"Billiards" in Celestial Mechanics

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Degenerate billiard

- Consider a Hamiltonian system with convex in p Hamiltonian $H(q, p)$ on T^*M.
- Let N be a submanifold in M (scatterer). Orbits γ colliding with N at x are reflected elastically:

$$\Delta p = p_+ - p_- \perp T_x N, \quad \Delta H = H_+ - H_- = 0.$$

- Changing direction condition: $\Delta p \neq 0$.
- No tangency: velocities $v_\pm \notin T_x N$.
- If $\operatorname{codim} N > 1$, then p_- does not determine p_+. Past of a collision orbit doesn't determine the future. Degenerate billiard (M, N, H) is not a dynamical system on the set of orbits with collisions.

Collision chains

- Billiard trajectories $\gamma : [a, b] \to M$ having multiple collisions with N (collision chains) are extremals of the action functional

$$A(\gamma) = \int_\gamma p\,dq - H\,dt$$

with constraints $\gamma(t_i) = x_i \in N$.

- A collision chain is a concatenation of collision orbits $\gamma|_{[t_j, t_{j+1}]}$ joining $x_i, x_{i+1} \in N$ such that $\Delta p(t_j) \perp T_{x_j}N$, $\Delta H(t_j) = 0$.
- Changing direction condition: $\Delta p(t_j) \neq 0$.
- No tangency to N at collisions.
- No early collisions: $\gamma(t) \notin N$ for $t_j < t < t_{j+1}$.

Generating function of the collision orbit

- Fix energy $H = E$. Maupertuis action:

$$J_E(\gamma) = \int \|\dot\gamma\|_E \, dt, \quad \|\dot q\|_E = \max_{H(q,p)=E} p \cdot \dot q.$$

- A collision orbit $\gamma : [t_-, t_+] \to M$ joining $x_-, x_+ \in N$ is nondegenerate if the end points are non-conjugate.
- Define the (local) action function on $U \subset N \times N$ by

$$L(x_-, x_+) = \int_\gamma p \, dq = J_E(\gamma).$$

- No tangency $\dot\gamma(t_\pm) \notin T_{x_\pm} N$ implies that L has nondegenerate twist $B(x_-, x_+) = \frac{\partial^2 L}{\partial x_- \partial x_+}$.
- L generates a (local) symplectic map $f : V^- \to V^+$,

$$f(x_-, y_-) = (x_+, y_+) \;\Leftrightarrow\; y_+ = \frac{\partial L}{\partial x_+}, \quad y_- = -\frac{\partial L}{\partial x_-}.$$

$y_\pm$ – projections to $T_x^* N$ of the initial and final momenta $p_\pm$.

Collision map

- In general there are several (maybe none) nondegenerate collision orbits with energy E joining a pair of points, then the generating function has several branches $\mathcal{L}_E = \{L_k\}$ defined on open sets $U_k \subset N \times N$.
- Collision map $\mathcal{F}_E = \{f_k\}$ has branches $f_k : V_k^- \to V_k^+$, $V_k^\pm \subset T^*N$.
- Collision chains correspond to orbits

$$\mathbf{k} = (k_j), \quad \mathbf{z} = (z_j), \quad z_j = (x_j, y_j) \in V_{k_{j-1}}^+ \cap V_{k_j}^-$$

of the skew product of the maps $\mathcal{F}_E = \{f_k\}$.
- $\mathbf{x} = (x_i)$ is an extremal of the discrete action functional

$$A_{\mathbf{k}}(\mathbf{x}) = \sum L_{k_i}(x_i, x_{i+1})$$

- $\mathbf{x}$ is a trajectory of the discrete Lagrangian system $(\mathcal{L}_E)$.

Newtonian singularities

- Consider a Hamiltonian system (H_μ) on $T^*(M \setminus N)$ with

$$H_\mu(q,p) = H(q,p) + \mu V(q) + \mu h_\mu(q,p), \quad \mu \ll 1,$$

$$H(q,p) = \frac{1}{2}\|p - a(q)\|^2 + W(q).$$

  H and h_μ are smooth on T^*M.

- Newtonian singularity: in a tubular neighborhood U of N,

$$V(q) = -\frac{\varphi(q)}{d(q,N)},$$

  where $\varphi > 0$ is a smooth function on M. The distance is defined by the Riemannian metric $\| \ \|$.

Shadowing

Nearly collision trajectories $\gamma_\mu : [a, b] \to M \setminus N$ of system (H_μ) which approach N as $\mu \to 0$ shadow collision chains of the degenerate billiard (M, N, H) with Hamiltonian H and scatterer N.

