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Homogeneous systems with quadratic integrals, Lie–Poisson quasi-brackets, and Kowalewski method

(joint with Ivan Bizyaev)

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Linear systems

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n. \tag{1}$$

Proposition. *If this linear system admits as a first integral a nondegenerate quadratic form*

$$H = \frac{1}{2}(Bx, x), \qquad \det B \neq 0,$$

then $\operatorname{div}(Ax) = \operatorname{tr} A = 0$*. Therefore, the phase flow of* (1) *preserves the standard measure in* $\mathbb{R}^n = \{x\}$*.*

Proof. H is a first integral of (1) if and only if

$$BA + AB = 0.$$

Hence, $A = -B^{-1}AB$, and $\operatorname{tr} A = -\operatorname{tr}(B^{-1}AB) = -\operatorname{tr} A$. Then $\operatorname{tr} A = 0$. □

The following result is not so obvious. If the operator A is also degenerate ($\det A \neq 0$), then the linear system (1) is a Hamiltonian system (with respect of some symplectic structure), and H is the Hamiltonian function. In particular, n is even.

You can easily find this symplectic structure yourselves.

If $\det A \neq 0$, and H is a positive definite form, then the system (1) admits $n/2$ independent quadratic first integrals.

Quadratic systems

$$\dot{x}_k = \sum_{i,j} a_{ij}^k x_i x_j, \qquad a_{ij}^k = \text{const}, \quad 1 \leqslant k \leqslant n. \tag{2}$$

This system is invariant under the action of the group of homotheties $x \mapsto \alpha x$, $t \mapsto t/\alpha$; $\alpha \in \mathbb{R} \setminus \{0\}$.

Example. **The Euler–Poincaré equations on the Lie algebra g:**

$$\dot{m}_k = \sum c_{jk}^i m_i \omega_j, \qquad k = 1, \ldots, n. \tag{3}$$

$\omega = (\omega_1, \ldots, \omega_n)$ is the velocity of the mechanical system ($\omega \in g$),
$m = (m_1, \ldots, m_n)$ is the momentum ($m \in g^*$),
$m_p = \sum I_{pq}\omega_q$, where $\|I_{pq}\| = I$ is the inertia tensor,
c_{ij}^k are the structure constants of g.
$H = \frac{1}{2}(I\omega, \omega) = \frac{1}{2}(I^{-1}m, m)$, the kinetic energy.

$$(3) \quad \Longleftrightarrow \quad \dot{m}_k = \{m_k, H\}, \quad 1 \leqslant k \leqslant n.$$

$\{\ ,\ \}$ is the Lie–Poisson bracket:

$$\{m_i, m_j\} = \sum c_{ij}^p m_p.$$

The Jacobi identity:

$$\{\{m_i, m_j\}, m_k\} + \{\{m_j, m_k\}, m_i\} + \{\{m_k, m_i\}, m_j\} = 0.$$

Homogeneous systems with quadratic integrals

$$F = \frac{1}{2}(Ax, x), \qquad H = \frac{1}{2}(Bx, x)$$

F is positive definite,

$$\det(B - \lambda A) = 0, \qquad \lambda_1, \dots, \lambda_n \in \mathbb{R}.$$

Theorem 1. *If $\lambda_i \neq \lambda_j$ for $i \neq j$, then after some linear transformation the system* (2) *takes the following form:*

$$\dot{x}_k = \sum_{i<j} c_{ij}^k (\lambda_i - \lambda_j) x_i x_j, \tag{4}$$

where $c_{ki}^j = c_{jk}^i = c_{ij}^k = -c_{ik}^j = -c_{kj}^i = -c_{ji}^k$.

In fact, Theorem 1 goes back to Vito Volterra (Sopra una classe di equazioni dinamiche // Atti della Reale Accademia delle Scienze di Torino. 1898. V. 33. P. 451–475).

$$\dot{x}_k = \sum c_{ij}^k \, \frac{\partial(H, F)}{\partial(x_i, x_j)} \, .$$

Theorem 2. *The system* (4) *has the Hamiltonian form*

$$\dot{x}_k = \{x_k, H\},$$

where $\{\ ,\ \}$ *is a Lie–Poisson quasi-bracket:*

$$\{x_i, x_j\} = \sum c_{ij}^k x_k.$$

F is a Casimir function: $\{x_j, F\} = 0$ for $j = 1, \ldots, n$.

Theorem 3. *If $\lambda_i \neq \lambda_j$ for $i \neq j$, then the phase flow of quadratic system* (2) *preserves the volume form*

$$dx_1 \wedge \cdots \wedge dx_n.$$

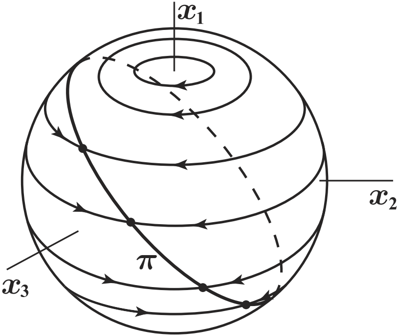
Theorem 3 is not valid if $\lambda_i = \lambda_j$ for some $i \neq j$.

