} \eta_j \\ \psi_j \end{pmatrix} \cos(jx) \Big\}$$

Symmetrization + action-angle variables (θ, y) on tangential sites:

$$\eta_j := \Lambda_j^{1/2} \sqrt{y_j} \cos(\theta_j), \quad \psi_j := \Lambda_j^{-1/2} \sqrt{y_j} \sin(\theta_j)\,,\ j \in \mathbb{S}\,,$$

$$\Lambda_j := \sqrt{(1 + \kappa j^2) j^{-1}}\,,\ j \in \mathbb{S}\,,$$

Linear problem: $\varepsilon = 0$

$$\dot{\theta} = \omega(\kappa)\,, \;\; \dot{y} = 0\,, \quad z_t = J\Omega z$$

Family of invariant tori filled by quasi-periodic solutions

$$\mathbb{T}^\nu \times \mathbb{R}^\nu \times \{0\}, \quad \theta = \omega(\kappa)t\,, \;\; I(t) = \xi \in \mathbb{R}^\nu\,, z(t) = 0$$

For $\varepsilon \neq 0$?

The frequency of the expected quasi-periodic solution $\tilde{\omega} = \omega(\kappa) + O(\varepsilon)$ changes with $\varepsilon, \xi \Longrightarrow$ consider the family of Hamiltonians

$$H_\alpha = \alpha \cdot I + \frac{1}{2}(\Omega z, z)_{L^2} + \varepsilon P_\varepsilon(\theta, I, z)\,, \quad \alpha \in \mathbb{R}^\nu\,,$$

where $\alpha \in \mathbb{R}^\nu$ is an unknown

Look for quasi-periodic solutions of X_{H_α} with Diophantine frequencies $\omega \in \mathbb{R}^\nu$

Embedded torus equation:

$$\partial_\omega i(\varphi) - X_{H_\alpha}(i(\varphi)) = 0$$

$$H_\alpha = \alpha \cdot I + \frac{1}{2}(\Omega z, z)_{L^2} + \varepsilon P_\varepsilon(\theta, I, z), \quad \alpha \in \mathbb{R}^\nu,$$

Functional setting

$$\mathcal{F}(\varepsilon, X) := \begin{pmatrix} \partial_\omega \theta(\varphi) - \alpha - \varepsilon \partial_I P_\varepsilon(i(\varphi)) \\ \partial_\omega I(\varphi) + \varepsilon \partial_\theta P_\varepsilon(i(\varphi)) \\ \partial_\omega z(\varphi) - J\Omega z - \varepsilon J \nabla_z P_\varepsilon(i(\varphi)) \end{pmatrix} = 0$$

unknowns: $X := (i, \alpha), \quad i(\varphi) := (\theta(\varphi), I(\varphi), z(\varphi))$

Theorem (Nash-Moser-Théorèm de conjugation hypothetique)

Let $\varepsilon \in (0, \varepsilon_0)$ small. Then there exists a smooth function

$$\alpha_\varepsilon : \mathbb{R}^\nu \mapsto \mathbb{R}^\nu \,, \quad \alpha_\varepsilon(\omega) = \omega + r_\varepsilon(\omega) \,, \quad \text{with} \quad r_\varepsilon = O(\varepsilon\gamma^{-1}) \,,$$

and torus embedding $\varphi \mapsto i_\infty(\varphi)$ defined for all $\omega \in \mathbb{R}^\nu$, satisfying $\|i_\infty(\varphi) - (\varphi, 0, 0)\|_{s_0} = O(\varepsilon)$, and a Cantor like set $\mathcal{C}_\infty$ such that, for all $\omega \in \mathcal{C}_\infty$, the embedded torus $\varphi \mapsto i_\infty(\varphi)$ solves

$$\omega \cdot \partial_\varphi i(\varphi) = X_{H_{\alpha_\varepsilon}}(i(\varphi))$$

i.e. it is invariant for the Hamiltonian system $H_{\alpha_\varepsilon(\omega)}$ and it is filled by quasi-periodic solutions with frequency ω

$\Longrightarrow$ for $\beta \in \alpha_\varepsilon(\mathcal{C}_\infty)$ the Hamiltonian system

$$H_\beta = \beta \cdot I + \frac{1}{2}(\Omega z, z)_{L2} + \varepsilon P(\theta, I, z)$$

has a quasi-periodic solution with frequency $\omega = \alpha_\varepsilon^{-1}(\beta)$. *Picture*

