Growth of Sobolev norms for the defocusing analytic NLS

Marcel Guardia, Emanuele Haus and Michela Procesi

June 4, 2015

The analytic gauge invariant NLS

- Consider the equation

$$-iu_t + \Delta u = |u|^{2(d-1)}u + G'(|u|^2)u, \quad d \in \mathbb{N},\ d \geq 2$$

  where $x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $t \in \mathbb{R}$ and $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$.
- $G(y)$ is an analytic function with a zero of degree at least $d + 1$.
- Defocusing implies well posed globally in time.
- Solutions of NLS conserve the quantities:
  - The Hamiltonian

$$E[u](t) = \int_{\mathbb{T}^2} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2d}|u|^{2d} + \frac{1}{2}G(|u|^2) \right) \frac{dx}{(2\pi)^2} dx(t)$$

  - The mass

$$\mathcal{M}[u](t) = \int_{\mathbb{T}^2} |u|^2 dx(t) = \int_{\mathbb{T}^2} |u|^2 dx(0),$$

    the square of the L^2-norm.

Transfer of energy

- Fourier series of u,

$$u(x,t) = \sum_{n\in\mathbb{Z}^2} a_n(t) e^{inx}.$$

- Can we have transfer of energy to higher and higher modes as $t \to +\infty$?
- We measure it with the growth of s-Sobolev norms ($s > 1$)

$$\|u(t)\|_{H^s(\mathbb{T}^2)} := \|u(t,\cdot)\|_{H^s(\mathbb{T}^2)} := \left(\sum_{n\in\mathbb{Z}^2} \langle n \rangle^{2s} |a_n(t)|^2 \right)^{1/2},$$

  where $\langle n \rangle = (1 + |n|^2)^{1/2}$.
- Thanks to mass and energy conservation,

$$\|u(t)\|_{H^1(\mathbb{T}^2)} \leq C \|u(0)\|_{H^1(\mathbb{T}^2)} \quad \text{for all} \quad t \geq 0.$$

How fast the energy transfer can be?

- Dimension 1 and $d = 2$ (cubic case), a priori bounds for all H^s.
- Dimension $D \geq 2$ or power $d > 2$: growth of H^s expected to happen.
- Polynomial upper bounds for the growth of H^s, $s > 1$:

  $$\|u(t)\|_{H^s} \leq t^A \|u(0)\|_{H^s} \quad \text{for} \qquad t \to +\infty.$$

  for some $A > 0$.
- Question by Bourgain (2000): Are there solutions u such that for $s > 1$,

  $$\|u(t)\|_{H^s} \to +\infty \qquad \text{as } t \to +\infty?$$

The cubic case

- We are interested in small initial data.
- Cubic case: $-iu_t + \Delta u = |u|^2 u$

Theorem (Colliander, Keel, Staffilani, Takaoka, Tao (2010))

Fix $s > 1$, $\mathcal{C} \gg 1$ and $\mu \ll 1$. Then there exists a global solution u of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(0)\|_{H^s} \leq \mu, \qquad \|u(T)\|_{H^s} \geq \mathcal{C}.$$

- Valid on any $\mathbb{T}^D$, $D \geq 2$.

The cubic case

- M. G. and V. Kaloshin: $T \sim e^{\left(\frac{\mathcal{C}}{\mu}\right)^A}$ for some $A > 0$.
- M. G. and V. Kaloshin also in the cubic case: Fix $\mathcal{K} \gg 1$, there exists a solution u of NLS on $\mathbb{T}^2$ and T satisfying that

$$\|u(T)\|_{H^s} \geq \mathcal{K}\|u(0)\|_{H^s}, \qquad T \sim \mathcal{K}^B, \quad \text{for some } B > 0$$

  and

$$\|u(t)\|_{L^2} \leq \mathcal{K}^{-\sigma} \quad \text{for some } \sigma > 0.$$

- E. Haus and M. Procesi generalized the I-team result to the quintic NLS ($d = 3$ and $D \geq 2$).

$$-iu_t + \Delta u = |u|^4 u$$

Main result

$$-iu_t + \Delta u = |u|^{2(d-1)}u + G'(|u|^2)u$$

Theorem (M. G. – E. Haus – M. Procesi)

