# New Techniques for the $n$–Body Problem

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Hamiltonian systems and their applications
Euler Institute (St. Petersburg) June 3, 2015

# Overview

- Measure of the collision orbits
- Measure of non-collision singularities (2 bodies, $n$ centers)

- Regularization of the Kepler problem
- Regularization of (simultaneous) binary collisions)
- Kinematics
- Train correspondences

- Classical scattering
- Open questions

with Stefan Fleischer, respectively with
Jacques Fejoz and Richard Montgomery

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- with collision set
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- Given masses $m_1, \ldots, m_n > 0$ and
- interaction parameters $I_{i,j} \in \mathbb{R} \setminus \{0\}$

we consider the Hamiltonian

$$\widehat{H} : T^*\widehat{M} \to \mathbb{R} \quad , \quad \widehat{H}(p,q) := K(p) + V(q), \tag{1}$$

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# Notions for $n$ body motion

- Phase space $\widehat{P} := T^*\widehat{M}$,
- Maximal flow $\Phi = (p, q) : D \to \widehat{P}$ with domain

$$D = \{(t, x) \in \mathbb{R} \times \widehat{P} \mid T^-(x) < t < T^+(x)\}.$$

## Definition

- Set $\mathrm{Sing} := \left\{x \in \widehat{P} \mid T^+(x) < +\infty\right\}$ of phase space points encountering a singularity in the future;
- Subset of phase space points encountering a collision singularity:

$$\mathrm{Coll} := \left\{x \in \mathrm{Sing} \mid \lim_{t \nearrow T^+(x)} q(t, x) \text{ exists}\right\}.$$

# Measure of collision orbits

## Theorem (Saari)

*For dimension $d = 3$ and forces $\nabla V_{i,j}(q) = -m_i m_j \, q/\|q\|^{1+\alpha}$*

- *collisions are improbable for $\alpha < 17/7$.*
- *Binary collisions are improbable for $\alpha < 3$.*

Don Saari: Improbability of Collisions in Newtonian Gravitational Systems. II. *Transactions of the AMS* **181**, *351–368 (1973)*

**Generalisation:** Dimension $d \geq 2$

$$V(q) = \sum_{1 \leq i < j \leq n} V_{i,j}(q_i - q_j) \quad \text{with} \quad V_{i,j} \in C^2(\mathbb{R}^d \setminus \{0\}, \mathbb{R})$$

## Definition

A potential $V$ is called

- *moderate* if $\|\nabla V_{i,j}(q)\| \leq C\|q\|^{-3+\varepsilon}$ for $\|q\| \leq 1$,
- *long ranged* if $\|\nabla V_{i,j}(q)\| \leq C\|q\|^{-1-\varepsilon}$ for $\|q\| \geq 1$.

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# Measure of collision orbits

**Theorem (St. Fleischer, AK)**

*For moderate (and long ranged) potentials $V$ collisions are improbable.*

**Proof.** Use family of global Poincaré surfaces of section $\mathcal{H}^{(m)}$ in phase space 'around' the collision set $\Delta$, with

- surface area $\int_{\mathcal{H}^{(m)}} \mathbf{i}_{X_H}\Omega$ (with phase space volume form $\Omega$) going to zero as $m \to \infty$,
- and any collision orbit intersecting all $\mathcal{H}^{(m)}$ with $m$ large.
- Cluster: set partition $\mathcal{C} = \{C_1, \ldots, C_k\}$ of $\{1, \ldots, n\}$
- $\mathrm{Coll}_{\mathcal{C}} := \{x \in \mathrm{Coll} \mid \lim_{t \nearrow T^+(x)} \left(q_i(t,x) - q_j(t,x)\right) = 0 \text{ iff } i \simeq_{\mathcal{C}} j\}$,

$$\mathrm{Coll} = \dot{\bigcup_{\mathcal{C}}} \mathrm{Coll}_{\mathcal{C}}.$$

- Use internal and external coordinates for cluster $\mathcal{C}$ to define $\mathcal{H}^{(m)}$: modulus of internal position as function of other coordinates. $\square$

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# Measure of non-collision singularities

## For an $n$-center – two-body system

- Non-collision singularities exist in the four-body problem.

