How to cross multiple strong resonance

Chong-Qing CHENG & Jinxin XUE

Nanjing University & University of Chicago

June 3-8, 2015 - St. Petersburg

Arnold diffusion

a priori unstable system: Bernard (08), C.-Yan (04), Delshames-de la Llave-Seara (06), Treschev (04)...

a priori stable system (nearly integrable systems)

$$H(x,y) = h(y) + \epsilon P(x,y), \qquad (x,y) \in \mathbb{T}^n \times \mathbb{R}^n.$$

Question: Under typical perturbation ϵP, $\exists$ orbit $(x(t), y(t))$ s.t. $y(t)$ connects any two (finitely many) small balls in the same energy level set $H^{-1}(E)$ for $E > \min h$.

If h is positive definite $\exists$ announcements and works

- n=3 (2.5): Mather, C., Kaloshin-Zhang, Marco,
- n=3.5: Kaloshin-Zhang
- arbitrary n: C.-Xue (2015)

Ingredients of the proof (for nearly integrable systems)

1. away from multiple-strong resonance $\Rightarrow$ 2-dimensional normally hyperbolic invariant cylinder (NHIC) (for time-1-map)
    1. higher energy $\Rightarrow$ KAM
    2. intermediate energy $\Rightarrow$ variational
    3. lower energy (very close the m-strong resonance) $\Rightarrow$ hyperbolic dynamics

    with 2-d NHIC one obtains *a priori* unstable system (C-Yan 04,09)
2. cross multiple-strong resonance (lack of 2-d NHIC), for $n = 3$, double resonance problem

    I shall focus on this issue in this talk, for details refer to C.-Xue: arXiv1503.04153 (109 pages)

Definitions and notations

- Tonelli Lagrangian L: $T\mathbb{T}^n \to \mathbb{R}$ if it is positive definite in $\dot{x}$ with super-liner growth and its Lagrangian flow is complete;
- A Tonelli Lagrangian is uniquely related to a Hamiltonian

$$H(x,y) = \max_{\dot{x}} \langle \dot{x}, y\rangle - L(x,\dot{x});$$

- Given $c \in H^1(\mathbb{T}^n, \mathbb{R}) = \mathbb{R}^n$, a holonomic probability measure μ_c on $T\mathbb{T}^n \times \mathbb{T}$ is called c-minimal if

$$\int (L - \langle c, \dot{x}\rangle) d\mu_c = \inf_{\nu \in \mathfrak{H}} \int (L - \langle c, \dot{x}\rangle) d\nu := -\alpha(c)$$

- the α-function α: $H^1(\mathbb{T}^n, \mathbb{R}) \to \mathbb{R}$ is convex with super-linear growth. If the system is integrable $H = h(y) \Rightarrow y = c$ and $\alpha(c) = h(c)$.

Definitions and notations (continued)

- Given $g \in H_1(\mathbb{T}^n, \mathbb{R})$, let $\beta(g) = \max_c \langle g, c\rangle - \alpha(c)$, called β-function;
- Legendre-Fenchel duality $g \to \mathscr{L}(g) \in H^1(\mathbb{T}^n, \mathbb{R})$,

$$c \in \mathscr{L}(g) \iff \alpha(c) + \beta(g) = \langle g, c\rangle.$$

- Both α- β-functions are usually not smooth. Usually, if g is in k-resonance, $\mathscr{L}(g)$ is k-dim.
  1. A cylinder of periodic orbits with type $\lambda g \Rightarrow \mathscr{L}(\lambda g)$ 1-codim, $\mathbb{C}(g) = \cup_{\lambda>0}\mathscr{L}(\lambda g)$ makes up a channel;
  2. hyperbolic fixed point $\mathscr{L}(0)$ full dimensional.
- Mather set $\tilde{\mathcal{M}}(c) = \cup \mathrm{supp}\mu_c$, $\mathcal{M}(c) = \pi\tilde{\mathcal{M}}(c)$;
- Mañé set $\tilde{\mathcal{N}}(c)$ ($\mathcal{N}(c)$): the set of c-minimal orbits (curves), Each weak KAM solution of $H(x, \partial u + c) = \alpha(c)$ produces c-minimal curves (orbits): $(x, \partial u(x))$ is the initial condition.