Theorem

For any finite orbit of the collision map $\mathcal{F}_E$ such that the corresponding collision chain γ changes direction at collisions ($\Delta p \neq 0$) and any small $\mu > 0$ there exists an almost collision trajectory on $H_\mu = E$ shadowing γ with error $O(\mu|\ln \mu|)$.

Hyperbolic shadowing

Theorem

Let Λ be a compact hyperbolic invariant set of $\mathcal{F}_E$. There exists $\mu_0 > 0$ such that for any $\mu \in (0, \mu_0]$ and any orbit in Λ such that the corresponding collision chain γ changes direction at collisions ($\|\Delta p\| \geq \delta > 0$) there exists an almost collision trajectory of system (H_μ) shadowing γ.

Regularization

To prove shadowing theorems, we need to regularize the singularity at N. In applications to Celestial Mechanics, $d = \operatorname{codim} N \leq 3$. For $d = 2$ we use the Levi-Civita regularization, and for $d = 3$ the KS-regularization given by the Hopf map $\mathbb{R}^4 \to \mathbb{R}^3$. For $d > 3$ a different method is needed.

Let U be a tubular neighborhood of N.

Theorem

There exist a manifold $\tilde{U}$, S^1 group action Φ_t on $\tilde{U}$, a surjective map $\pi : \tilde{U} \to U$ commuting with Φ_t, and a Φ_t-invariant Hamiltonian $\tilde{H}$ on $T^*\tilde{U}$ such that:

- *Φ_t is trivial on $\tilde{N} = \pi^{-1}(N)$ and free on $\tilde{U} \setminus \tilde{N}$.*
  $\pi : \tilde{N} \to N$ is a diffeomorphism.
- *Let G be the momentum integral of the symmetry group Φ_t. π takes trajectories of system $(\tilde{H})$ in $\tilde{U} \setminus \tilde{N}$ with $\tilde{H} = \mu$, $G = 0$ to trajectories of system (H_μ) with $H_\mu = E$ (with different time parametrization).*
- *The Hamiltonian $\tilde{H}$ has a normally hyperbolic critical manifold $\mathcal{M} \subset \{\tilde{H} = 0\}$ with real eigenvalues, and trajectories asymptotic to $\mathcal{M}$ are projected by π to orbits of system (H) colliding with N.*

Proof of shadowing

- Collision orbits of the billiard (M, N, H) with $H = E$ correspond to orbits of system $(\tilde{H})$ heteroclinic to $\mathcal{M}$.
- Collision chains of the billiard correspond to heteroclinic chains.
- Orbits of system (H_μ) with energy E passing $O(\mu)$-close to a collision with N correspond to orbits on $\tilde{H} = \mu$ passing $O(\sqrt{\mu})$-close to $\tilde{N}$.
- Almost collision orbits correspond to orbits shadowing heteroclinic chains.
- We use a version of Shilnikov's lemma for normally hyperbolic critical manifolds (B-Negrini 2013) to construct shadowing orbits.

Example: Plane 3-body problem

- Suppose m_3 is much larger than m_1, m_2:

$$m_1/m_3 = \mu\alpha_1, \quad m_2/m_3 = \mu\alpha_2, \quad \alpha_1 + \alpha_2 = 1, \quad \mu \ll 1.$$

- Let $q = (q_1, q_2)$ be positions of m_1, m_2 with respect to m_3. The Hamiltonian of the 3 body problem:

$$H_\mu(q,p) = H_1 + H_2 - \frac{\mu\alpha_1\alpha_2}{|q_1 - q_2|} + \frac{\mu|p_1 + p_2|^2}{2},$$

$$H_i = \frac{|p_i|^2}{2\alpha_i} - \frac{\alpha_i}{|q_i|}.$$

- The Hamiltonian $H = H_1 + H_2$ describes 2 uncoupled Kepler problems.
- The configuration space of the billiard is $M = (\mathbb{R}^2 \setminus \{0\})^2$, and the scatterer is $N = \{q \in M : q_1 = q_2\}$. Collisions with m_3 are excluded.

Second species solutions

- 3-body problem (H_μ) has integrals of energy and angular momentum
$$H_\mu = E, \quad G = G_1 + G_2.$$
- Unperturbed system (H) has integrals H_1, H_2 and G_1, G_2. The flow is quasiperiodic in
$$\mathcal{P} = \{(q,p) : E_1, E_2 < 0,\ G_1, G_2 \neq 0\},$$
- H_μ is a regular perturbations of H in
$$\mathcal{R} = \{(q,p) \in \mathcal{P} : \text{Kepler ellipses do not cross}\}$$
- In the singular part $\mathcal{S} = \mathcal{P} \setminus \mathcal{R}$, every solution of system (H) with incommensurable frequencies approaches the singular set N – perturbation becomes large.
- m_1, m_2 move nearly along ellipses and after many revolutions they almost collide. Then m_1, m_2 move near a new pair of ellipses until they nearly collide again, ...