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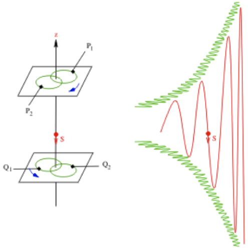

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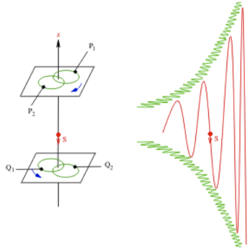


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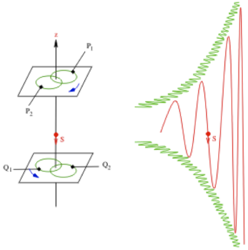


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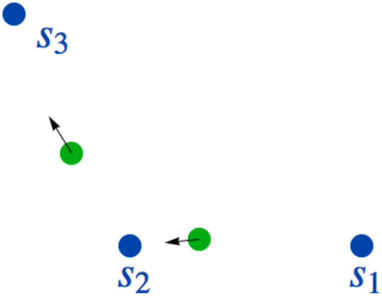
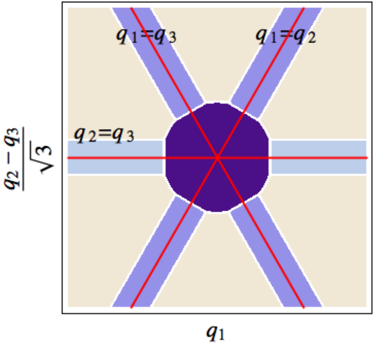
*For moderate pair potentials non-collision singularities are improbable.*




- Fixed centers $s_1, \ldots, s_n \in \mathbb{R}^d$
- two moving particles

# Poincaré surfaces for collision set




Picture in configuration space, for

- $d = 1$ dimension,
- $n = 3$ bodies,
- in center of mass system.

Collision set $\Delta$

## Why another regularisation?

There are already regularisations by
Levi-Civita, Kustaanheimo-Stiefel, Moser, Souriau...

Advantages:

- All dimensions $d \geq 1$
- simultaneously for all energies
- no change of time parameter
- symplectic extension of phase space.

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# Regularisation

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- regularity of a Poincaré map (block regularity)
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# Regularisation by phase space extension: Kepler case

- Hamiltonian, attractive case ($I < 0$):

$$\widehat{H} : T^*\left(\mathbb{R}^d \setminus \{0\}\right) \to \mathbb{R} \quad , \quad \widehat{H}(p,q) = \frac{\|p\|^2}{2m} + \frac{I}{\|q\|}$$

- $\varepsilon$–neighborhood of the excluded fiber $T_0^*\mathbb{R}^d$

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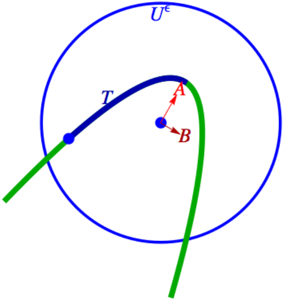
Together we obtain a smooth map

$$\widehat{\Psi} := \left(\widehat{H}, \widehat{T}, \widehat{A}, \widehat{B}\right) : \widehat{U}^{\varepsilon} \longrightarrow T^*(\mathbb{R} \times S^{d-1}),$$

- all entries except time $\widehat{T}$ are constants of the motion.
- $\mathrm{im}(\widehat{\Psi})$ misses the zero section $\cong \mathbb{R} \times S^{d-1}$ of the cotangent bundle, since the collision orbits are characterized by zero angular momentum and thus $\widehat{B} = \widehat{L}\widehat{A} = 0$, and the point of collision on such an orbit corresponds to $\widehat{T} \to 0$.



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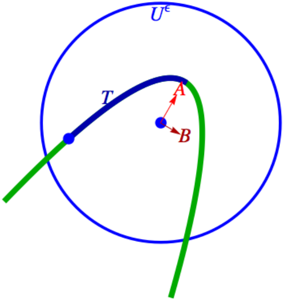

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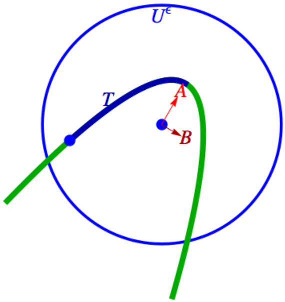

# Regularisation: Kepler case

- We now complete phase space by setting

$$P := T^*\big(\mathbb{R}^d \setminus \{0\}\big) \;\cup\; (\mathbb{R} \times S^{d-1})$$

and introducing a second chart $\Psi$ with domain

$$U^\varepsilon := \widehat{U}^\varepsilon \cup (\mathbb{R} \times S^{d-1}) \;\subseteq P,$$

$$\Psi \equiv \big(H, T, A, B\big) : U^\varepsilon \to T^*(\mathbb{R} \times S^{d-1}),$$

$$\Psi\restriction_{\widehat{U}^\varepsilon} := \widehat{\Psi} \quad \text{and} \quad \Psi\restriction_{\mathbb{R}\times S^{d-1}}(h,a) := (h, 0, a, 0).$$

