On new phenomenon of chaotic behavior of non-smooth Hamiltonian systems coming from optimal control

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Based on joined work with M.I. Zelikin and R. Hildebrand

2015

Layout

Consider a Hamiltonian system on $\mathcal{M}$ with non-smooth Hamiltonian H

$$H \in C^{\infty}(\mathcal{M} \setminus S) \cap C^{0}(\mathcal{M})$$

Here $\mathcal{M}$ is a symplectic C^{∞}-manifold and S is a stratified submanifold.

$$\operatorname{Int} S = \emptyset.$$

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Let $i : T^*\mathcal{M} \to T\mathcal{M}$ denote the isomorphism induced by symplectic form.

Solutions are defined in the following way. Trajectory $x(t) \in AC$ is a solution if for a.e. t

$$\dot{x}(t) \in i\left(\overline{\operatorname{conv}}\{\text{all limits of } dH(y) \text{ while } y \to x(t)\} \right)$$

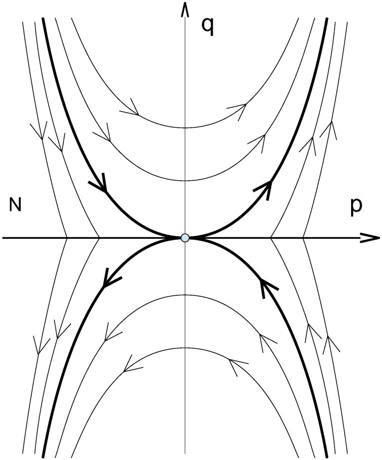
This definition is obviously coincide with classical one if $x(t) \notin S$.

Consider the following simple example with non-smooth potential

$$H = \frac{1}{2}\,p^2 - |q|$$

$$\begin{cases} \dot{q} = p \\ \dot{p} = \operatorname{sign} q \end{cases}$$

There is no uniqueness in the origin. Two trajectories arrive in the origin and two trajectories leave it in finite time.


q
N
p

Consider an optimal control problem for $q \in M$

$$\dot{q} = f(q,u) \qquad q(0) = q_0 \qquad l\big(q(T)\big) \to \inf$$

Here $u = u(t)$ is a control which takes values in a set $U \subset \mathbb{R}^k$ and $l : M \to \mathbb{R}$ is a terminal functional.

Pontryagin's maximum principle for $p \in T_q^*M$ gives

$$H(p,q) = \max_{u \in U} \langle\, p, f(q,u)\,\rangle$$

Any optimal trajectory $\widehat{q}(t)$ has a lift $\big(\widehat{q}(t), \widehat{p}(t)\big)$ to a trajectory of the Hamiltonian system with Hamiltonian H.

Some basic properties of non-smooth Hamiltonian systems.

- Solution exists for every initial conditions (by the Filippov theorem).
- Generally there is no uniqueness of solutions intersecting the manifold of discontinuity S.
- Usually there exists trajectories lying in S. These trajectories are called singular.

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- Solution exists for every initial conditions (by the Filippov theorem).
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Consider simplest case when S is a hypersurface (locally).

- Singular trajectories on a hypersurface are smooth and form a symplectic submanifold of codimension $2h$. The number h is called order of singular trajectories.
- Uniqueness of the solution does not usually hold in points of singular trajectories.
- Uniqueness does hold in other points of the hypersurface S.

Fuller's example

$$\frac{1}{2}\int_0^\infty q^2(t)dt \to \min$$

$$\ddot{q} = u \qquad |u| \leq 1$$

Denote $q_1 = q$, $q_2 = \dot{q}$. Pontryagin's maximum principle gives

$$H(p,q) = -\frac{1}{2}q_1^2 + p^1 q_2 + |p^2|$$

Here $p^1 = p^2 = q_1 = q_1 = u = 0$ is a singular trajectory of second order.

In this case conjugation theorem says that if a trajectory $x(t)$ hits the origin in finite time $t_0 > 0$ i.e.

$$x(t) \neq 0 \quad \text{for} \quad t < t_0 \qquad \text{and} \qquad x(t_0) = 0$$

then its velocity is not partially continuous:

$$\nexists \lim_{t \to t_0 - 0} \dot{x}(t)$$

Absence of uniqueness in the fuller problem: there are two-dimensional integral (Lagrangian) manifolds M^+ and M^-.

- M^+ consists of trajectories which hits the origin in finite (positive) time.
- M^- consists of trajectories starting from the origin.

