Invariant tori in a neighborhood of a degenerate resonance

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Main objects

- Cylinder
$$Z = \{(I,\varphi) \mid I \in \mathbb{R}, \varphi \in \mathbb{T}^1\}$$
- Symplectic self-maps $P_\varepsilon : (I,\varphi) \mapsto (\bar{I},\bar{\varphi})$
- P_ε is defined by the generating function
$$S(\bar{I},\varphi,\varepsilon) = f(\bar{I}) + \varepsilon W(\bar{I},\varphi,\varepsilon)$$
f and W are real-analytic, W is periodic in φ.
- The map takes the form
$$\bar{\varphi} = \varphi + f'(I) + \varepsilon \frac{\partial W}{\partial \bar{I}}(\bar{I},\varphi,\varepsilon), \qquad I = \bar{I} - \varepsilon \frac{\partial W}{\partial \varphi}(\bar{I},\varphi,\varepsilon)$$
- In the unperturbed ($\varepsilon = 0$) system Z is foliated by the circles
$$T_\varkappa = \{(I,\varphi) \mid I = I_0, f'(I_0) = \varkappa, \varphi \in \mathbb{T}^1\}$$

Problem

- Consider a resonant circle $T_\varkappa$.
- After perturbation all resonant circles are as a rule destroyed.
- However diophantine ones survive for small ε. This means that in the perturbed system in an neighborhood of $T_\varkappa$ we could find invariant curve and motion on it would be with the same frequency.

Problem

Find invariant circles with frequencies as close as possible to $\varkappa$ which survive the perturbation.

Technical remark

We can assume that I on $T_\varkappa$ vanishes and (little bit more trickly) $\varkappa = 0$.

Non-degenerate results: Conditions

Twist condition

$$f''(0) \neq 0$$

Twist condition shows that near $I = 0$ the frequency changes in linear approximation.

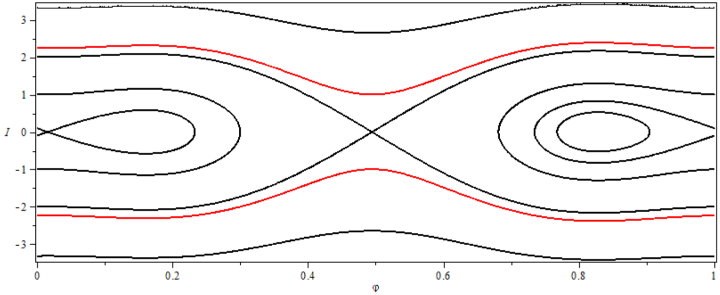
Non-degeneracy for perturbation

Minimum of function $f''(0)W(0, q, 0)$ is unique and non-degenerate.

Non-degenerate case: Theorem

Theorem (Treschev and Zubelevich, 1998)

If Twist condition holds and perturbation is non-degenerate then there exists two invariant circles with frequencies $\varkappa_1 > 0$ and $\varkappa_2 < 0$ which are close to zero: $|\varkappa_1|, |\varkappa_2| \leqslant C\varepsilon$.

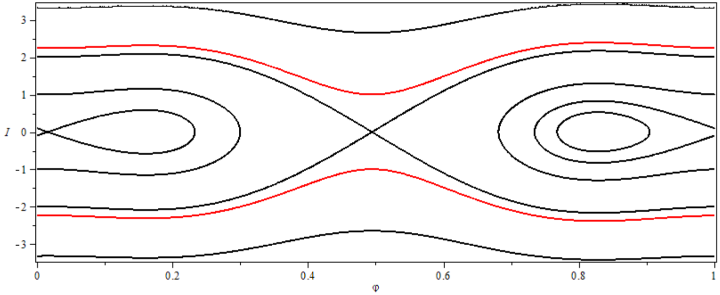


Non-degenerate case: Phase portrait

We can find invariant circle in neighborhood of original one with frequency of order ε. This result is obtained via following computation

$$\varepsilon \sim \frac{\ln \mu}{\ln |I|} \sim \frac{\sqrt{\varepsilon}}{\ln e^{-\frac{\alpha}{\sqrt{\varepsilon}}}}.$$

$\mu \sim e^{\beta\sqrt{\varepsilon}}$ multiplier at the hyperbolic point.