Example $(n = 3)$.

$$\dot{x}_1 = x_2(\alpha x_1 + \beta x_2 + \gamma x_3), \quad \dot{x}_2 = -x_1(\alpha x_1 + \beta x_2 + \gamma x_3), \quad \dot{x}_3 = 0$$

$$F = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2), \quad H = \frac{1}{2}(x_1^2 + x_2^2);$$

$$\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = 0.$$



$\mathrm{div} \not\equiv 0$ if $\alpha^2 + \beta^2 \neq 0$

$\pi = \{x : \alpha x_1 + \beta x_2 + \gamma x_3 = 0\}$

The density of an invariant measure has singularities:

$\rho = |\alpha x_1 + \beta x_2 + \gamma x_3|^{-1}$.

Case $n = 3$

According to Theorem 1

$$\dot{x}_1 = \mu(\lambda_2 - \lambda_3)x_2x_3, \quad \dot{x}_2 = \mu(\lambda_3 - \lambda_1)x_3x_1, \quad \dot{x}_3 = \mu(\lambda_1 - \lambda_2)x_1x_2,$$

where $\mu = c^1_{23} = -c^2_{13} = c^3_{12}$.

If $\mu \neq 0$, then after substitution $x_k \mapsto -x_k/\mu$ we obtain the famous Euler equation from rigid body dynamics.
The Lie–Poisson 'quasi-bracket' is the real bracket on the Lie group SO(3).

Kowalewski method

Let $x_k = \dfrac{c_k}{t}$ $(k = 1, \ldots, n)$ be a solution of (2).

Then $\sum a_{ij}^k c_i c_j = -c_k$ $(1 \leqslant k \leqslant n)$.

Let $(c_1, \ldots, c_n) = c$ be a non-zero solution $(c \in \mathbb{C}^n)$.

Set $v_k = \sum a_{ij}^k x_i x_j$, and $K = \left\| \dfrac{\partial v_i}{\partial x_j}(c) + \delta_{ij} \right\|$.

K is the Kowalewski matrix; eigenvalues $\rho_1, \ldots, \rho_n$ of K are called the Kowalewski exponents: $\det(K - \rho E) = 0$.

Theorem 4. *Kowalewski exponents contain the numbers*

$$-1, 2, 2,$$

and

$$\rho_1 + \rho_2 + \cdots + \rho_n = n.$$

Example $(n = 3)$. $\rho_1 = -1$, $\rho_2 = 2$, $\rho_3 = 2$, $\sum \rho_i = 3$.

Case $n = 4$

$$
\begin{aligned}
\dot{x}_1 &= \mu_2(\lambda_3 - \lambda_4)x_3x_4 + \mu_3(\lambda_2 - \lambda_4)x_2x_4 + \mu_4(\lambda_2 - \lambda_3)x_2x_3, \\
\dot{x}_2 &= \mu_1(\lambda_3 - \lambda_4)x_3x_4 + \mu_3(\lambda_4 - \lambda_1)x_4x_1 + \mu_4(\lambda_3 - \lambda_1)x_3x_1, \\
\dot{x}_3 &= \mu_1(\lambda_4 - \lambda_2)x_4x_2 + \mu_2(\lambda_4 - \lambda_1)x_4x_1 + \mu_4(\lambda_1 - \lambda_2)x_1x_2, \\
\dot{x}_4 &= \mu_1(\lambda_2 - \lambda_3)x_2x_3 + \mu_2(\lambda_1 - \lambda_3)x_1x_3 + \mu_3(\lambda_1 - \lambda_2)x_1x_2.
\end{aligned}
\tag{5}
$$

Here μ_1, μ_2, μ_3, μ_4 are c_{ij}^k (with signs).

Theorem 5.

1°. ***The system* (5) *admits an additional linear integral***

$$G = \mu_1 x_1 - \mu_2 x_2 + \mu_3 x_3 - \mu_4 x_4.$$

2°. ***The system* (5) *admits the representation***

$$\dot{x}_j = \frac{\partial(x_j, G, F, H)}{\partial(x_1, x_2, x_3, x_4)}\,, \qquad 1 \leqslant j \leqslant 4.$$

3°. ***The bracket*** $\{x_i, x_j\} = \sum c_{ij}^k x_k$ ***is the Lie–Poisson bracket of the Lie algebra*** $\mathbb{R} \oplus \mathrm{so}(3)$ ***with the Casimir functions*** G ***and*** F***.***
4°. ***Solutions of* (5) *are elliptic functions of time*** t***.***

The existence of an additional linear integral is predicted by Theorem 4: for $n = 4$ **the Kowalewski exponents are** -1**, 2, 2, and 1.**