Let $x(t)$ belongs to M^+ and $x(0) = 0$. Then there exists $t_1 < t_2 < \ldots < 0$, $t_k \to 0$ such that

$$u(t) = \begin{cases} 1, & t \in (t_{2k}, t_{2k+1}); \\ -1, & t \in (t_{2k+1}, t_{2k}). \end{cases}$$

This phenomenon is called chattering.

Consider general non-smooth Hamiltonian structure near a hypersurface. Assume $x_0 \in S$ and S is a hypersurface. In the neighbourhood of x_0

$$H(x) = \begin{cases} F_0(x) + F_1(x) \text{ on the one side of } S; \\ F_0(x) - F_1(x) \text{ on the other side of } S. \end{cases}$$

Here $F_0, F_1 \in C^\infty(\mathcal{M})$, and $S = \{x : F_1(x) = 0\}$.

We are interested in Poisson brackets of F_0, F_1 evaluated at x_0:

$$\{F_i, F_j\}(x_0), \quad \{F_i\{F_j, F_k\}\}(x_0), \quad \ldots$$

Theorem (Kupka, Zelikin-Borisov)

Suppose all brackets up to 5th order vanish in x_0 except

$$\{F_1, (\mathrm{ad}\, F_0)^3 F_1\}(x_0) < 0$$

Then if

$$\frac{(\mathrm{ad}\, F_0)^4 H_1}{\{F_1, (\mathrm{ad}\, F_0)^3 F_1\}} \in [-1; 1]$$

then there exists integral 2-dimensional manifolds $M^+(x_0)$, and $M^-(x_0)$ such that

- *Trajectories in $M^+(x_0)$ hit x_0 in finite time with chattering.*
- *Same picture in $M^-(x_0)$ with backwards time.*

Layout

Suppose that S divides $\mathcal{M}$ into finite number of open domains $\Omega_1, \ldots \Omega_k$: $(\mathcal{M} = \overline{\bigcup \Omega_i})$ and denote $H_i = H|_{\Omega_i}$.

Consider an open set $\mathcal{U} \subset \mathcal{M}$ such that $\mathcal{U}$ contains some parts of only three hypersurfaces $S_{ij} \subset S$, $(i, j = 1, 2, 3)$ which divides the domains Ω_i and Ω_j.

Let S_{ij} be joined by the stratum $S_{123} = \overline{S_{12}} \bigcap \overline{S_{23}} \bigcap \overline{S_{31}}$ of codimension 2.

Here

$$H = \max\{H_1, H_2, H_3\}$$

and

$$\Omega_i = \big\{x : H_i(x) > \max\{H_j(x), H_k(x)\}\big\}.$$

We have a model example from optimal control point of view:

$$\int_0^\infty |q|^2 \, dt \to \min$$

$$\ddot{q} = u \quad u \in U$$

Here $q, u \in \mathbb{R}^2$, U is a triangle, and $0 \in \operatorname{Int} U$

Similar to the Fuller example we receive

$$\begin{aligned} H(p,q) &= \max_{u\in U}\left(-\frac{1}{2}\langle q_1, q_1\rangle + \langle p^1, q_2\rangle + \langle p^2, u\rangle\right) \\ &= -\frac{1}{2}\langle q_1, q_1\rangle + \langle p^1, q_2\rangle + \max_{u\in U}\langle p^2, u\rangle. \end{aligned}$$

Consequently, we see for the model problem that S_{ij} is the set of points where p^2 is perpendicular to the face (ij) of triangle U and

$$S_{123} = \{\psi = 0\}$$

The point $x_0 \in S_{123}$, $H_1(x_0) = H_2(x_0) = H_3(x_0)$, is called strange if the following conditions are fulfilled. Denote

$$F_0 = H_1 + H_2 + H_3;$$

$$F_1 = H_2 - H_3, \qquad F_2 = H_3 - H_1, \qquad F_3 = H_1 - H_2.$$

(i) The commutators of the functions F_i of the fourth order or less vanish at the point x_0 (except $F_0(x_0)$). Their differentials are linearly independent at x_0 (taking into account the conditions of anti-commutativity and the Jacobi conditions).

(ii) Then the symmetrical bilinear form

$$B_{ij} = (\mathrm{ad}F_i)(\mathrm{ad}F_0)^3 F_j|_{x_0},\ i,j = 1,2,3$$

has the rank 2. It is non-positive definite, and proportional to the bilinear form

$$B = \lambda \begin{pmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{pmatrix} \qquad \lambda > 0.$$

All other commutators of the functions F_i of the fifth order (independent from the mentioned above) vanish at x_0.