Choice of diffusion path

Given two frequency vectors $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ and $\omega' = (\omega'_1, \omega'_2, \cdots, \omega'_n)$, we choose a path

$$
\begin{aligned}
&(\omega_1, \omega_2, \cdots, \omega_n) \to (\omega'_1, \omega_2, \cdots, \omega_n) \to \\
&(\omega'_1, \omega'_2, \cdots, \omega_n) \to \cdots \to \\
&(\omega'_1, \omega'_2, \cdots, \omega'_{n-1}, \omega_n) \to (\omega'_1, \omega'_2, \cdots, \omega'_{n-1}, \omega'_n)
\end{aligned}
$$

For the segment $(\omega_1, \omega_2, \cdots, \omega_n) \to (\omega'_1, \omega_2, \cdots, \omega_n)$ we use approximation of rational $(a, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \hat{\omega}_{n-3})$.

1. we assume $\hat{\omega}_{n-3}$ is irrational (Diophantine) $\Rightarrow$ as a increases from ω_1 to ω'_1, $\exists$ single and double resonance,
2. coordinate change $(a, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \hat{\omega}_{n-3}) \to (a, 0, \frac{p'_3}{q'_3}, \frac{p'_4}{q'_4}, \hat{\omega}_{n-4})$
3. reduction of order, around single resonance, reduce n degrees of freedom to $n-1$ (Delshame-de la Llave-Seare 08)

Multiple-strong resonance

- after k-th step of reduction, further reduction can be done along single resonance;
- after $n-3$ steps of reduction, we get a system of 3 degrees of freedom;
- once a point becomes strong double resonant at k-th step of reduction, we call it $(n-k)$-multiple strong resonance;
- around multiple strong resonance, the dynamics turns out to be complicated.

Normal form

After $n-2$ steps of transformation we get a normal form around strong $(n-1)$-resonance from $H(x,y,-s,G)=E$

$$\begin{aligned}G(x,y,s)=&\frac{1}{2}\langle Ay,y\rangle - V_2(x_1,x_2)-\sum_{j=3}^{n-1}\delta_j V_j(x_1,\cdots,x_j)\\&-\epsilon^\sigma R(x,y,s),\\L(x,\dot{x},s)=&\frac{1}{2}\langle A^{-1}\dot{x},\dot{x}\rangle + V_2(x_1,x_2)+\sum_{j=3}^{n-1}\delta_j V_j(x_1,\cdots,x_j)\\&+\epsilon^\sigma R'(x,\dot{x},s),\end{aligned}$$

where $(x,y)\in\mathbb{T}^{n-1}\times\mathbb{R}^{n-1}$, $\tau\in\mathbb{T}$,

$$0\ll\epsilon^\sigma\ll\delta_{n-1}\ll\cdots\ll\delta_3\ll 1.$$

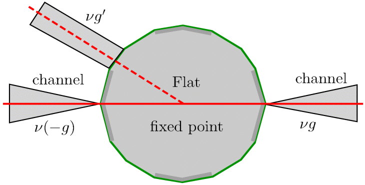
The matrix A has some singularities, but it does not cause trouble.

Special case: $n = 3$

As the first step, let us consider the simplest case $n = 3$ (double resonance) and assume V_2 has a unique minimal point at $x = 0$, non-degenerate. Ignore the term $\epsilon^\sigma R$ we have

$$L = \frac{1}{2}\langle A^{-1}\dot{x}, \dot{x}\rangle + V_2(x)$$

The point $(\dot{x}, x) = (0, 0)$ is a hyperbolic fixed point, and there is a disk $\mathbb{F}_0 = \alpha_G(\min \alpha_G) \subset H^1(\mathbb{T}^2, \mathbb{R}) = \mathbb{R}^2$



For $c \in \text{int}\mathbb{F}_0$, the Mather set $\tilde{\mathcal{M}}(c) = (0, 0)$.

Dynamics around the fixed point

Heuristics:

- the stable and unstable manifolds intersect "transversally" along homoclinics, it destructs invariant tori around $\{\dot{x} = 0\}$;
- it seems $\exists$ Birkhoff instability region, any two Aubry-Mather sets are connected, but the proof turns out to be in another way.