- Nearly collision periodic orbits of the 3 body problem were named by Poincaré second species solutions.
- Many works of Astronomers.
- Rigorous existence proof of periodic second species was given by Marco-Niderman (1995) for the circular restricted problem.
- Chaotic second species solutions for the circular problem B-MacKay (2000), Font, Nunes, Simo (2002).
- Chaotic and periodic second species solutions for the elliptic restricted problem with small eccentricity. Fast "diffusion" of the Jacobi constant. B (2006).

A collision chain $\gamma = (\gamma_1, \gamma_2)$ of the degenerate billiard of the 3 body problem is a sequence of pairs of Kepler arcs joining a sequence $x_j \in \mathbb{R}^2 \setminus \{0\}$:

- $\gamma(t_j) = (x_j, x_j)$.
- $\gamma|_{[t_j, t_{j+1}]}$ is a pair of Kepler arcs joining x_j, x_{j+1}.
- Total momentum $y = \alpha_1\dot\gamma_1 + \alpha_2\dot\gamma_2$ is continuous: $\Delta y(t_j) = 0$.
- Direction change: $\Delta v(t_j) \neq 0$, $v = \dot\gamma_1 - \dot\gamma_2$.
- No early collisions: $\gamma_1(t) \neq \gamma_2(t)$ for $t_j < t < t_{j+1}$.
- Total energy $E_1 + E_2 = E$ and angular momentum $G = G_1 + G_2$ are constant along γ.

In the discrete Lagrangian $\mathcal{L}_E = \{L_k\}$, the index $k = (k_1, k_2) \in \mathbb{Z}^2$ gives the numbers of revolutions of m_1, m_2 between collisions.

Shadowing collision chains

Theorem

Any finite orbit of the collision map $\mathcal{F}_E$ of the billiard (M, N, H) such that the corresponding collision chain γ satisfies direction change condition $\Delta v(t_j) \neq 0$ is shadowed for small $\mu > 0$ by an orbit of the 3 body problem with $H_\mu = E$ with error $O(\mu|\ln \mu|)$.

Periodic collision chains with energy E correspond to critical points $\mathbf{x} = (x_i)$ of the discrete functional

$$A_{\mathbf{k}}(\mathbf{x}) = \sum_{i=1}^{n} L_{k_i}(x_i, x_{i+1}), \quad x_{n+1} = x_1.$$

Due to rotational symmetry, $A_{\mathbf{k}}(\mathbf{x}) = A_{\mathbf{k}}(e^{i\theta}\mathbf{x})$.

Theorem

Let $\mathbf{x}$ be a nondegenerate modulo rotation critical point of $A_{\mathbf{k}}$. If the corresponding collision chain satisfies changing direction condition, then for small $\mu > 0$ it is $O(\mu|\ln\mu|)$ shadowed by a periodic orbit of the 3 body problem with energy E.

Routh reduction

To construct chaotic shadowing orbits we reduce rotational symmetry. Let G be the integral of angular momentum for the collision map $\mathcal{F}$. Define the Routh Lagrangian

$$R_k(r_-, r_+) = \min_\theta (L_k(r_-, r_+ e^{i\theta}) - G\theta).$$

Reduced Routh system $\mathcal{R}_{E,G} = \{R_k\}$ has one degree of freedom.

Theorem

For small $\mu > 0$ and any hyperbolic orbit of the discrete Routh system, if the corresponding collision chain saties uniform changing direction condition, it is shadowed modulo rotation and time translation by an almost collision hyperbolic orbit of the 3 body problem with angular momentum G and energy E.

Lambert's problem

To study dynamics of the billiard corresponding to the 3 body problem we need to compute the collision generating functions $\mathcal{L}_E = \{L_k\}_{k\in\mathbb{Z}^2}$, $E < 0$. This is reduced to the classical Lambert's problem in Celestial Mechanics.

- Lambert's Theorem gives an explicit formula for the Maupertuis action $f(x_-, x_+)$ of a Kepler elliptic arc with major semiaxis 1 joining x_-, x_+.
- The action of a Kepler orbit γ with energy $h < 0$ joining $z = (x_-, x_+)$ and making $n = [\gamma]$ full revolutions is

$$J_n(z,h) = (-2h)^{-1/2}(2\pi|n| + (\operatorname{sgn} n) f(-2hz)).$$

- The action of a collision orbit $\gamma = (\gamma_1, \gamma_2)$, $[\gamma_i] = k_i$, with energy E joining x_-, x_+ is

$$L_k(z) = \min_{\alpha_1 h_1 + \alpha_2 h_2 = E} (\alpha_1 J_{k_1}(z, h_1) + \alpha_2 J_{k_2}(z, h_2)), \qquad k \in \mathbb{Z}^2.$$

No explicit formula: need to solve Kepler's equation.