Let $d \geq 2$ and $s > 1$. There exists $c > 0$ with the following property: for any large $\mathcal{C} \gg 1$ and small $\mu \ll 1$, there exists a global solution $u(t) = u(t, \cdot)$ of NLS and a time T satisfying

$$T \leq e^{\left(\frac{\mathcal{C}}{\mu}\right)^c}$$

such that

$$\|u(0)\|_{H^s} \leq \mu \ \text{ and } \ \|u(T)\|_{H^s} \geq \mathcal{C}.$$

- Valid on any $\mathbb{T}^D$, $D \geq 2$.
- If we do not assume small initial Sobolev norm, we do not get better time estimates.

The I-team approach for the cubic case

- Cubic NLS as an ode (of infinite dimension) for the Fourier coefficients of u:

$$-i\dot{a}_n = |n|^2 a_n + \sum_{\substack{n_1,n_2,n_3 \in \mathbb{Z}^2 \\ n_1 - n_2 + n_3 = n}} a_{n_1} \overline{a_{n_2}} a_{n_3}, \qquad n \in \mathbb{Z}^2.$$

- Drift through resonances.
- Resonant monomial

$$n_1 - n_2 + n_3 - n = 0 \quad \text{and} \quad |n_1|^2 - |n_2|^2 + |n_3|^2 - |n|^2 = 0$$

- Non-degenerate resonances form a rectangle in $\mathbb{Z}^2$.

The I-team approach for the cubic case

They choose carefully a finite set of modes which interact through the resonances in a very particular way.

Roughly speaking:

- Consider a finite set of modes $\Lambda \subset \mathbb{Z}^2$ of large size $|\Lambda| = N2^{N-1}$ with $N \sim \log(\mathcal{C}/\mu) \gg 1$.
- Impose each mode only interacts with other modes through two rectangles.
- It receives energy from one of them and pumps energy into the other.

The I-team approach for the cubic case

- Doing further reductions: finite dimensional (toy) model

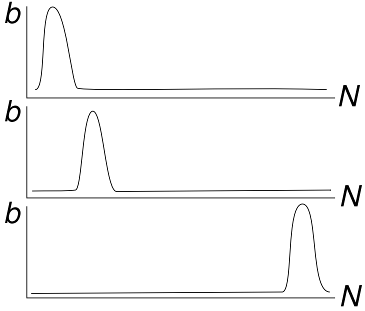
$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 1, \dots N.$$

  which approximates well certain solutions of NLS.
- Each b_j represents 2^{N-1} modes.
- It can be seen as a Hamiltonian system on a lattice $\mathbb{Z}$ with nearest neighbor interactions.
- Hamiltonian:

$$h(b) := \frac{1}{4}\sum_{j=1}^{N}|b_j|^4 - \frac{1}{2}\sum_{j=1}^{N}\left(\overline{b}_j^2 b_{j-1}^2 + b_j^2\overline{b}_{j-1}^2\right).$$

- It has the mass as a first integral $M = \sum_{i=1}^{N}|b_i|^2$.

Transfer of energy for the cubic toy model



- There are orbit $b(t)$ of the toy model such that at $t = 0$ is localized in b_1 and at a certain $t = T \gg 1$ is localized in b_N.

Dynamics of the cubic toy model

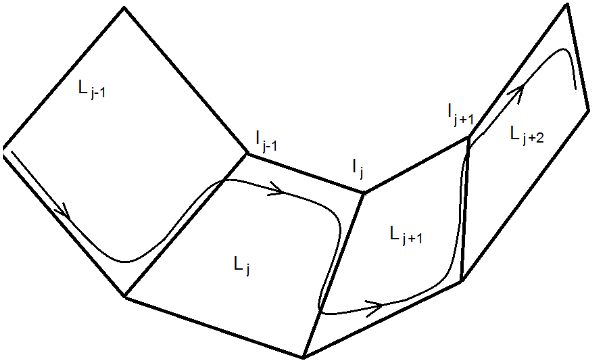
- We analyze the dynamics of the toy model

$$\dot{b}_j = -ib_j^2\overline{b}_j + 2i\overline{b}_j\left(b_{j-1}^2 + b_{j+1}^2\right),\ j = 0, \dots N,$$

- Each 4-dimensional plane

$$L_j = \{b_1 = \cdots = b_{j-1} = b_{j+2} = \cdots = b_N = 0\}$$

is invariant.