Layout

The set of trajectories reaching a strange point is described by the following theorem

Theorem (Zelikin, Hildebrand, L.)

Consider a strange point x_0 *of the Hamiltonian system with the piece-wise smooth Hamiltonian* H.

Then in any sufficiently small neighbourhood of the point x_0 *there exists a set* Ξ *such that*

$\boxed{I.}$ For any $y \in \Xi$ there exists a unique trajectory $X(t, y)$ in an interval $t \in [0; T(y)]$, and it reaches x_0 in the finite time $T(y)$:

$$X\big(T(y), y\big) = x_0.$$

Moreover, $X(t, y) \in \Xi$ for $t \in [0, T(y))$ and $X(t, y)$ has the countable number of successive intersections with S:

$$0 < t_1 < t_2 < \ldots < t_k < \ldots < T(y)$$

Moreover $\lim\limits_{k \to \infty} t_k = T(y)$.

$\boxed{II.}$ Denote by $f : \Xi \cap S \to \Xi \cap S$ – the mapping that transfer points $y \in \Xi \cap S$ into points of the following intersection of the trajectory $X(t, y)$ with S, that is $f(y) = X(t_1, y)$. The right-side topological Markov chain Σ_Γ^+ on the graph Γ, that does not depends on x_0 and H, is a quotient of the dynamical system f:

$$\begin{tikzcd}
\Xi \cap S \arrow[r, "f"] \arrow[d, "\Phi_\Gamma"] & \Xi \cap S \arrow[d, "\Phi_\Gamma"] \\
\Sigma_\Gamma^+ \arrow[r, "l"] & \Sigma_\Gamma^+
\end{tikzcd}$$

Φ_Γ is a continuous surjective mapping. The pre-image $\Phi_\Gamma^{-1}(\sigma)$ of each point $\sigma \in \Sigma_\Gamma^+$ is homeomorphic to an open two-dimensional disc D^2, and the diameter of $f^k\big(\Phi_\Gamma^{-1}(\sigma)\big)$ tends to 0 as $k \to +\infty$.

$\boxed{III.}$ If $dF_0(x_0) = 0$ then the Hausdorff and box dimensions do not depend on x_0 and H. The following inequality is valid

$$3,204762 \leq \dim_H \Xi \leq \overline{\dim_B}\Xi \leq 3,407495$$

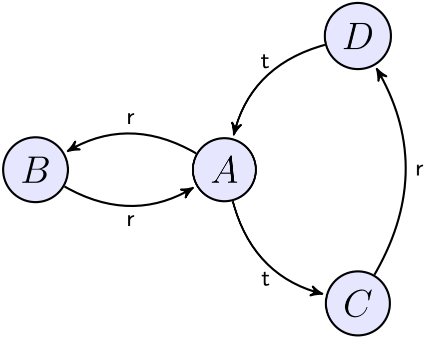
$\boxed{IV.}$ The topological entropy of the Bernoulli shift on l equal

$$h_{\text{top}}(l) = \log_2 r$$

where r is the positive solution of $r^3 - r - 1 = 0$:

$$r = \sqrt[3]{\frac{1}{2} + \frac{\sqrt{69}}{18}} + \sqrt[3]{\frac{1}{2} - \frac{\sqrt{69}}{18}}$$

$\boxed{V.}$ The similar picture with the inversion of the time current takes place for trajectories passing from the strange point point x_0.



In the picture above is shown a prototype of graph Γ.

To construct Γ one should take 24 vertices A_{ij}, B_{ij}, C_{ij} and D_{ij} where $i \neq j$, and $i, j \in \{1, 2\}$ and then connect them by the following rule depending on "r" or "t".

- There is an arrow from $A_{ij} \to B_{kl}$ iff $j = k$ and $l \neq i$ $(A_{12} \to B_{23})$.
- There is an arrow from $A_{ij} \to C_{kl}$ iff $j = k$ and $l = i$ $(A_{12} \to C_{21})$.

Layout

Note

The theorem starts working for Hamiltonian systems with 16 degrees of freedom.
Nonetheless this phenomenon is observed in system with 4 degrees of freedom.

Note

The set $\mathcal{W} \subseteq S_{123}$ *of all strange points of* S_{123} *generate a sub-manifold of the codimension* $\operatorname{codim} \mathcal{W} = 76$ *in* S_{123} *in general situation.*

Note

The definition of the strange point follows immediately the structural stability.