Back to rigorous way

1. the boundary $\partial\mathbb{F}_0 = \partial^*\mathbb{F}_0 \cup (\partial\mathbb{F}_0 \backslash \partial^*\mathbb{F}_0)$ where

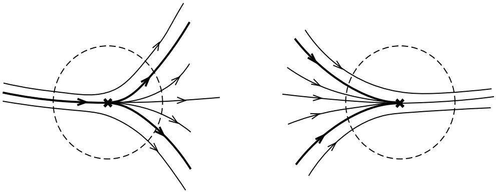
$$\partial^*\mathbb{F}_0 = \{c \in \partial\mathbb{F}_0 : \mathcal{M}(c) \backslash \{0\} \neq \varnothing\},$$

2. $\partial\mathbb{F}_0 \backslash \partial^*\mathbb{F}_0$ contains countably many edges $\{\mathbb{E}_i\}$, $\forall\ c \in \mathbb{E}_i \Rightarrow$ c-minimal orbits are either the fixed point or minimal homoclinics with a homological type $g_i \in H_1(\mathbb{T}^2, \mathbb{Z})$;
3. the set $\partial^*\mathbb{F}_0$ may not be empty.

Generic potential destructs everything

Goal: $\exists$ residual set $\mathfrak{V} \subset C^r(\mathbb{T}^2, \mathbb{R})$ s.t. $\forall\ V \in \mathfrak{V}$ it holds $\forall$ $c \in \partial\mathbb{F}_0$ that $\mathcal{N}(c) \subsetneq \mathbb{T}^2$ (no invariant torus for each $c \in \partial\mathbb{F}_0$).

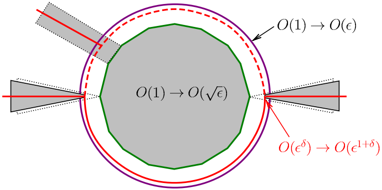
1. each edge $\mathbb{E}_i \subset \partial\mathbb{F}_0 \backslash \partial^*\mathbb{F}_0$ determines one barrier function $\Rightarrow$ countably many perturbations on V_2 $\Rightarrow$ open-dense in C^r
2. $\forall\ c \in \partial^*\mathbb{F}_0$, if the Mañé set is an invariant torus $\mathcal{N}(c) = \mathbb{T}^2$, one of the two cases occurs



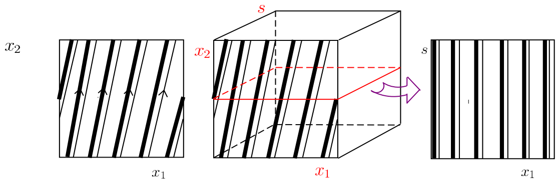
the orbits in the sector lie on stable (unstable) manifold, the angle $\geq \frac{\pi}{2}$ $\Rightarrow$ there are at most 4 points (edges) of first cohomology class when $\mathcal{N}(c) = \mathbb{T}^2$, destruct one by one!

Candidates of Birkhoff instability region

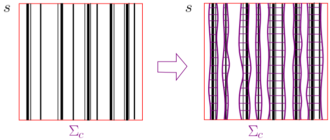
$\partial\mathbb{F}_0$ is compact, the upper semi-continuity of Mañé set on the first cohomology class $\Rightarrow$ $\exists$ an annulus around $\mathbb{A}$ around $\mathbb{F}_0$



$\forall$ $c \in \mathbb{A}$, Mañé set is not an invariant torus. For positive energy, the dynamics on $G^{-1}(E)$ is similar to a twist map. $E > 0 \Rightarrow \forall$ $c \in \mathbb{A}$ non-zero rotation vector



Recover the small perturbation $\frac{1}{2}\langle A^{-1}\dot{x}, \dot{x}\rangle + V(x) + \epsilon^\sigma R(x, \dot{x}, s)$.
By the upper semi-continuity, for small $\epsilon^\sigma R$, we have



$\mathcal{N}(c) \subsetneq \mathbb{T}^2 \Rightarrow$ section Σ_c transversal to the flow s.t.