# Kepler regularization: Smooth & complete

## Lemma

*$\widehat{\Psi}$ is a diffeomorphism onto its image. On $U^\varepsilon$ for all $1 \le i, j \le d$*

$$\{H, T\} = 1 \quad , \quad \{A_i, B_j\} = \delta_{i,j} - A_i A_j \quad , \quad \{B_i, B_j\} = L_{i,j},$$

*all other Poisson brackets being zero.*
*The Hamiltonian system $(P, \omega, H)$ is a smooth complete extension of $\left(T^*(\mathbb{R}^d \setminus \{0\}) \, , \, dq \wedge dp \, , \, \widehat{H}\right)$.*

# $n$-body simultaneous binary regularisation:
## non-smooth dynamics

- McGehee (1974): Triple collisions are not regularizable.
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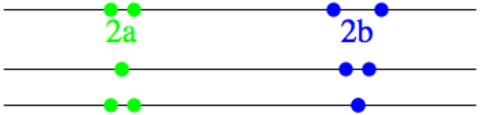
# Intuition for the Martínez–Simó $C^{8/3}$ result

This is for special one-parameter solutions, the parameter $\Delta t \in \mathbb{R}$ being the time between the two binary collisions.

**Example:** Four equal masses on a line, distance one at simultaneous binary collision.

Distance between first pair: $2a(t)$, between second pair: $2b(t)$.




Force on second pair: $\ddot{b} = -\frac{1}{2}b^{-2}+$

$$\frac{1}{2}\Big((1+a-b)^{-2}+(1-a-b)^{-2}-(1+a+b)^{-2}-(1-a+b)^{-2}\Big)$$
$$= \quad -\frac{1}{2}b^{-2}+4b+8b^3+24a^2b+\ldots$$

Perturbation theory: $b(t)=(\frac{3}{2}t)^{2/3}+\ldots$, $a(t)=\big(\frac{3}{2}(t+\Delta t)\big)^{2/3}+\ldots$

# Intuition for the Martínez–Simó $C^{8/3}$ result

- Rate of change in energy of $b$ pair, for $t$ of same order as $\Delta t$:
  $\ddot{b}\dot{b} = \ldots + k a^2 b\dot{b} = \ldots + k\,\mathrm{sign}(\Delta t)\,\Delta t^{5/3}$.
- Integrate for time $\sim \Delta t$: $k\,\mathrm{sign}(\Delta t)\,\Delta t^{8/3}$
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..after additional coordinate change...

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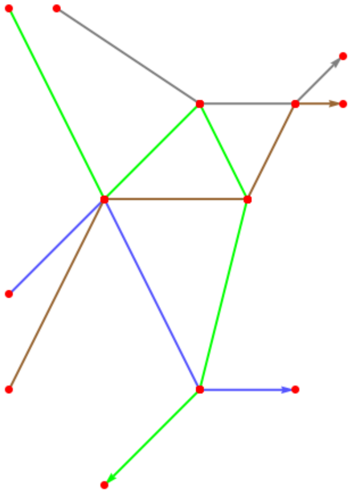
# Train correspondences

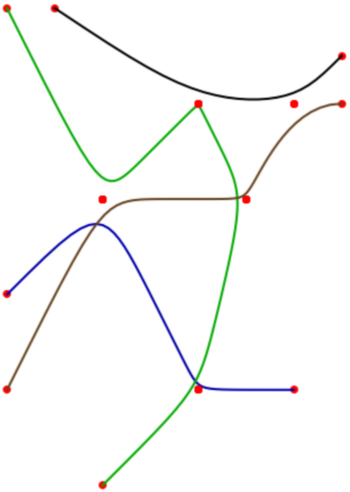

Two Wood Burners in Kamloops Train Yards: Circa 1888

# Specific idea:

## find solution where the celestial bodies meet at kinematically prescribed points

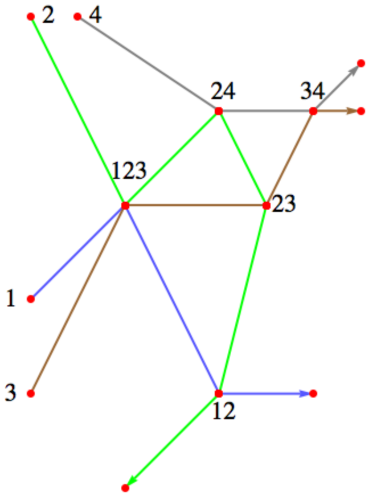


Ideal



reality (for Coulomb repulsion)