The Cantor set $\mathcal{C}_\infty$ expressed in terms of "final torus"

$\exists$ smooth functions $\mu_j^\infty : \mathbb{R}^\nu \to \mathbb{R}$,

$$\mu_j^\infty(\omega) = \lambda_3^\infty(\omega) j^{\frac{1}{2}}(1+\kappa j^2)^{\frac{1}{2}} + \lambda_1^\infty(\omega) j^{\frac{1}{2}} + r_j^\infty(\omega)\,,\ j \notin \mathbb{S}^c\,,$$

satisfying $|\lambda_3^\infty - 1|$, $|\lambda_1^\infty|$, $\sup_{j\in\mathbb{S}^c} |r_j^\infty| \leq C\varepsilon$ such that

$\mathcal{C}_\infty := \Big\{ \omega \in \mathbb{R}^\nu :$ *diophantine and*

$$|\omega\cdot\ell + \mu_j^\infty(\omega)| \geq \gamma j^{\frac{3}{2}} \langle \ell\rangle^{-\tau},\ \forall \ell \in \mathbb{Z}^\nu, j \in S^c$$

$$|\omega\cdot\ell + \mu_j^\infty(\omega) \pm \mu_{j'}^\infty(\omega)| \geq \gamma |j^{\frac{3}{2}} \pm j'^{\frac{3}{2}}| \langle \ell\rangle^{-\tau},\ \forall \ell \in \mathbb{Z}^\nu,\ j, j' \in S^c \Big\}$$

Goal

Prove that for 'most" $\kappa \in [\kappa_1, \kappa_2]$ the vector of unperturbed linear frequencies $\omega(\kappa) := j^{1/2}(1+\kappa j^2)^{1/2} \in \alpha_\varepsilon(\mathcal{C}_\infty)$

Use κ ($=$ surface tension) as a parameter

Degenerate KAM theory for PDEs, Bambusi-Berti-Magistrelli

1. *Analyticity* $\kappa \mapsto \omega(\kappa) := (\omega_j(\kappa)) \in \mathbb{R}^{\mathbb{S}}$, $\omega_j(\kappa) := \sqrt{j(1+\kappa j^2)}$
2. *Non-degeneracy:* $\kappa \mapsto \omega(\kappa) \in \mathbb{R}^{\mathbb{S}}$ is **not** contained in any hyperplane (**torsion**); also $(\omega(\kappa), \omega_j(\kappa)), (\omega(\kappa), \omega_j(\kappa), \omega_{j'}(\kappa))$
3. *Asymptotic:* $\omega_j(\kappa) := \sqrt{\kappa} j^{3/2} + \dots$

$\Longrightarrow$ There exist $k_0 \in \mathbb{N}$, $\rho > 0$ such that: $\forall \ell, j, \kappa \in [\kappa_1, \kappa_2]$,

1. $\max_{k \le k_0} |\partial_\kappa^k \{\omega(\kappa) \cdot \ell\}| \ge \rho \langle \ell \rangle$
2. $\max_{k \le k_0} |\partial_\kappa^k \{\omega(\kappa) \cdot \ell + \omega_j(\kappa)\}| \ge \rho \langle \ell \rangle$
3. $\max_{k \le k_0} |\partial_\kappa^k \{\omega(\kappa) \cdot \ell + \omega_j(\kappa) \pm \omega_{j'}(\kappa)\}| \ge \rho \langle \ell \rangle$

$\rho =$ *amount of non-degeneracy*, $k_0 =$ *index of non-degeneracy*

By perturbation the same bounds are true for

$$\omega_\varepsilon(\kappa) := \alpha_\varepsilon^{-1}(\omega(\kappa)) = \omega(\kappa) + O(\varepsilon)$$