- There are solutions that stay close to the planes $\{L_j\}_{j=2}^{N-1}$ and go from one intersection $l_j = L_j \cap L_{j+1}$ to the next one $l_{j+1} = L_{j+1} \cap L_{j+2}$ consequently for $j = 3, \ldots, N-1$.
- In the intersections l_j only b_j is nonzero.
- The planes L_j have normal positive Lyapunov exponents.

The resonant sets in the general case

- Resonant monomials

$$\sum_{i=1}^{2d} (-1)^i n_i = 0 \quad \text{and} \quad \sum_{i=1}^{2d} (-1)^i |n_i|^2 = 0.$$

- Geometry and combinatorics of resonances are far more complicated.
- Idea from Procesi and Haus (2014): use still rectangles as building blocks.
- They seem to be better suited to lead to transfer of energy.

The resonant sets in the general case

- We need to impose further conditions to avoid non-desired resonances.
- Some resonances are unavoidable: take two rectangles with a common vertex

$$n_1 - n_2 + n_3 - n_4 = 0 \qquad |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 = 0$$
$$n_4 - n_5 + n_6 - n_7 = 0 \qquad |n_4|^2 - |n_5|^2 + |n_6|^2 - |n_7|^2 = 0$$

  They create the resonant sextuple

$$n_1 - n_2 + n_3 - n_5 + n_6 - n_7 = 0$$
$$|n_1|^2 - |n_2|^2 + |n_3|^2 - |n_5|^2 + |n_6|^2 - |n_7|^2 = 0.$$

- Each mode receives and pumps energy through more than two rectangles.

The toy model in the general case

- We imposing combinatorial restrictions on the interactions between modes to avoid undesired resonances.
- Hamiltonian of the toy model

$$h(b) = \left(\sum_{i=1}^{N} |b_i|^2\right)^{d-2} \left[\frac{1}{4}\sum_{i=1}^{N} |b_i|^4 - \sum_{i=1}^{N-1} \mathrm{Re}\,(b_i^2 \bar{b}_{i+1}^2)\right] + \frac{1}{2^N}\mathcal{P}\left(b, \bar{b}, \frac{1}{2^N}\right).$$

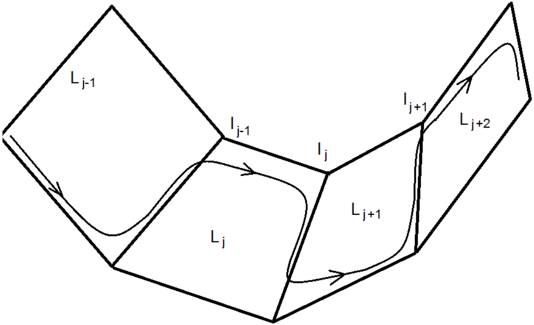
- Recall $N \sim \log(\mathcal{C}/\mu) \gg 1$.
- $M = \sum_{i=1}^{N} |b_i|^2$ is a first integral.
- Roughly speaking: unavoidable “higher order” resonances only appear at higher order.
- The drift obtained for the cubic toy model takes long time.
- We have to analyze carefully $\mathcal{P}$ to see that we have a behavior similar to the cubic case.

Properties on $\mathcal{P}$

- Using combinatorics, we can impose extra properties on $\mathcal{P}$.
- These properties imply that we have the same structure as in the cubic case.
- All monomials in $\mathcal{P}$ are of even degree in $(b_j, \overline{b_j})$.
- The subspaces $\{b_j = 0\}$ are invariant.
- Now we do not have nearest neighbor interaction.
- The strongest non-nearest neighbor interaction is integrable: a monomial depending on two modes i, j, $|i - j| \neq 1$, is of the form $|b_i||b_j|^{d-2}$.

Shadowing the invariant planes

- The planes $L_j = \{b_1 = \cdots = b_{j-1} = b_{j+2} = \cdots = b_N = 0\}$ are still invariant.



- We proceed as in M. G.– V. Kaloshin.
- We construct solutions that drift through the planes
- These shadowing orbits are a good first order of orbits of NLS undergoing growth of Sobolev norms