Case $n = 5$

In general, for $n = 5$ the quadratic equations are not Euler–Poincaré equations on any Lie algebra g, $\dim g = 5$.
But if

$$\begin{aligned}
\tilde{c}_1 &= c^1_{23}c^1_{45} - c^1_{24}c^1_{35} + c^1_{25}c^1_{34} = 0,\\
\tilde{c}_2 &= c^1_{23}c^2_{45} - c^1_{24}c^2_{35} + c^1_{25}c^2_{34} = 0,\\
\tilde{c}_3 &= c^1_{23}c^3_{45} - c^1_{34}c^2_{35} + c^1_{35}c^2_{34} = 0,\\
\tilde{c}_4 &= c^1_{24}c^3_{45} - c^1_{34}c^2_{45} + c^1_{45}c^2_{34} = 0,\\
\tilde{c}_5 &= c^1_{25}c^3_{45} - c^1_{35}c^2_{45} + c^1_{45}c^2_{35} = 0,
\end{aligned}$$

then there are two linear Casimir functions

$$\begin{aligned}
G_1 &= -c^3_{45}x_1 + c^1_{45}x_3 - c^1_{35}x_4 + c^1_{34}x_5,\\
G_2 &= -c^3_{45}x_2 + c^2_{45}x_3 - c^2_{35}x_4 + c^2_{34}x_5.
\end{aligned}$$

The quadratic equations are integrable Euler–Poincaré equations on the Lie algebra $\mathbb{R}^2 \oplus \mathrm{so}(3)$.

Let

$$N = (\tilde{c}_1 x_1 + \tilde{c}_2 x_2 + \tilde{c}_3 x_3 + \tilde{c}_4 x_4 + \tilde{c}_5 x_5)^{-1}$$

and

$$\{\widetilde{x_i, x_j}\} = N\{x_i, x_j\} = N \sum\nolimits'' c_{ij}^k x_k.$$

The bracket $\{\widetilde{,}\}$ satisfies the Jacobi identity.

Therefore, the equations are presented in conformally Hamiltonian form

$$\dot{x}_k = N^{-1}\{\widetilde{x_k, H}\}, \qquad 1 \leqslant k \leqslant n.$$

After the change of time

$$d\tau = N\, dt$$

we obtain the Hamiltonian equations.

In general case for $n = 5$ the quadratic equations are not integrable.

Case $n = 6$

In general, for $n = 6$ there is no reducing factor N.

Example. $G = E(3)$, $\dim G = 6$. **The Euler–Poincaré equations on g are the well-known Kirchhoff equations (describing the motion of the rigid body in ideal fluid). They admit one more Casimir function, but in general case they are not integrable.**

The case, when Casimir function is not positive definite

First integrals

$$F = \frac{1}{2}(x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2), \quad H = \frac{1}{2}(\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2).$$

Theorem 6. *If $\lambda_i \neq \lambda_j$ for $i < j \leqslant p$ and $p < i < j$, and $\lambda_i + \lambda_j \neq 0$ for $i \leqslant p < j$, then after a linear transformation the quadratic system takes the following form:*

$$\begin{aligned}
\dot{x}_k = {} & \sum_{i<j\leqslant p} c_{ij}^k(\lambda_i - \lambda_j)x_i x_j + \sum_{i\leqslant p<j} c_{ij}^k(\lambda_i + \lambda_j)x_i x_j \\
& + \sum_{p<i<j} c_{ij}^k(\lambda_i - \lambda_j)x_i x_j \quad \textit{for } k \leqslant p, \\
\dot{x}_k = {} & -\sum_{i<j\leqslant p} c_{ij}^k(\lambda_i - \lambda_j)x_i x_j + \sum_{i\leqslant p<j} c_{ij}^k(\lambda_i + \lambda_j)x_i x_j \\
& - \sum_{p<i<j} c_{ij}^k(\lambda_i - \lambda_j)x_i x_j \quad \textit{for } k > p.
\end{aligned}$$

Example (from a letter of S. Kowalewski to G. Mittag-Leffler, 1884).

$$\dot{y}_1 = y_1(-y_1 + y_2 + y_3), \quad \dot{y}_2 = y_2(-y_2 + y_3 + y_1),$$
$$\dot{y}_3 = y_3(-y_3 + y_1 + y_2).$$
$$2F = y_1y_3 + y_2y_3 - 2y_1y_2, \qquad 2H = y_2y_3 - y_1y_2.$$
$$\lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = -\frac{1}{2}\,.$$

In 'canonical' coordinates

$$\dot{x}_1 = -\frac{\mu}{2}x_2x_3, \quad \dot{x}_2 = -\frac{\mu}{2}x_3x_1, \quad \dot{x}_3 = -\mu x_1x_2.$$
$$x_1 \mapsto \frac{\sqrt{2}\,x_1}{\mu}\,, \quad x_2 \mapsto \frac{\sqrt{2}\,x_2}{\mu}\,, \quad x_3 \mapsto -\frac{2x_3}{\mu}\,.$$
$$\dot{x}_1 = x_2x_3, \quad \dot{x}_2 = x_3x_1, \quad \dot{x}_3 = x_1x_2$$

(M. Petrera and Y. Suris, 2015).

These equations are the Euler–Poincaré equations on so(1, 2).