$\implies$ Using Russmann's lemma

Lemma: measure estimates

For τ large, the Melnikov non-resonance conditions

1. $|\omega_\varepsilon(\kappa) \cdot \ell| \geq \gamma \langle \ell \rangle^{-\tau}$, $\forall \ell \in \mathbb{Z}^{\mathbb{S}} \setminus \{0\}$
2. $|\omega_\varepsilon(\kappa) \cdot \ell + (\omega_\varepsilon)_j(\kappa)| \geq \gamma \langle \ell \rangle^{-\tau}$, $\forall \ell \in \mathbb{Z}^{\mathbb{S}}$, $j \notin \mathbb{S}$
3. $|\omega_\varepsilon(\kappa) \cdot \ell + (\omega_j)_\varepsilon(\kappa) \pm (\omega_\varepsilon)_{j'}(\kappa)| \geq \gamma |j^{3/2} \pm (j')^{3/2}| \langle \ell \rangle^{-\tau}$, $\forall (\ell, j, j') \in \mathbb{Z}^{\mathbb{S}} \times \mathbb{S}^c \times \mathbb{S}^c$,

hold for all $\kappa \in [\kappa_1, \kappa_2]$ except a set of small measure $O(\gamma^{1/k_0})$

Proof of non-degeneracy

Geometric Lemma:

$\forall N$, $\forall j_1, \ldots, j_N$, the curve

$$[\kappa_1, \kappa_2] \ni \kappa \mapsto \left(\omega_{j_1}(\kappa), \ldots, \omega_{j_N}(\kappa)\right) \in \mathbb{R}^N$$

is not contained in any hyperplane of $\mathbb{R}^N$

Computation: the vectors

$$\begin{pmatrix} \omega_{j_1}(\kappa) \\ \partial_\kappa \omega_{j_1}(\kappa) \\ \vdots \\ \partial_\kappa^{N-1} \omega_{j_1}(\kappa) \end{pmatrix}, \ldots, \begin{pmatrix} \omega_{j_N}(\kappa) \\ \partial_\kappa \omega_{j_N}(\kappa) \\ \vdots \\ \partial_\kappa^{N-1} \omega_{j_N}(\kappa) \end{pmatrix},$$

are linearly independent

by analyticity it is sufficient to prove it only at one $\kappa \neq 0$

Ideas of Proof:

1. Nash-Moser implicit function theorem for a torus embedding $\varphi \mapsto i(\varphi)$ formulated as a "Théorème de conjugaison hypothétique" à la Herman
2. **Degenerate KAM theory: measure estimates**
3. **Analysis of linearized PDE on approximate solutions**
    - **Symplectic reduction of linearized operator to "normal" directions** developed with P. Bolle for NLW on $\mathbb{T}^d$:

      "*Existence of invariant torus* $\iff$ *Normal form near the torus*"

      (Action-angle variables, more refined than Lyapunov-Schmidt)
    - Reduction of linearized PDE in normal directions:
        1. **Step 1.** **Pseudo-differential theory** in **original physical coordinates** (*not* in Fourier space).
           Advantage: pseudo-differential structure is more evident
           First steps similar to Alazard-Baldi + Egorov type analysis + more steps of decoupling
        2. **Step 2. KAM reducibility scheme**. Imply stability

Reduction of linearized operator in normal directions

After approximate-inverse transformation we have to analyze

(L): linearized equation $\partial_t h = J\partial_u \nabla H(u(\omega t, x))h$

$$\mathcal{L}_\omega =$$

$$\omega \cdot \partial_\varphi + \Pi_S^\perp \begin{pmatrix} \partial_x V + G(\eta)B & -G(\eta) \\ (1 + BV_x) + BG(\eta)B - \kappa\partial_x c\partial_x & V\partial_x - BG(\eta) \end{pmatrix} \Pi_S^\perp$$

GOAL:

Conjugate $\mathcal{L}_\omega$ to a diagonal operator (Fourier multiplier):

$$\Phi^{-1} \circ \mathcal{L}_\omega \circ \Phi = \mathrm{diag}\{\mathrm{i}\mu_j(\varepsilon)\}_{j \in S^\perp \subset \mathbb{Z}}$$

where

$$\mu_j(\varepsilon) = \lambda_3 j^{\frac{1}{2}}(1 + \kappa j^2)^{\frac{1}{2}} + \lambda_1 j^{\frac{1}{2}} + r_j(\omega), \quad \sup_j r_j = O(\varepsilon)$$

usual KAM scheme to diagonalize $\mathcal{L}_\omega$ is clearly unbounded

1. **"REDUCTION IN DECREASING SYMBOLS"**

$$\mathcal{L}_1 := \Phi^{-1}\mathcal{L}_\omega\Phi = \omega \cdot \partial_\varphi + \lambda_3 T(D) + \lambda_1 |D_x|^{1/2} + R_0$$

$$T(D) := \sqrt{|D|(1+\kappa\partial_x^2)}$$

- $R_0(\varphi, x) \in OPS^0$ on diagonal (and OPS^{-M} off-diagonal)
- $\lambda_1, \lambda_3 \in \mathbb{R}$, **constants**