$$H_1(\mathcal{N}(c) \cap \Sigma_c, \mathbb{Z}) = \mathrm{span}(0, 0, 1)$$

α-functions for 3-d and 2.5-d

1. The Hamiltonian $G(x,y)=\frac{1}{2}\langle Ay,y\rangle - V(x) - \epsilon^{\sigma}R$ is obtained from the Hamiltonian H

$$H=\frac{1}{2}\langle \bar{A}(y,y_3),(y,y_3)\rangle + \frac{y_3}{\sqrt{\epsilon}} - V(x) + \epsilon^{\sigma}\bar{R} = \frac{E}{\epsilon}$$

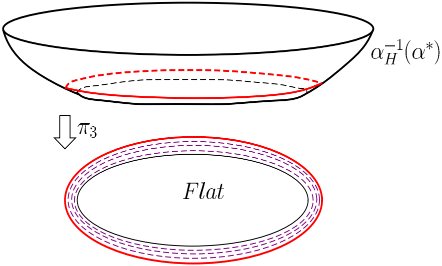
where $(x_3,y_3)=(-s,\sqrt{\epsilon}G)$

2. α_H for H, on the energy level set $(c,c_3)\in\alpha_H(\frac{E}{\epsilon})\Rightarrow$

$$c_3=\sqrt{\epsilon}\alpha_G(c)$$

where α_G for G.

The definition was introduced in C-Li 10



$$\Gamma = \{(c, c_3) : \alpha_H(c, c_3) = \alpha^*, \ c_3 = \text{const.}\} \Rightarrow (c, c_3), (c', c_3') \in \Gamma$$

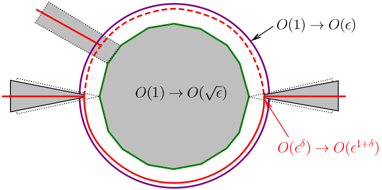
$$(c, c_3) - (c', c_3') = (c - c', 0) \in V_c^{\perp}$$

$$V_c = \text{span}\{(0, 0, 1)\}.$$

Any two classes on Γ are cohomologically equivalent.

A way to cross double resonance

1. around the flat $\mathbb{F}_0$ $\exists$ an annulus $\mathbb{A}$ with thickness ϵd, admits a foliation of circles $\{\Gamma_E\}$, $\forall$ $c, c' \in \Gamma_E \Rightarrow$ $(c, \alpha_G(c))$ and $(c', \alpha_G(c'))$ are equivalent $\Rightarrow$ connecting orbits (C-Li 10)
2. corresponding to each channel, $\exists$ NHIC which can reach $\epsilon^{1+\delta}$-close to $\mathbb{F}_0$ (in terms of energy) $\Rightarrow$ connecting orbits along cylinder (C-Yan 04, 09)



3. As two channels extend into the annulus, we obtain a diffusion path crossing the double resonance.

What do the diffusion orbits look like?

1. assume the boundary $\partial\mathbb{F}_0$ consists of k edges, each of them corresponds to a homoclinic orbit, they stay on $G^{-1}(0)$;
2. in the energy level slightly higher than zero, there are k hyperbolic periodic orbits close to the homoclinic orbits, getting close to the hyperbolic fixed point;
3. the stable manifold of one periodic orbit intersects transversally the unstable manifold of another one;
4. in the phase space, by the λ-lemma, we get orbits moving from one periodic orbit to another. Indeed, if we label these periodic orbits by $1, 2, \cdots, k$, then for all prescribed bi-infinite symbolic sequence in $\{1, 2, \cdots, k\}^{\mathbb{Z}}$, there is an orbit visiting these periodic orbits according to the given sequence;
5. if $\partial\mathbb{F}_0$ contains infinitely many edges, some of them correspond to Aubry-Mather set, ...

