KAM for Derivative Nonlinear Schrödinger Equation with Periodic Boundary Conditions

Jianjun Liu

Sichuan University, Chengdu, P R China

St.Petersburg, June 3-8, 2015

Outline

Equation and result

Consider derivative nonlinear Schrödinger equation with periodic boundary conditions

$$\mathbf{i}u_t + u_{xx} + \mathbf{i}\Big(f(x, u, \bar{u})\Big)_x = 0, \qquad x \in \mathbb{T}, \tag{1.1}$$

where f is an analytic function of the form

$$f(x, u, \bar{u}) = |u|^2 u + f_{\geq 4}(x, u, \bar{u}), \tag{1.2}$$

here $f_{\geq 4}$ denotes the higher order terms.

Equation and result

Moreover, we require that (1.1) can be written into the Hamiltonian form

$$\frac{\partial u}{\partial t} = -\frac{d}{dx}\frac{\partial H}{\partial \bar{u}}, \tag{1.3}$$

$$H = -\mathbf{i}\int_{\mathbb{T}} u_x \bar{u} dx + \frac{1}{2}\int_{\mathbb{T}} |u|^4 dx + \int_{\mathbb{T}} g_{\geq 5}(x, u, \bar{u}) dx. \tag{1.4}$$

Main result

The above equation possesses plenty of smooth quasi-periodic solutions of small amplitude.

Introduction to infinite dimensional KAM theory

Consider an infinite dimensional Hamiltonian

$$H = N + P = \sum_{1\le\nu\le n} \omega_\nu(\xi) y_\nu + \frac{1}{2}\sum_{j\ge 1} \Omega_j(\xi)(u_j^2 + v_j^2) + P(x, y, u, v; \xi)$$

on a phase space

$$\mathcal{P}^{a,p} = \mathbb{T}^n \times \mathbb{R}^n \times \ell^{a,p} \times \ell^{a,p} \ni (x, y, u, v),$$

where $\ell^{a,p}$ is the Hilbert space of all real sequences $w = (w_1, w_2, \cdots)$ with

$$||w||_{a,p}^2 = \sum_{j\ge 1} e^{2aj} j^{2p} |w_j|^2 < \infty.$$

Introduction to infinite dimensional KAM theory

When $P \equiv 0$, the system is integrable and its equations of motion are

$$\dot{x}_\nu = \omega_\nu(\xi), \quad \dot{y}_\nu = 0, \quad \dot{u}_j = -\Omega_j(\xi)v_j, \quad \dot{v}_j = \Omega_j(\xi)u_j.$$

Hence, for each $\xi \in \Pi$, there is an invariant torus $\mathcal{T}_0 = \mathbb{T}^n \times \{0\} \times \{0\} \times \{0\}$. This is a lower dimensional torus with tangential frequencies $(\omega_\nu)_{1\le\nu\le n}$ and normal frequencies $(\Omega_j)_{j\ge1}$. The aim is to prove the persistence of invariant tori for most of the parameters $\xi \in \Pi$.

Introduction to infinite dimensional KAM theory

Roughly,

$$\Omega_j = j^d + \cdots, \qquad d \geq 1, \tag{2.1}$$

and the Hamiltonian vector field $X_P := (P_y, -P_x, -P_v, P_u)$ defines a real analytic map

$$\mathcal{P}^{a,p} \to \mathcal{P}^{a,q}, \qquad \text{let } \delta = p - q. \tag{2.2}$$

Classification of infinite dimensional KAM theory:
when $\delta \leq 0$, P is called bounded perturbation, and corresponds to bounded KAM theory;
when $\delta > 0$, P is called unbounded perturbation, and corresponds to unbounded KAM theory.

Introduction to infinite dimensional KAM theory

Examples of bounded case:

$$(NLW) \qquad u_{tt} - u_{xx} + mu + f(u) = 0, \tag{2.3}$$

$$d = 1, \quad \delta = -1;$$

$$(NLS) \qquad \mathbf{i}u_t - u_{xx} + mu + f(|u|^2)u = 0, \tag{2.4}$$

$$d = 2, \quad \delta = 0.$$

Introduction to infinite dimensional KAM theory

Examples of unbounded case:

$$(KdV) \qquad u_t + u_{xxx} + 6uu_x + \text{ perturbation } = 0, \tag{2.5}$$

$$d = 3, \quad \delta = 1;$$

$$(DNLS) \qquad \mathbf{i}u_t + u_{xx} + f(x, u, \bar{u}, u_x, \bar{u}_x) = 0, \tag{2.6}$$

$$d = 2, \quad \delta = 1;$$

Previous results of unbound KAM theory

- Kuksin, ZAMP 1997, Analysis of Hamiltonian PDEs 2000
  Kuksin's Lemma, $0 < \delta < d - 1$
  Persistence of the finite-gap solutions of KdV equation (2.5)
- Kappeler-Pöschel, KdV&KAM 2003
- Bambusi-Graffi, CMP 2001
  A class of time dependent Schrödinger equation

$$\mathbf{i}\partial_t \psi(x,t) = \Big(-\frac{d^2}{dx^2} + Q(x) + \epsilon V(x, \omega t) \Big) \psi(x,t)$$

$$Q(x) \sim |x|^{\alpha}, |V(x,\phi)| \sim |x|^{\beta}, \beta < \frac{\alpha - 2}{2}$$