Use Egorov type theorem

2. **"REDUCTION OF THE SIZE of R_0"**

$$\mathcal{L}_n := \Phi_n^{-1}\mathcal{L}_1\Phi_n = \omega \cdot \partial_\varphi + \lambda_3 T(D) + \lambda_1 |D_x|^{1/2} + r^{(n)} + \mathcal{R}_n$$

- *KAM quadratic scheme:* $\mathcal{R}_n = O(\varepsilon^{2^n})$, $r^{(n)} = \mathrm{diag}_{j\in\mathbb{Z}}(r_j^{(n)})$, As Alazard-Baldi, after introducing a linearized good unknown of Alinhac and symmetrizing

Linearized system $h = \eta + i\psi$, $\bar{h} = \eta - i\psi$

$$\mathcal{L}(h, \bar{h}) = \omega \cdot \partial_\varphi h + i a_0(\varphi, x) T(D) h + a_1(\varphi, x)\partial_x h + b_1(\varphi, x)\partial_x \bar{h} + \dots$$

where $T(D) := \sqrt{|D|(1 + \kappa \partial_x^2)}$

Eliminate the x, φ dependence at highest order

Under $x \mapsto x + \beta(\varphi, x)$ like Alazard-Baldi and $\varphi \mapsto \varphi + \alpha(\varphi)\omega$

$$\mathcal{L}_1(h, \bar{h}) = \omega \cdot \partial_\varphi h + i m_3 T(D) h + a_1(\varphi, x)\partial_x h + b_1(\varphi, x)\partial_x \bar{h} + \dots$$

Block-diagonalize up to smoothing operators

$$\mathcal{L}_2(h, \bar{h}) = \omega \cdot \partial_\varphi h + i m_3 T(D) h + a_1(\varphi, x)\partial_x h + O(\partial_x^{-M})\bar{h} + \dots$$

Egorov approach

Eliminate $a_1(\varphi, x)\partial_x$

Evolve with the flow Φ of $u_t = \mathrm{i}a(x)|D|^{1/2}u$

$$\mathcal{L} = \omega \cdot \partial_\varphi \mathbb{I}_2 + P_0(\varphi, x, D)$$

where we denote the diagonal part

$$P_0(\varphi, x, D) := \mathrm{i}(\lambda_3 T(D) + a_{11}(\varphi, x)D)$$

where $T(D) = |D|^{1/2}(1 + \kappa D^2)^{1/2}$

The flow $\Phi(\varphi, \tau) : H^s \mapsto H^s$ of

$$\partial_t u = \mathrm{i}a(\varphi, x)|D|^{\frac{1}{2}}u$$

is well defined in Sobolev spaces and is tame

The conjugated operator $P(\varphi,\tau) := \Phi(\varphi,\tau)P_0\Phi(\varphi,\tau)^{-1}$ solves

Heisenberg equation

$$\begin{cases} \partial_\tau P(\varphi,\tau) = \mathrm{i}[a(\varphi,x)|D|^{\frac{1}{2}}, P(\varphi,\tau)] \\ P(\varphi,0) = p_0(\varphi,x,D) \end{cases}$$

We look for an approximate solution $Q(\varphi,\tau) := q(\tau,\varphi,x,D)$ with a symbol of the form (expanded in decreasing symbols)

$$q(\tau,\varphi,x,\xi) = q_0 + q_1 + \dots, \quad q_0 \in S^{\frac{3}{2}}, q_1 \in S^1 \dots$$

$q_0 = p_0$ then $\partial_\tau Q_1 = \mathrm{i}[a(\varphi,x)|D|^{\frac{1}{2}}, q_0(D)] \in OPS^1, \dots$

$$q_0 + q_1 + \dots = \mathrm{i}\lambda_3 T(\xi) + \mathrm{i}(a_{11} - \tfrac{3}{4}\lambda_3\sqrt{\kappa}\, a_x)\xi + \dots$$