Non-degenerate case: Ideas of the proof

- Use Neishtadt averaging method we come to the following form of P_ε:
  $$P_\varepsilon = \widetilde{P} + \psi,$$
  where $\widetilde{P}$ is integrable, and ψ is exponentially small: $\|\psi\| \leqslant e^{-\frac{\widetilde{\alpha}}{\sqrt{\varepsilon}}}$
- So there is an invariant curve in an exponentially small neighborhood of separatrices of the hyperbolic critical point.
- For the integrable map $\widetilde{P}$ near the hyperbolic point the frequency has asymptotic:
  $$\varkappa(J) \sim \frac{\ln \mu}{\ln |J|\, s},$$
  where J is an action variable for the map $\widetilde{P}$ and $J = 0$ at hyperbolic point.

Degenerate case

No Twist

$$f''(I_0) = \varkappa'(I_0) = 0, \qquad f'''(I_0) \neq 0$$

The frequency is a non-degenerate minimum or maximum.

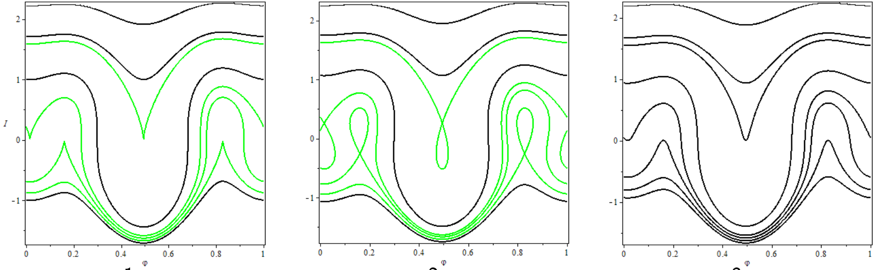
Consider arbitrary critical point φ^* of function $W(I_0, \varphi, 0)$

Conditions for perturbation

- $\frac{\partial^2 W}{\partial \varphi^2}(0, \varphi^*, 0) \neq 0$ for all k. (non-degenerate critical point)
- $\frac{\partial W}{\partial I}(0, \varphi^*, 0) \neq 0$ for all k. (special non-degenerate condition)

Degenerate case: Phase space pictures

Consider different perturbation types in a neighborhood of the resonance.



1. 2. 3.

Phase portraits 2 and 3 are deformations of 1.

1. First order approximation
2. Hyperbolic fixed point $\left(f'''(0)\frac{\partial W}{\partial I}(0, \varphi^*, 0) < 0\right)$
3. Separatrices disappear $\left(f'''(0)\frac{\partial W}{\partial I}(0, \varphi^*, 0) > 0\right)$

Degenerate case: Previous results

Exists many publications on this theme: C. Simo, M. Herman, A. Morosov.
But mostly all of them studies bifurcations and dynamics ”not very close,, to separatrices.

Degenerate case: Results

Theorem

If conditions above holds then

- *if at the critical point* φ^* *that* $f'''(0)\frac{\partial W}{\partial I}(0, \varphi^*, 0) < 0$ *then there exists an invariant curve near* T_0 *with frequency* $|\varkappa| \leqslant C\varepsilon^{\frac{3}{4}+\frac{2}{3}}$.
- *if* $f'''(0)\frac{\partial W}{\partial I}(0, \varphi^*, 0) > 0$ *then separatrix disappears. Invariant curves in an exponentially small neighborhood of it have frequencies larger than* $c_1\varepsilon^{\frac{3}{4}}$, *and there exists an invariant curve with frequency smaller than* $c_2\varepsilon^{\frac{3}{4}}$

Thank you for your attention