Asymptotics for many revolutions

- Computation of L_k simplifies in case of many revolutions:

$$\|k\| = (\alpha_1 k_1^{2/3} + \alpha_2 k_2^{2/3})^{3/2} \gg 1, \qquad \nu = (\nu_1, \nu_2) = \frac{k}{\|k\|}.$$

  Without loss of generality let $E = -1/2$. Then

$$L_k(z) = 2\pi\|k\| + S(\nu, z) + O(\|k\|^{-1}),$$
$$S = \alpha_1 \nu_1^{-1/3} f(\nu_1^{2/3} z) + \alpha_2 \nu_2^{-1/3} f(\nu_2^{2/3} z).$$

- Suppose the bodies move counterclockwise: $k_i > 0$. Set $c_i = \nu_i^{2/3}$ and $c(k) = (c_1, c_2)$. Then

$$S = S_c(z) = \alpha_1 c_1^{-1/2} f(c_1 z) + \alpha_2 c_2^{-1/2} f(c_2 z), \qquad \alpha_1 c_1 + \alpha_2 c_2 = 1.$$

- In the first approximation the discrete Lagrangian system L_k, $k \in \mathbb{Z}^2$, is replaced by a much simpler Lagrangian S_c, $c \in I$, where I is the segment $\alpha_1 c_1 + \alpha_2 c_2 = 1$, $c_i > 0$.
- The major semiaxis of the elliptic arcs forming the collision orbit are c_1^{-1} and c_2^{-1}.

Limit action functional

Let $\mathbf{k} = (k^1, \dots, k^n) \in \mathbb{Z}^{2n}$ with $\|k^j\| > \varepsilon^{-1}$. Discrete action functional for a finite orbit is

$$A_{\mathbf{k}}(\mathbf{x}) = 2\pi \sum_{j=1}^{n} \|k^j\| + \sum_{j=1}^{n} \phi_{c^j}(x_j, x_{j+1}) + O(\varepsilon).$$

If ε is small, we can replace $A_{\mathbf{k}}$ with

$$\Phi_c(\mathbf{x}) = \sum_{j=1}^{n} S_{c^j}(x_j, x_{j+1}) = \alpha_1 \Psi_{c_1}(\mathbf{x}) + \alpha_2 \Psi_{c_2}(\mathbf{x}), \quad \mathbf{c}_i = (c_i^1, \dots, c_i^n)$$

$$\Psi_{\mathbf{b}}(\mathbf{x}) = \sum_{j=1}^{n} b_j^{-1/2} f(b_j(x_j, x_{j+1})), \quad \mathbf{b} = (b_1, \dots, b_n).$$

The functional $\phi_{\mathbf{b}}$ is the sum of Maupertuis actions of a chain of simple arcs of Kepler ellipses with major semiaxis $a_j = b_j^{-1}$ and energies $h_j = -b_j/2$ joining x_j and x_{j+1}.

- Take a finite sequence $\mathbf{c} = (c^1, \dots, c^n) \in I^n$. To satisfy no tangency condition, we need $c_1^j \neq c_2^j$. To satisfy the changing direction condition, we need $c^j \neq c^{j+1}$.
- Let $\mathbf{x}$ be a trajectory of the discrete Lagrangian system $\{S_c\}_{c \in I}$, i.e. a critical point of $\Phi_{\mathbf{c}}$. There exists $\varepsilon > 0$ such that for any $\mathbf{k} = (k^1, \dots, k^n) \in \mathbb{Z}^{2n}$ with $\|k^j\|^{-1} \leq \varepsilon$ and $|c(k^j) - c^j| \leq \varepsilon$, the discrete Lagrangian system $(\mathcal{L})$ has a trajectory shadowing $\mathbf{x}$.
- For small $\mu > 0$ he corresponding collision chain is shadowed by an orbit of the 3 body problem (H_μ).

Conclusion

In the limit of many revolutions, we don't have to worry about synchronizing collision times. Fix E, G and take a sequence $x_j \in \mathbb{R}^2$ and a sequence of pairs Γ_1^j, Γ_2^j of Kepler ellipses with energy and angular momentum E, G and intersecting at x_j, x_{j+1}. We need $\Gamma_1^{j+1} \neq \Gamma_1^j$ and the total momentum $p_1 + p_2$ to be continuous at collisions x_j. Then there exist iterated arcs $\gamma_{1,2}^j$ of $\Gamma_{1,2}$ such that the corresponding collision chain is shadowed by an orbit of the 3 body problem. This partly explains the construction of second species solutions in the third volume of New Methods of Celestial Mechanics of Poincaré.

References

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