Choose $a(\varphi,x)$ such that $a_{11} - \frac{3}{4}\lambda_3\sqrt{\kappa}\, a_x = 0$. a_{11} is odd in x (reversibility) as in Alazard-Baldi

Conjugating $\omega \cdot \partial_\varphi$ gives

$$\Phi(\varphi,\tau) \circ \omega\cdot\partial_\varphi \circ \Phi(\varphi,\tau)^{-1} = \omega\cdot\partial_\varphi + \Phi(\varphi,\tau)\omega\cdot\partial_\varphi\{\Phi(\varphi,\tau)^{-1}\}$$

Analysis of $\Psi(\varphi,\tau) := \Phi(\varphi,\tau)\omega\cdot\partial_\varphi\{\Phi^{-1}(\varphi,\tau)\}$

It solves

$$\partial_\tau \Psi(\varphi,\tau) = -\mathrm{i}\Phi(\varphi,\tau)\Big(\omega\cdot\partial_\varphi a(\varphi)|D_x|^{1/2}\Big)\Phi^{-1}(\varphi,\tau)$$

Hence $S_\omega(\varphi,\tau) := \Phi(\varphi,\tau)\Big(\omega\cdot\partial_\varphi a(\varphi)|D_x|^{1/2}\Big)\Phi^{-1}(\varphi,\tau)$ solves the Heisenberg equation

$$\begin{cases} \partial_\tau S_\omega(\varphi,\tau) = \mathrm{i}[a(\varphi,x)|D|^{\frac{1}{2}}, S_\omega(\varphi,\tau))] \\ S_\omega(\varphi,0) = \omega\cdot\partial_\varphi a(\varphi)|D_x|^{1/2} \end{cases}$$

$\Longrightarrow$ analyze it as in the previous Egorov analysis

The evolution of the off-diagonal terms is completely different: they evolve according to

$$\begin{cases} \partial_\tau P = AP + PA\,, \quad A := \mathrm{i}a(\varphi, x)|D|^{1/2} \\ P(0) = \mathrm{Op}(p_0(D))\,. \end{cases}$$

$\Longrightarrow$ if $p_0 \in S^{-M}$ then $p(\tau) \in S^{-M}_{\frac{1}{2},\frac{1}{2}}$.

We get a conjugated operator

$$\mathcal{L}(h, \bar{h}) = \omega \cdot \partial_\varphi h + \mathrm{i} m_3 T(D) h + \Phi^{-1} O(\partial_x^{-M}) \Phi \bar{h} + \dots$$

and $\Phi^{-1} O(\partial_x^{-M}) \Phi \in S^{-M}_{\frac{1}{2},\frac{1}{2}}$ is smoothing for M large

KAM transformations are of the same type:

$$\mathcal{L} = \omega \cdot \partial_\varphi + D + \varepsilon P \,, \quad D := \mathrm{diag}(\mu_j) \,, \quad P \text{ bounded} \,.$$

Transform $\mathcal{L}$ under the flow $\Phi(\varphi, \tau)$ of a linear equation

$$\partial_\tau u = \varepsilon W(\varphi) u$$

Expand the solution of Heisenberg equation in size of ε:

$$\mathcal{L}(\tau) = \Phi(\varphi,\tau)\mathcal{L}\Phi(\varphi,\tau)^{-1} = \omega\cdot\partial_\varphi + D + \varepsilon(\omega\cdot\partial_\varphi W + [D, W] + P) + O(\varepsilon^2)$$

Homological equation

Linear map $W \mapsto \omega \cdot \partial_\varphi W + [D, W]$ has eigenvalues

$$\omega \cdot \ell + \mu_j - \mu_i \,, \quad \omega \cdot \ell + \mu_j + \mu_i \,.$$

To kill the $O(\varepsilon)$ term we need Melnikov non-resonance conditions

$$|\omega \cdot \ell + \mu_j \pm \mu_i| \geq |j^{3/2} \pm j'^{3/2}| \gamma \langle \ell \rangle^{-\tau}$$

KAM reducibility for operators which satisfy tame estimates