# Kinematics: Setup




- between collisions, the $n$ particles (figure: $n = 4$) move with constant velocity
- at collision, there is energy and momentum conservation for each cluster.
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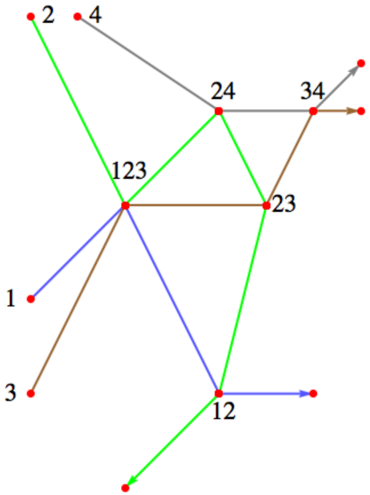


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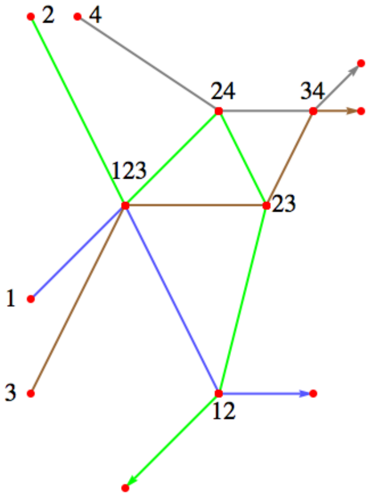


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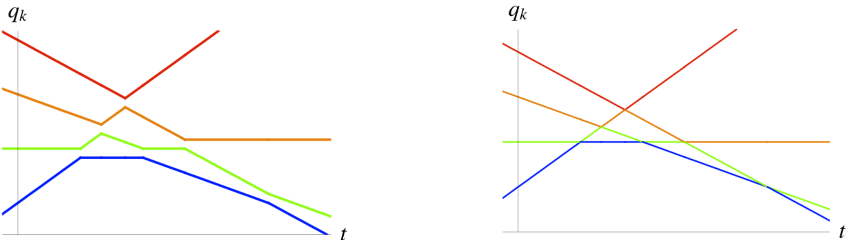
# Kinematics of $n$-body scattering

(the hard problems!)

- **Question:** How many collisions for $n$ *balls* on the line, having equal mass? (deterministic for binary collisions)




**Answer:** $\binom{n}{2}$, look at position $q_k(t)$ of ball no. $k$ at time $t$!

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  **Answers: 1)** Best upper bound is super-exponential in $n$

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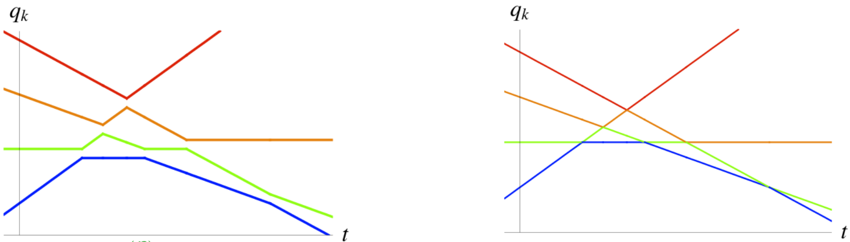


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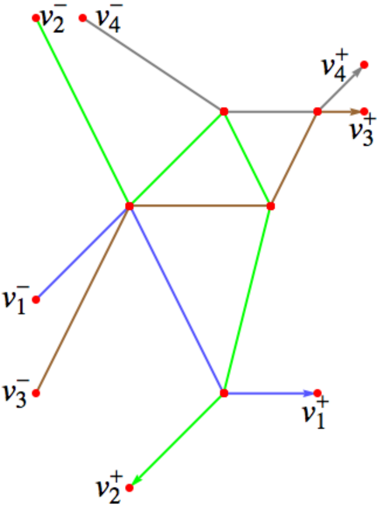


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$$(v^-, v^+) \in S^D \times S^D$$

of initial/final velocities $v^\pm$, with $D := d(n-1) - 1$ (conservation of total kinetic energy and of total momentum)
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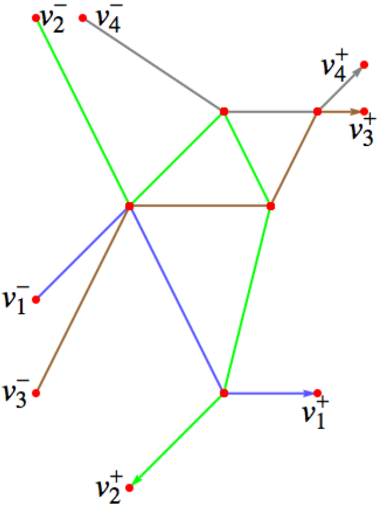