Previous results of unbound KAM theory

- L.-Yuan, CPAM 2010, CMP 2011
  Solve homological equations, $0 < \delta = d - 1$
  Quantum Duffing oscillator

$$\mathbf{i}\partial_t\psi(x,t) = \Big(-\frac{d^2}{dx^2} + x^4 + \epsilon x V(\omega t)\Big)\psi(x,t)$$

  DNLS

$$\mathbf{i}u_t + u_{xx} - V * u + \mathbf{i}f(u,\bar{u})u_x = 0, x \in [0,\pi]$$

  Perturbed Benjamin-Ono equation

$$u_t + \mathcal{H}u_{xx} - uu_x + \text{ perturbation } = 0, x \in \mathbb{T}$$

- Zhang-Gao-Yuan, Nonlinearity 2011
  DNLS (reversible system)

$$\mathbf{i}u_t + u_{xx} + |u_x|^2 u = 0, x \in [0,\pi]$$

Previous results of unbound KAM theory

- Baldi-Berti-Montalto,
  Math. Ann 2014, C. R. Math. Acad. Sci. Paris 2014

  $$u_t + u_{xxx} + 6uu_x + \text{ quasi-linear or fully nonlinear perturbation } = 0$$

- Feola-Procesi

  $$\mathbf{i}u_t + u_{xx} + \epsilon f(\omega t, x, u, u_x, u_{xx}) = 0$$

Previous results of unbound KAM theory

Naturally, we consider DNLS with periodic boundary conditions (1.1). However, the multiplicity $\Omega_j^\sharp = 2$ and thus the previous estimates for homological equations seem to be invalid.

- L.-Yuan, JDE 2014

$$\mathbf{i}u_t + u_{xx} + \mathbf{i}\Big(f(|u|^2)u\Big)_x = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{T}$$

the nonlinear $\mathbf{i}\Big(f(|u|^2)u\Big)_x$ does not contain the space variable x explicitly, so that momentum is conserved; consequently, passing to Fourier coefficients, the corresponding Hamiltonian consists of monomials $q_{n_1}\bar{q}_{n_2}q_{n_3}\bar{q}_{n_4}\cdots q_{n_{2r-1}}\bar{q}_{n_{2r}}$ with

$$n_1 - n_2 + n_3 - n_4 + \cdots + n_{2r-1} - n_{2r} = 0.$$

Some ideas on the proof

Passing to Fourier coefficients, (1.1) can be rewritten as

$$\dot{q}_j = -\mathbf{i}\sigma_j \frac{\partial H}{\partial \bar{q}_j}, \quad \sigma_j = \begin{cases} 1, & j \geq 1 \\ -1, & j \leq -1 \end{cases}$$

with the Hamiltonian

$$H = \Lambda + G + K,$$

where

$$\Lambda = \sum_{j \neq 0} \sigma_j j^2 |q_j|^2,$$

$$G = \frac{1}{4\pi} \sum_{\substack{j,k,l,m \neq 0 \\ j-k+l-m=0}} \sqrt{|jklm|}\, q_j \bar{q}_k q_l \bar{q}_m,$$

$$|K| = O(\|q\|^5).$$

Some ideas on the proof

Eliminating 4 order non-resonant terms, we thus get a Birkhoff normal form up to order four

$$H = \Lambda + B + R,$$

where

$$B = \frac{1}{4\pi}\sum_{j\neq 0} j^2|q_j|^4 + \frac{1}{2\pi}\sum_{j,l,j-l\neq 0}|jl||q_j|^2|q_l|^2,$$

$$|R| = O(||q||^5).$$

Some ideas on the proof

For a given index set $J = \{j_1 < j_2 < \cdots < j_n\} \subset \bar{\mathbb{Z}} = \mathbb{Z} \setminus \{0\}$, introduce new symplectic coordinates $(x, y, z, \bar{z})$ by setting

$$\begin{cases} q_{j_b} = \sqrt{\xi_b + y_b} e^{-\mathbf{i}x_b}, \quad \bar{q}_{j_b} = \sqrt{\xi_b + y_b} e^{\mathbf{i}x_b}, \quad b = 1, \cdots, n, \\ q_j = z_j, \quad \bar{q}_j = \bar{z}_j, \quad j \in \mathbb{Z}_* = \bar{\mathbb{Z}} \setminus J, \end{cases}$$

where $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n_+$. Then

$$H = \sum_{1 \le b \le n} \sigma_{j_b} \omega_b y_b + \sum_{j \in \mathbb{Z}_*} \sigma_j \Omega_j z_j \bar{z}_j + \cdots,$$

where

$$\Omega_j = j^2 + \frac{j}{\pi} \sum_{1 \le b \le n} |j_b| \xi_b.$$

$\Omega_j - \Omega_{-j}$ large!

Thank You!