Diffusion path when $n > 3$

The truncated normal form $(1 \gg \delta_3 \gg \cdots \gg \delta_{n-1})$

$$G(x, y, s) = \frac{1}{2}\langle Ay, y\rangle - V_2(x_1, x_2) - \sum_{j=3}^{n-1} \delta_j V_j(x_1, \cdots, x_j)$$

- if we ignore the terms $\delta_j V_j$, the flat $\tilde{\mathbb{F}}_0 = \alpha_G^{-1}(\min \alpha_G)$ is a 2-dim disk, $\hat{y} = (y_3, \cdots, y_{n-1})$ keep constant;
- recover the terms $\delta_j V_j$, the flat $\mathbb{F}_0 = \alpha_G^{-1}(\min \alpha_G)$ looks like a pizza, stay in $O(\sqrt{\delta_3})$-neighbourhood of $\tilde{\mathbb{F}}_0$;
- for $g \in H_1(\mathbb{T}^{n-1}, \mathbb{Z})$ let $\mathbb{C}(g) = \cup_{\lambda>0}\mathscr{L}(\lambda g) \subset H^1(\mathbb{T}^{n-1}, \mathbb{R})$, $\forall c \in \mathscr{L}(\lambda g)$ with $\lambda \geq \lambda_0 > 0$ sufficiently small $\Rightarrow$ Mather set is a periodic orbit with type g;
- these channels $\{\mathbb{C}(g_i)\}$ are connected to $\mathbb{F}_0$ at $\lambda = 0$.

What cohomology equivalence do we have for $n > 3$?

- what cohomology equivalence do we have? If we ignore the terms $\{\delta_j V_j\}$, the Mañé set admits a product structure $c = (\tilde{c}, \hat{c})$, $y = (\tilde{y}, \hat{y})$, $\tilde{c}, \tilde{y} \in \mathbb{R}^2$, $\hat{c}, \tilde{c} \in \mathbb{R}^{n-3}$

$$\mathcal{N}(c) = \mathcal{N}_{\hat{y}}(\tilde{c}) \times \mathbb{T}^{n-3}$$

- $\exists$ a section Σ_c of $\mathbb{T}^{n-1}$ s.t. $e_i \in \mathbb{R}^{n-1}$ standard unit vector

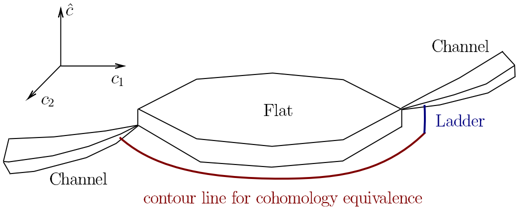
$$V_c = H_1(\mathcal{N}(c) \cap \Sigma_c, \mathbb{Z}) = \mathrm{span}\{e_3, e_4, \cdots, e_{n-1}\}$$

- using upper semi-continuity of Mañé set, for small $\delta_j V_j$, $\exists$ annulus $\mathbb{A}$ around the pizza $\mathbb{F}_0$ which admits a foliation of curves of cohomology equivalence $(c, c_n), (c', c_n') \in \{(c, c_n) : \alpha_H(c, c_n) = \alpha^*, \ c_n = \mathrm{const.}\} \Rightarrow$

$$(\tilde{c}, \hat{c}, c_n) - (\tilde{c}', \hat{c}', c_n') = (\tilde{c} - \tilde{c}', 0) \in V_c^{\perp}$$

What is the new difficulty when $n > 3$

- ♣ along each curve of cohomology equivalence, $\hat{c}$ keeps constant;
- ♣ different channels $\{\mathbb{C}(g_i)\}$ may be connected to the flat $\mathbb{F}_0$ with different "height" (different $\tilde{c}$-coordinate)



these curves of cohomology equivalence may not connect that two channels. How to solve this problem?

$\Rightarrow$ **Ladder climbing**

Construction of ladder

- the following system can be treated as two degrees of freedom

$$\frac{1}{2}\langle Ay, y\rangle - V_2(x_1, x_2)$$

$\exists$ normally hyperbolic invariant cylinder (NHIC) close to the *m*-strong resonance $\Rightarrow$ NHIM of 2(n-1)-d, (Delshams-de la Llave-Seare 08) $\Rightarrow$ reduction of order;

$$\tilde{h}(I + \tau_3 y_3) + \frac{1}{2}A_3 y_3^2 + \frac{1}{2}\langle \hat{A}_{n-4}\hat{y}_{n-4}, \hat{y}_{n-4}\rangle$$