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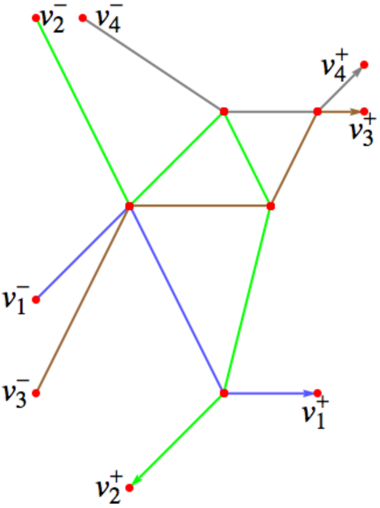


- Consider for given succession of clusters all pairs

$$(v^-, v^+) \in S^D \times S^D$$

of initial/final velocities $v^\pm$, with $D := d(n-1) - 1$ (conservation of total kinetic energy and of total momentum)

- Whereas the $n$-body collision has codimension zero,
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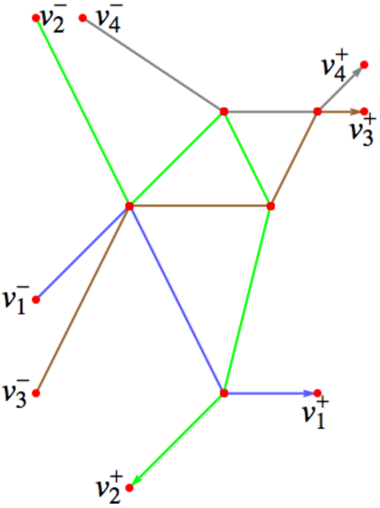


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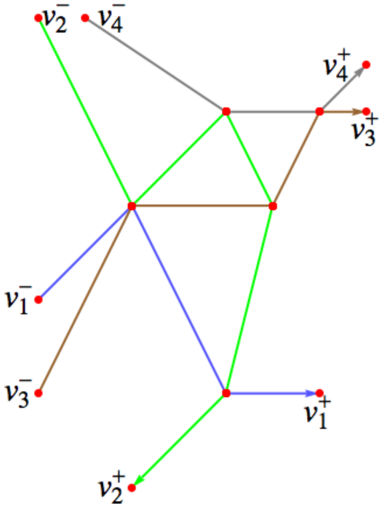


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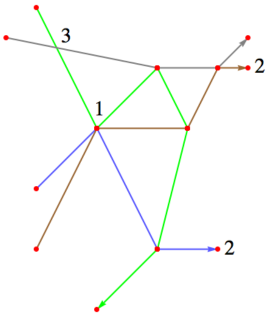
# Transversality; long range of gravity

Transversality: We only consider kinematic solutions for which

1. all collisions are binary
2. the initial velocities of all particles are different (same for final velocities)
3. forward scattering (that is, collision without scattering) does not occur.
4. No particle is allowed to go through without any deflection.




Long range of the $1/r$ potential: already Kepler hyperbolae are not asymptotic in time to *any* straight line.
Thus kinematic solutions can at most be approximated by $n$-body solutions locally uniformly in time.

# Main result

Rescale all interactions: $I_{k,\ell} \mapsto \varepsilon I_{k,\ell}$.

## Theorem

*In the limit $\varepsilon \searrow 0$ any transversal kinematic solution is approximated locally uniformly by n-body solutions.*

## Corollary

*Homogeneity of the kinetic and the potential part in $H$ leads to corresponding n-body solutions for any given positive energy, with interaction $I_{k,\ell}$ unchanged but spatial scale $\nearrow \infty$.*

# Asymptotic velocity
(discussion for smooth long-ranged pair potentials)

$$\overline{v} : T^*\mathbb{R}^{nd} \to \mathbb{R}^{nd} \quad , \quad \overline{v}(x) := \lim_{t \to +\infty} \frac{q(t,x)}{t}$$

**Known:**

- Asymptotic velocity $\overline{v}$ exists.
- $\overline{v}$ is discontinuous.

**Theorem (FKM)**

*On the free phase space region $P_{free} := \overline{v}^{-1}(\mathbb{R}^{nd} \setminus \Delta)$ it is smooth.*

# Questions

1. Can we obtain a global topological picture of three-body scattering in celestial mechanics?
2. Can we show for $m$ electrons in a molecule with $n$ fixed nuclei existence of solution $x : \mathbb{R} \to P$ for a.e. initial condition $x_0 \in P$?

# THANK YOU