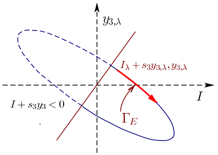
- turn on one small term $\delta_3 V_3$

$$\frac{1}{2}\langle Ay, y\rangle - V_2(x_1, x_2) - \delta_3 V_3(x_1, x_2, x_3) \;\Rightarrow$$

$$\tilde{h}(I + \tau_3 y_3) + \frac{1}{2}A_3 y_3^2 - \delta_3 V_3'(\theta, x_3, I + \tau_3 y_3) + \frac{1}{2}\langle \hat{A}_{n-4}\hat{y}_{n-4}, \hat{y}_{n-4}\rangle$$

Construction of ladder (continued)

- without $\delta_3 V_3$, the stable manifold of NHIM intersects its unstable manifold, but not transversally;
- turn on the term $\delta_3 V_3$, we still have regularity of weak KAM for 2-d system restricted on energy level $E > \min \alpha$;
- by normal hyperbolicity, we extend this regularity to the whole space $\Rightarrow$ transversality for system with 3 degrees of freedom (treat $(y_4, \cdots, y_{n-1})$ as parameter);
- move (I, y_3) along energy level (using hyperbolic structure: stable and unstable manifold of NHIC)



- connecting orbits with incomplete intersection for the follow-up construction of ladder

Regularity of weak KAM for 2-d system

- why do we want NHIC of 2-d? $\Rightarrow$ with which we obtained the $\frac{1}{2}$-Hölder regularity (C.-Yan 04, Zhou 11);
- why do we need the regularity? it seems the way available only to show the intersection transversality of stable and unstable "set" of all Mather sets, there are uncountably many;
- is there another way to get the regularity? a Hamiltonian with 2-degrees of freedom, restricted on energy level set $H^{-1}(E)$ with $E > \min \alpha$. (C. 11)
  - one Aubry class determines one elementary weak KAM
  - barrier function is defined by elementary weak KAM

$$B_{i,j}(x) = u_i^-(x) - u_j^+(x)$$

  - all elementary weak KAM can be parameterized by "volume" σ so that $\sigma \to u_\sigma^\pm$ is $\frac{1}{3}$-Hölder in C^0-topology

New mechanism of local connecting orbit

Before this work there are two types of local connecting orbts

- by cohomology equivalence: $\tilde{\mathcal{M}}(c)$ can be connected to $\tilde{\mathcal{M}}(c')$ if $c \sim c'$;
- by Arnold's mechanism: if the stable "set" of $\tilde{\mathcal{M}}(c)$ intersects its unstable "set" transversally, then $\tilde{\mathcal{M}}(c)$ can be connected to $\tilde{\mathcal{M}}(c')$ if c' is close to c.

To construct ladder, we need to generalize the second one

- ♣ weaker condition: intersection may not be transversal, may contain some circles $\{\ell_i\}$,
- ♣ weaker result: some more restriction $\Rightarrow \langle c - c', [\ell_i]\rangle = 0$.

This version of local connecting orbit is good enough for the construction of ladders.

Composition of simple ladders

The 1-st step: on $2(n-1)d$ NHIC the normal form

$$\tilde{h}(I+\tau_3 y_3)+\frac{1}{2}A_3 y_3^2-\delta_3 V_3'(\theta,x_3,I+\tau_3 y_3)+\frac{1}{2}\langle \hat{A}_{n-4}\hat{y}_{n-4},\hat{y}_{n-4}\rangle$$

a simple ladder: $\mathbb{L}_3$ $c_3\to c_3'$

The 2-nd step: on $2(n-2)d$ NHIC the normal form

$$\tilde{h}(I+\tau_4 y_4)+\frac{1}{2}A_4 y_4^2-\delta_4 V_4'(\theta,x_4,I+\tau_4 y_4)+\frac{1}{2}\langle \hat{A}_{n-5}\hat{y}_{n-5},\hat{y}_{n-5}\rangle$$

a simple ladder: $\mathbb{L}_4$ $c_4\to c_4'$

....

We finally construct the ladder (with small size)

$$\mathbb{L}=\mathbb{L}_{n-1}*\cdots*\mathbb{L}_3:\quad (c_3,\cdots,c_{n-1})\to(c_3',\cdots,c_{n-1}').$$